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# ON COMPLEX CONFORMAL CONNECTIONS

Dedicated to Professor Yûsaku Komatu on his sixtieth birthday

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## § 0. Introduction.

Let $M$ be an $n$-dimensional differentiable manifold in which a system of paths is given by

$$\frac{d^2\xi^h}{dt^2}+\Gamma_{ji}^h(\xi)\frac{d\xi^j}{dt}\frac{d\xi^i}{dt}=0.$$

A change of $\Gamma_{ji}^h(=\Gamma_{ij}^h)$ which does not change the system of paths is given by

$$\bar{\Gamma}_{ji}^h=\Gamma_{ji}^h+\delta_j^h p_i+\delta_i^h p_j,$$

where $p_i$ is an arbitrary covector field, and is called a projective change of $\Gamma$. If there exists a covector field $p_i$ such that the curvature tensor of $\bar{\Gamma}_{ji}^h$ vanishes, the manifold is said to be projectively flat.

It is well known (Weyl [6]) that the so-called Weyl projective curvature tensor

$$P_{kji}{}^h=R_{kji}{}^h+\delta_k^h P_{ji}-\delta_j^h P_{ki}-(P_{kj}-P_{jk})\delta_i^h$$

is invariant under a projective change of $\Gamma$, where $R_{kji}{}^h$ is the curvature tensor of $\Gamma$ and

$$P_{ji}=-\frac{n}{n^2-1}R_{ji}+\frac{1}{n^2-1}R_{ij},\qquad R_{ji}=R_{tji}{}^t,$$

and a necessary and sufficient condition for $M$ to be projectively flat is that

$$P_{kji}{}^h=0 \qquad \text{for} \quad n>2$$

and

$$\nabla_k P_{ji}-\nabla_j P_{ki}=0 \qquad \text{for} \quad n=2,$$

$\nabla_k$ denoting the operator of covariant differentiation with respect to $\Gamma$.

If a Riemannian manifold is projectively flat, then it is of constant sectional curvature.

A complex analogue of the above is the following. In an almost complex manifold with structure tensor $F_i{}^h$, an affine connection $\Gamma$ is called an $F$-connection if the almost complex structure tensor $F$ is covariantly constant with

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respect to this connection.

In a complex manifold with a symmetric $F$-connection, we consider a curve $\xi^h(t)$ satisfying differential equations

$$\frac{d^2\xi^h}{dt^2}+\Gamma^h_{ji}(\xi)\frac{d\xi^j}{dt}\frac{d\xi^i}{dt}=\alpha(t)\frac{d\xi^h}{dt}+\beta(t)F_t{}^h\frac{d\xi^t}{dt},$$

where $\alpha(t)$ and $\beta(t)$ are certain functions of the parameter $t$. We call such a curve a holomorphically planar curve. If two symmetric $F$-connections $\Gamma$ and $\bar{\Gamma}$ have all the holomorphically planar curves in common, they are said to be $H$-projectively related to each other.

It is known (Ishihara [3], [4]) that two symmetric $F$-connections $\Gamma$ and $\bar{\Gamma}$ are $H$-projectively related to each other when and only when

$$\bar{\Gamma}^h_{ji}=\Gamma^h_{ji}+\delta^h_j p_i+\delta^h_i p_j+F_j{}^h q_i+F_i{}^h q_j$$

holds for a certain covector field $p_i$, where

$$q_i=-p_t F_i{}^t.$$

We call such a change of $\Gamma$ an $H$-projective change of symmetric $F$-connections. If there exists a covector field $p_i$ such that the curvature tensor of $\bar{\Gamma}$ vanishes, the complex manifold with symmetric $F$-connection is said to be $H$-projectively flat.

It is also known that the $H$-projective curvature tensor of a symmetric $F$-connection $\Gamma$ defined by

$$P_{kji}{}^h=R_{kji}{}^h+\delta^h_k P_{ji}-\delta^h_j P_{ki}-(P_{kj}-P_{jk})\delta^h_i$$
$$+F_k{}^h Q_{ji}-F_j{}^h Q_{ki}-(Q_{kj}-Q_{jk})F_i{}^h$$

is invariant under an $H$-projective change of symmetric $F$-connections, where

$$P_{ji}=-\frac{1}{n+2}\left\{R_{ji}+\frac{2}{n-2}O^{ts}_{ji}(R_{ts}+R_{st})\right\},$$

$$O^{ts}_{ji}=\frac{1}{2}(\delta^t_j\delta^s_i-F_j{}^tF_i{}^s)$$

and

$$Q_{ji}=-P_{jt}F_i{}^t$$

and a necessary and sufficient condition for $M$ $(n\geqq 4)$ with a symmetric $F$-connection to be $H$-projectively flat is that

$$P_{kji}{}^h=0.$$

If a Kähler manifold is $H$-projectively flat, then it is of constant holomorphic sectional curvature.

Let $M$ be an $n$-dimensional Riemannian manifold with metric tensor $g_{ji}$. The change of the metric

$$\bar{g}_{ji}=e^{2p}g_{ji},$$

where $p$ is a certain scalar function, does not change the angle between two vectors at a point and so is called a conformal change of the metric.

If there exists a function $p$ such that the Riemannian manifold with the metric tensor $e^{2p}g_{ji}$ is locally Euclidean, the Riemannian manifold is said to be conformally flat.

It is well known (Weyl [6]) that the so-called Weyl conformal curvature tensor

$$C_{kji}{}^{h}=K_{kji}{}^{h}+\delta_k^h C_{ji}-\delta_j^h C_{ki}+C_k{}^h g_{ji}-C_j{}^h g_{ki}$$

is invariant under a conformal change of $g$, where $K_{kji}{}^{h}$ is the Riemann-Christoffel curvature tensor of $M$ and

$$C_{ji}=-\frac{1}{n-2}K_{ji}+\frac{1}{2(n-1)(n-2)}Kg_{ji},$$

$$C_k{}^h=C_{kt}g^{th}, \qquad K_{ji}=K_{tji}{}^{t}, \qquad K=g^{ji}K_{ji}$$

and a necessary and sufficient condition for $M$ to be conformally flat is that

$$C_{kji}{}^{h}=0 \qquad \text{for} \quad n>3$$

and

$$\nabla_k C_{ji}-\nabla_j C_{ki}=0 \qquad \text{for} \quad n=3,$$

$\nabla_k$ denoting the operator of covariant differentiation with respect to Christoffel symbols formed with $g$.

A complex analogue of the above is not yet known. The main purpose of the present paper is to try to find the complex analogue of the above. It seems to the author that in the complex analogue a curvature tensor introduced by Bochner (Bochner [1], Tachibana [5], Yano and Bochner [8]) in a Kähler manifold plays the rôle of the Weyl conformal curvature tensor in a Riemannian manifold.

In § 1, we state some of fundamental formulas in Riemannian and Kählerian manifolds to fix our notations and in § 2 we study the curvature tensor introduced by Bochner in a Kähler manifold.

In § 3, we introduce what we call complex conformal connections and in § 4 we study the condition for a Kähler manifold to admit a complex conformal connection whose curvature tensor vanishes.

## § 1. Preliminaries.

We consider an $n$-dimensional Kähler manifold $M$ covered by a system of coordinate neighborhoods $\{U;\xi^h\}$, where here and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, n\}$ $(n\geqq 4)$, and denote by $g_{ji}$ and $F_i{}^h$ the components of the Hermitian metric tensor and those of the complex structure tensor of $M$ respectively.

We denote by $\nabla_j$ the operator of covariant differentiation with respect to the Christoffel symbols $\left\{ {h \atop ji} \right\}$ formed with $g_{ji}$, then we have

$$\nabla_k g_{ji}=0, \qquad \nabla_k F_i{}^h=0, \qquad \nabla_k F_{ji}=0, \tag{1.1}$$

where $F_{ji}=F_j{}^t g_{ti}$ and consequently $F_{ji}=-F_{ij}$.

We denote by

$$K_{kji}{}^h=\partial_k\left\{ {h \atop ji} \right\}-\partial_j\left\{ {h \atop ki} \right\}+\left\{ {h \atop kt} \right\}\left\{ {t \atop ji} \right\}-\left\{ {h \atop jt} \right\}\left\{ {t \atop ki} \right\}, \tag{1.2}$$

where $\partial_k=\partial/\partial\xi^k$, the components of the Riemann-Christoffel curvature tensor of $M$.

It is well known that $K_{kji}{}^h$ and $K_{kjih}=K_{kji}{}^t g_{th}$ satisfy

$$K_{kjih}=-K_{jkih}, \qquad K_{kjih}=-K_{kjhi}, \tag{1.3}$$

$$K_{kjih}=K_{ihkj}, \tag{1.4}$$

$$K_{kjih}+K_{jikh}+K_{ikjh}=0 \tag{1.5}$$

and

$$\nabla_l K_{kji}{}^h+\nabla_k K_{jli}{}^h+\nabla_j K_{lki}{}^h=0, \tag{1.6}$$

$$\nabla_t K_{kji}{}^t=\nabla_k K_{ji}-\nabla_j K_{ki}, \tag{1.7}$$

$$2\nabla_t K_k{}^t=\nabla_k K, \tag{1.8}$$

where

$$K_{ji}=K_{ij}=K_{tji}{}^t \quad \text{and} \quad K=g^{ji}K_{ji}$$

are the Ricci tensor and the scalar curvature of $M$ respectively.

In the Kähler manifold $M$, from the Ricci identity

$$\nabla_k\nabla_j F_i{}^h-\nabla_j\nabla_k F_i{}^h=K_{kjt}{}^h F_i{}^t-K_{kji}{}^t F_t{}^h,$$

we have (see, Yano [7], Chapter IV)

$$K_{kjt}{}^h F_i{}^t-K_{kji}{}^t F_t{}^h=0, \tag{1.9}$$

$$K_{kji}{}^h+K_{kjs}{}^t F_i{}^s F_t{}^h=0, \tag{1.10}$$

or

$$K_{kjht}F_i{}^t-K_{kjit}F_h{}^t=0, \tag{1.11}$$

$$K_{kjih}-K_{kjst}F_i{}^s F_h{}^t=0 \tag{1.12}$$

and

$$K_j{}^t F_t{}^h-K_t{}^h F_j{}^t=0, \tag{1.13}$$

$$K_j{}^h + K_s{}^t F_j{}^s F_t{}^h = 0\,, \tag{1.14}$$

or

$$K_{jt} F_i{}^t + K_{it} F_j{}^t = 0\,, \tag{1.15}$$

$$K_{ji} - K_{ts} F_j{}^t F_i{}^s = 0\,, \tag{1.16}$$

where $K_j{}^t = K_{ji} g^{it}$.

If we define $H_i{}^h$ by

$$2H_i{}^h = -K_{kji}{}^h F^{kj}\,, \tag{1.17}$$

where $F^{kj} = g^{kt} F_t{}^j$, (in Yano [7], the $H_i{}^h$ here is denoted by $-H_i{}^h$), then $H_{ih} = H_i{}^t g_{th}$ is given by

$$2H_{ih} = -K_{tsih} F^{ts} = -K_{ihts} F^{ts}\,. \tag{1.18}$$

From (1.5) and (1.18), we find

$$H_{ih} = K_{tihs} F^{ts}\,. \tag{1.19}$$

We also have, from (1.12) and (1.19),

$$K_{ji} = H_{js} F_i{}^s\,, \tag{1.20}$$

$$H_{ji} = -K_{jt} F_i{}^t\,, \tag{1.21}$$

$$H_{ts} F^{ts} = K\,. \tag{1.22}$$

From (1.6) and (1.18), we have

$$\nabla_j H_{ih} + \nabla_i H_{hj} + \nabla_h H_{ji} = 0 \tag{1.23}$$

and from (1.8) and (1.21)

$$2\nabla_t H_i{}^t = (\nabla_t K) F_i{}^t\,. \tag{1.24}$$

If the Kähler manifold $M$ has a constant holomorphic sectional curvature $k$ at each point of the manifold, then we have (Yano [7], Chapter IV)

$$Z_{kji}{}^h = K_{kji}{}^h - \frac{k}{4}(\delta_k^h g_{ji} - \delta_j^h g_{ki} + F_k{}^h F_{ji} - F_j{}^h F_{ki} - 2F_{kj} F_i{}^h) = 0 \tag{1.25}$$

and consequently

$$Z_{kjih} = K_{kjih} - \frac{k}{4}(g_{kh} g_{ji} - g_{jh} g_{ki} + F_{kh} F_{ji} - F_{jh} F_{ki} - 2F_{kj} F_{ih}) = 0\,, \tag{1.26}$$

where $k$ is an absolute constant. A Kähler manifold of constant holomorphic sectional curvature is an Einstein space:

$$K_{ji} = \frac{n+2}{4} k g_{ji}\,. \tag{1.27}$$

## § 2. Bochner curvature tensor.

We now consider the so-called Bochner curvature tensor (Bochner [1], Tachibana [5], Yano and Bochner [8]) deflned by

$$B_{kji}{}^{h}=K_{kji}{}^{h}+\delta_{k}^{h}L_{ji}-\delta_{j}^{h}L_{ki}+L_{k}{}^{h}g_{ji}-L_{j}{}^{h}g_{ki} \\ +F_{k}{}^{h}M_{ji}-F_{j}{}^{h}M_{ki}+M_{k}{}^{h}F_{ji}-M_{j}{}^{h}F_{ki} \\ -2(M_{kj}F_{i}{}^{h}+F_{kj}M_{i}{}^{h}), \tag{2.1}$$

where

$$L_{ji}=-\frac{1}{n+4}K_{ji}+\frac{1}{2(n+2)(n+4)}Kg_{ji}, \tag{2.2}$$

$$M_{ji}=-L_{jt}F_{i}{}^{t}, \tag{2.3}$$

that is,

$$M_{ji}=-\frac{1}{n+4}H_{ji}+\frac{1}{2(n+2)(n+4)}KF_{ji}, \tag{2.4}$$

and

$$L_{k}{}^{h}=L_{kt}g^{th}, \qquad M_{k}{}^{h}=M_{kt}g^{th}. \tag{2.5}$$

From (2.2) and (2.4), we have, using (1.22),

$$g^{ji}L_{ji}=-\frac{1}{2(n+2)}K \tag{2.6}$$

and

$$F^{ji}M_{ji}=-\frac{1}{2(n+2)}K \tag{2.7}$$

respectivelly. It will be easily seen that $B_{kji}{}^{h}$ and

$$B_{kjih}=B_{kji}{}^{t}g_{th} \\ =K_{kjih}+g_{kh}L_{ji}-g_{jh}L_{ki}+L_{kh}g_{ji}-L_{jh}g_{ki} \\ +F_{kh}M_{ji}-F_{jh}M_{ki}+M_{kh}F_{ji}-M_{jh}F_{ki} \\ -2(M_{kj}F_{ih}+F_{kj}M_{ih}) \tag{2.8}$$

satisfy

$$B_{kjih}=-B_{jkih}, \qquad B_{kjih}=-B_{kjhi}, \tag{2.9}$$

$$B_{kjih}=B_{ihkj}, \tag{2.10}$$

$$B_{kjih}+B_{jikh}+B_{ikjh}=0, \tag{2.11}$$

$$B_{tji}{}^{t}=0, \tag{2.12}$$

$$B_{kjt}{}^{h}F_{i}{}^{t}-B_{kji}{}^{t}F_{t}{}^{h}=0, \tag{2.13}$$

$$B_{kji}{}^{h}+B_{kjs}{}^{t}F_{i}{}^{s}F_{t}{}^{h}=0\,, \tag{2.14}$$

or

$$B_{kjht}F_{i}{}^{t}-B_{kjit}F_{h}{}^{t}=0\,, \tag{2.15}$$

$$B_{kjih}-B_{kjts}F_{i}{}^{t}F_{h}{}^{s}=0 \tag{2.16}$$

and

$$B_{kjts}F^{ts}=0\,,\qquad B_{tjis}F^{ts}=0\,. \tag{2.17}$$

From (2.2), we have

$$\nabla_k L_{ji}-\nabla_j L_{ki}=-\frac{1}{n+4}(\nabla_k K_{ji}-\nabla_j K_{ki}) + \frac{1}{2(n+2)(n+4)}[(\nabla_k K)g_{ji}-(\nabla_j K)g_{ki}]\,, \tag{2.18}$$

from which, using (1.8) and (2.6),

$$\nabla_t L_j{}^t=-\frac{n+1}{2(n+2)(n+4)}\nabla_j K\,. \tag{2.19}$$

From (2.4), we have

$$\nabla_k M_{ji}-\nabla_j M_{ki}=-\frac{1}{n+4}(\nabla_k H_{ji}-\nabla_j H_{ki}) + \frac{1}{2(n+2)(n+4)}[(\nabla_k K)F_{ji}-(\nabla_j K)F_{ki}]\,, \tag{2.20}$$

from which, using (1.24),

$$\nabla_t M_j{}^t=-\frac{n+1}{2(n+2)(n+4)}(\nabla_t K)F_j{}^t\,. \tag{2.21}$$

On the other hand, we have, from (2.4),

$$\nabla_t M_{ji}=-\frac{1}{n+4}\nabla_t H_{ji}+\frac{1}{2(n+2)(n+4)}(\nabla_t K)F_{ji}\,,$$

from which,

$$F_k{}^t\nabla_t M_{ji}=-\frac{1}{n+4}F_k{}^t\nabla_t H_{ji}+\frac{1}{2(n+2)(n+4)}F_k{}^t(\nabla_t K)F_{ji}\,,$$

or, using

$$\nabla_t H_{ji}=-(\nabla_j H_{it}-\nabla_i H_{jt})$$

obtained from (1.23),

$$F_k{}^t\nabla_t M_{ji}=\frac{1}{n+4}F_k{}^t(\nabla_j H_{it}-\nabla_i H_{jt})+\frac{1}{2(n+2)(n+4)}F_k{}^t(\nabla_t K)F_{ji}\,,$$

or using (1.20),

$$F_k{}^t\nabla_t M_{ji}=\frac{1}{n+4}(\nabla_j K_{ik}-\nabla_i K_{jk})+\frac{1}{2(n+2)(n+4)}F_k{}^t(\nabla_t K)F_{ji}\,. \tag{2.22}$$

We now compute $\nabla_t B_{kji}{}^t$. From (2.1), we have, using (2.19) and (2.21),

$$\begin{aligned}(2.23)\qquad \nabla_t B_{ki}{}^t =& \nabla_t K_{kji}{}^t + \nabla_k L_{ji} - \nabla_j L_{ki}\\ &-\frac{n+1}{2(n+2)(n+4)}[(\nabla_k K)g_{ji}-(\nabla_j K)g_{ki}]+F_k{}^t\nabla_t M_{ji}-F_j{}^t\nabla_t M_{ki}\\ &-\frac{n+1}{2(n+2)(n+4)}[(\nabla_t K)F_k{}^tF_{ji}-(\nabla_t K)F_j{}^tF_{ki}]\\ &-2\Big[(\nabla_t M_{kj})F_i{}^t-\frac{n+1}{2(n+2)(n+4)}F_{kj}(\nabla_t K)F_i{}^t\Big].\end{aligned}$$

But, using (2.22), we have

$$\begin{aligned}&F_k{}^t\nabla_t M_{ji}-F_j{}^t\nabla_t M_{ki}-2(\nabla_t M_{kj})F_i{}^t\\ &=\frac{1}{n+4}(\nabla_j K_{ik}-\nabla_i K_{jk})+\frac{1}{2(n+2)(n+4)}F_k{}^t(\nabla_t K)F_{ji}\\ &\quad-\frac{1}{n+4}(\nabla_k K_{ij}-\nabla_i K_{kj})-\frac{1}{2(n+2)(n+4)}F_j{}^t(\nabla_t K)F_{ki}\\ &\quad-\frac{2}{n+4}(\nabla_k K_{ji}-\nabla_j K_{ki})-\frac{1}{(n+2)(n+4)}F_i{}^t(\nabla_t K)F_{kj},\end{aligned}$$

that is,

$$\begin{aligned}(2.24)\qquad &F_k{}^t\nabla_t M_{ji}-F_j{}^t\nabla_t M_{ki}-2(\nabla_t M_{kj})F_i{}^t\\ &=-\frac{3}{n+4}(\nabla_k K_{ji}-\nabla_j K_{ki})\\ &\quad+\frac{1}{2(n+2)(n+4)}[F_k{}^t(\nabla_t K)F_{ji}-F_j{}^t(\nabla_t K)F_{ki}-2F_{kj}(\nabla_t K)F_i{}^t].\end{aligned}$$

Consequently, (2.23) can be written as

$$\begin{aligned}\nabla_t B_{kji}{}^t =& \nabla_t K_{kji}{}^t + \nabla_k L_{ji} - \nabla_j L_{ki}\\ &-\frac{3}{n+4}(\nabla_k K_{ji}-\nabla_i K_{ki})\\ &-\frac{n+1}{2(n+2)(n+4)}[(\nabla_k K)g_{ji}-(\nabla_j K)g_{ki}]\\ &-\frac{n}{2(n+2)(n+4)}[(\nabla_t K)F_k{}^tF_{ji}-(\nabla_t K)F_j{}^tF_{ki}-2F_{kj}(\nabla_t K)F_i{}^t]\end{aligned}$$

or, by (1.7)

$$\begin{aligned}(2.25)\qquad \nabla_t B_{kji}{}^t =& -n\Big[\nabla_k L_{ji}-\nabla_j L_{kj}\\ &+\frac{1}{2(n+2)(n+4)}(F_k{}^hF_{ji}-F_j{}^hF_{ki}-2F_{kj}F_i{}^h)(\nabla_h K)\Big]\end{aligned}$$

(Tachibana [5]).

## § 3. Complex conformal connections.

We consider an affine connection in a Kähler manifold $M$ and denote by $\Gamma^h_{ji}$ the components of the connection and by $D_j$ the operator of covariant differentiation with respect to $\Gamma^h_{ji}$.

We notice first of all that an affine connection which is metric, that is, which satisfies

$$D_k g_{ji}=0 \tag{3.1}$$

and whose torsion tensor is a given tensor

$$\frac{1}{2}(\Gamma^h_{ji}-\Gamma^h_{ij})=S_{ji}{}^h \tag{3.2}$$

is uniquely determined and is given by

$$\Gamma^h_{ji}=\left\{ {h \atop ji} \right\}+S_{ji}{}^h+S^h{}_{ji}+S^h{}_{ij}\,, \tag{3.3}$$

where

$$S^h{}_{ji}=S_{tj}{}^s g^{th} g_{si}\,, \tag{3.4}$$

(Hayden [2]).

We consider a conformal change of Hermitian metric

$$\bar{g}_{ji}=e^{2p}g_{ji}\,,\qquad \bar{F}_i{}^h=F_i{}^h\,,\qquad \bar{F}_{ji}=e^{2p}F_{ji}\,, \tag{3.5}$$

where $p$ is a scalar function and we look for an affine connection such that

$$D_k\bar{g}_{ji}=0 \tag{3.6}$$

and the torsion tensor $S_{ji}{}^h$ is given by

$$S_{ji}{}^h=-F_{ji}q^h\,, \tag{3.7}$$

where $q^h$ are components of a vector field.

By the remark above, we have, for the components $\Gamma^h_{ji}$ of this affine connection,

$$\Gamma^h_{ji}=\left\{ {\bar{h} \atop ji} \right\}-F_{ji}q^h-F^h{}_jq_i-F^h{}_iq_j\,, \tag{3.8}$$

where $\left\{ {\bar{h} \atop ji} \right\}$ are the Christoffel symbols formed with $\bar{g}_{ji}=e^{2p}g_{ji}$ and

$$F^h{}_j=g^{ht}F_{tj}\,,\qquad q_i=q^tg_{ti}\,,$$

or

$$\Gamma^h_{ji}=\left\{ {h \atop ji} \right\}+\delta^h_j p_i+\delta^h_i p_j-g_{ji}p^h+F_j{}^hq_i+F_i{}^hq_j-F_{ji}q^h\,, \tag{3.9}$$

where

$$p_i=\partial_i p\,,\qquad p^h=p_t g^{th}\,.$$

We now compute $D_k\bar{F}_{ji}$ and find

$$D_k\bar{F}_{ji}=D_k e^{2p}F_{ji}$$
$$=e^{2p}[-g_{kj}(p_t F_i{}^t+q_i)+g_{ki}(p_t F_j{}^t+q_j)$$
$$+F_{kj}(p_i-q_t F_i{}^t)-F_{ki}(p_j-q_t F_j{}^t)]\,.$$

Thus, in order that $D_k\bar{F}_{ji}=0$, we must have

$$-g_{kj}(p_t F_i{}^t+q_i)+g_{ki}(p_t F_j{}^t+q_j)+F_{kj}(p_i-q_t F_i{}^t)-F_{ki}(p_j-q_t F_j{}^t)=0\,,$$

from which, transvecting with $g^{kj}$, we find

$$(n-2)(p_t F_i{}^t+q_i)=0\,,$$

that is, since $n\geqq 4$,

(3.10) $$q_i=-p_t F_i{}^t\,,\qquad p_i=q_t F_i{}^t\,.$$

The converse being evident, we have

PROPOSITION 3.1. *In a Kähler manifold with Hermitian metric tensor $g_{ji}$, and complex structure tensor $F_i{}^h$, the affine connection which satisfies*

$$D_k e^{2p}g_{ji}=0\,,\qquad D_k e^{2p}F_{ji}=0\,,$$

*and*

$$\Gamma^h_{ji}-\Gamma^h_{ij}=-2F_{ji}q^h\,,$$

*where $p$ is a scalar function and $q^h$ is a vector field, is given by*

$$\Gamma^h_{ji}=\left\{ {h \atop ji} \right\}+\delta^h_j p_i+\delta^h_i p_j-g_{ji}p_h+F_j{}^h q_i+F_i{}^h q_j-F_{ji}q^h\,,$$

*where*

$$p_i=\partial_i p\,,\qquad p^h=p_t g^{th}\,,\qquad q_i=-p_t F_i{}^t\,,\qquad q^h=q_t g^{th}\,.$$

We call such an affine connection a *complex conformal connection.*

## § 4. Curvature tensor of a complex conformal connection.

We consider a complex conformal connection

(4.1) $$\Gamma^h_{ji}=\left\{ {h \atop ji} \right\}+\delta^h_j p_i+\delta^h_i p_j-g_{ji}p^h+F_j{}^h q_i+F_i{}^h q_j-F_{ji}q^h\,,$$

where $p_i=\partial_i p$, $p^h=p_t g^{th}$, $q_i=-p_t F_i{}^t$ and $q^h=q_t g^{th}$, $p$ being a scalar function, in a Kähler manifold and compute the curvature tensor of $\Gamma^h_{ji}$:

(4.2) $$R_{kji}{}^h\partial={}_k\Gamma^h_{ji}-\partial_j\Gamma^h_{ki}+\Gamma^h_{kt}\Gamma^t_{ji}-\Gamma^h_{jt}\Gamma^t_{ki}\,.$$

By a straightforward computation, we find

$$R_{kji}{}^{h}=K_{kji}{}^{h}-\delta_k^h p_{ji}+\delta_j^h p_{ki}-p_k{}^h g_{ji}+p_j{}^h g_{ki} -F_k{}^h q_{ji}+F_j{}^h q_{ki}-q_k{}^h F_{ji}+q_j{}^h F_{ki} +(\nabla_k q_j-\nabla_j q_k)F_i{}^h-2F_{kj}(p_i q^h-q_i p^h)\,, \tag{4.3}$$

where

$$p_{ji}=\nabla_j p_i-p_j p_i+q_j q_i+\frac{1}{2}p_t p^t g_{ji}\,, \tag{4.4}$$

$$q_{ji}=\nabla_j q_i-p_j q_i-q_j p_i+\frac{1}{2}p_t p^t F_{ji}\,, \tag{4.5}$$

consequently

$$q_{ji}=-p_{jt}F_i{}^t\,,\qquad p_{ji}=q_{jt}F_i{}^t \tag{4.6}$$

and

$$p_k{}^h=p_{kt}g^{th}\,,\qquad q_k{}^h=q_{kt}g^{th}\,. \tag{4.7}$$

Thus if we assume that

$$R_{kji}{}^h=0\,, \tag{4.8}$$

then we have

$$K_{kji}{}^h=\delta_k^h p_{ji}-\delta_j^h p_{ki}+p_k{}^h g_{ji}-p_j{}^h g_{ki} +F_k{}^h q_{ji}-F_j{}^h q_{ki}+q_k{}^h F_{ji}-q_j{}^h F_{ki} +\alpha_{kj}F_i{}^h+F_{kj}\beta_i{}^h\,, \tag{4.9}$$

where

$$\alpha_{kj}=-(\nabla_k q_j-\nabla_j q_k)\,, \tag{4.10}$$

$$\beta_i{}^h=2(p_i q^h-q_i p^h) \tag{4.11}$$

and consequently

$$\beta_{ih}=\beta_i{}^t g_{th}=2(p_i q_h-q_i p_h)\,. \tag{4.12}$$

From (4.10) and (4.12), we have respectively

$$\alpha=F^{kj}\alpha_{kj}=-2\nabla_t p^t\,, \tag{4.13}$$

and

$$\beta=F^{kj}\beta_{kj}=4p_t p^t\,. \tag{4.14}$$

From (4.9), we have

$$K_{kjih}=g_{kh}p_{ji}-g_{jh}p_{ki}+p_{kh}g_{ji}-p_{jh}g_{ki} +F_{kh}q_{ji}-F_{jh}q_{ki}+q_{kh}F_{ji}-q_{jh}F_{ki} \tag{4.15}$$

$$+\alpha_{kj}F_{ih}+F_{kj}\beta_{ih}.$$

From (1.4) and (4.15), using

$$p_{ji}-p_{ij}=0\,, \tag{4.16}$$

we find

$$\begin{aligned} F_{kh}(q_{ji}+q_{ij})-F_{jh}(q_{ki}+q_{ik}) & \\ +(q_{kh}+q_{hk})F_{ji}-(q_{jh}+q_{hj})F_{ki} & \\ +(\alpha_{kj}-\beta_{kj})F_{ih}-F_{kj}(\alpha_{ih}-\beta_{ih})=0\,. & \end{aligned} \tag{4.17}$$

Transvecting (4.17) with $F^{kh}$, we find

$$(n-2)(q_{ji}+q_{ij})=0\,,$$

and consequently

$$q_{ji}+q_{ij}=0\,, \tag{4.18}$$

which can also be written as

$$p_{jt}F_i{}^t+p_{it}F_j{}^t=0\,, \tag{4.19}$$

from which

$$p_{ji}=p_{ts}F_j{}^tF_i{}^s\,. \tag{4.20}$$

From (4.17) and (4.18), we find

$$(\alpha_{kj}-\beta_{kj})F_{ih}-F_{kj}(\alpha_{ih}-\beta_{ih})=0\,,$$

from which, by transvection with $F^{kj}$,

$$(\alpha-\beta)F_{ih}-n(\alpha_{ih}-\beta_{ih})=0\,,$$

or

$$\alpha_{ih}-\beta_{ih}=\frac{1}{n}(\alpha-\beta)F_{ih}\,,$$

or, using (4.13) and (4.14),

$$\alpha_{ih}-\beta_{ih}=-\frac{2}{n}(\nabla_t p^t+2p_t p^t)F_{ih}\,. \tag{4.21}$$

On the other hand, from (4.5), (4.10) and (4.18), we find

$$\alpha_{ji}=-2q_{ji}+p_t p^t F_{ji}\,, \tag{4.22}$$

from which, using

$$F^{ji}q_{ji}=F^{ji}(-p_{jt}F_i{}^t)=p_t{}^t\,, \tag{4.23}$$

we have

$$\alpha=-2p_t{}^t+np_t p^t\,. \tag{4.24}$$

From (4.21) and (4.22), we find

$$\beta_{ji}=-2q_{ji}+p_tp^tF_{ji}+\frac{2}{n}(\nabla_tp^t+2p_tp^t)F_{ji}$$

$$=-2q_{ji}+\left[\frac{2}{n}\nabla_tp^t+\frac{n+4}{n}p_tp^t\right]F_{ji},$$

from which, using

$$p_t{}^t=\nabla_tp^t+\frac{n}{2}p_tp^t, \tag{4.25}$$

we find

$$\beta_{ji}=-2q_{ji}+\frac{2}{n}(p_t{}^t+2p_tp^t)F_{ji}. \tag{4.26}$$

Now, from (1.5) and (4.15), we find, using (4.16) and (4.18),

$$\begin{aligned}2(F_{kh}q_{ji}+F_{jh}q_{ik}+F_{ih}q_{kj}+q_{kh}F_{ji}+q_{jh}F_{ik}+q_{ih}F_{kj})\\+F_{kh}\alpha_{ji}+F_{jh}\alpha_{ik}+F_{ih}\alpha_{kj}\\+\beta_{kh}F_{ji}+\beta_{jh}F_{ik}+\beta_{ih}F_{kj}=0.\end{aligned} \tag{4.27}$$

Substituting (4.22) and (4.26) into (4.27), we obtain

$$[2p_t{}^t+(n+4)p_tp^t](F_{kh}F_{ji}+F_{jh}F_{ik}+F_{ih}F_{kj})=0,$$

from which

$$2p_t{}^t+(n+4)p_tp^t=0. \tag{4.28}$$

In (4.9), we contract with respect to $h$ and $k$ and use (4.6), then we obtain

$$K_{ji}=np_{ji}+p_t{}^tg_{ji}-\alpha_{jt}F_i{}^t-\beta_{it}F_j{}^t,$$

from which, substituting (4.22) and (4.26),

$$K_{ji}=np_{ji}+p_t{}^tg_{ji}-(-2q_{jt}+p_sp^sF_{jt})F_i{}^t$$

$$-\left[-2q_{it}+\frac{2}{n}(p_s{}^s+2p_sp^s)F_{it}\right]F_j{}^t,$$

or, using (4.6),

$$K_{ji}=(n+4)p_{ji}+\left(\frac{n-2}{n}p_t{}^t-\frac{n+4}{n}p_tp^t\right)g_{ji},$$

or, using (4.28),

$$K_{ji}=(n+4)p_{ji}+p_t{}^tg_{ji}, \tag{4.29}$$

from which

$$K=2(n+2)p_t{}^t. \tag{4.30}$$

Substituting

$$p_t{}^t=\frac{1}{2(n+2)}K$$

obtained from (4.30) into (4.29), we find

$$p_{ji}=-L_{ji}\,, \tag{4.31}$$

where $L_{ji}$ is the tensor defined by (2.2). From (4.31), we find, using (4.6),

$$q_{ji}=-M_{ji}\,, \tag{4.32}$$

where $M_{ji}$ is the tensor defined by (2.4).

From (4.22) and (4.32), we find

$$\alpha_{ji}=2M_{ji}+p_t p^t F_{ji}\,,$$

or, using (4.28),

$$\alpha_{ji}=2M_{ji}-\frac{2}{n+4}p_t{}^t F_{ji}\,, \tag{4.33}$$

or, using (4.30),

$$\alpha_{ji}=2M_{ji}-\frac{K}{(n+2)(n+4)}F_{ji}\,. \tag{4.34}$$

From (4.26) and (4.32), we find

$$\beta_{ji}=2M_{ji}+\frac{2}{n}(p_t{}^t+2p_t p^t)F_{ji}\,,$$

or, using (4.28),

$$\beta_{ji}=2M_{ji}+\frac{2}{n+4}p_t{}^t F_{ji}\,, \tag{4.35}$$

or, using (4.30),

$$\beta_{ji}=2M_{ji}+\frac{K}{(n+2)(n+4)}F_{ji}\,. \tag{4.36}$$

Substituting (4.31), (4.32), (4.34) and (4.36) into (4.9), we find

$$\begin{aligned}K_{kji}{}^h=&-\delta_k^h L_{ji}+\delta_j^h L_{ki}-L_k{}^h g_{ji}+L_j{}^h g_{ki}\\&-F_k{}^h M_{ji}+F_j{}^h M_{ki}-M_k{}^h F_{ji}+M_j{}^h F_{ki}\\&+2(M_{kj}F_i{}^h+F_{kj}M_i{}^h)\,,\end{aligned}$$

that is,

$$B_{kji}{}^h=0\,. \tag{4.37}$$

Thus, we have

THEOREM 4.1. *If, in an n-dimensional Kähler manifold ($n\geqq 4$), there exists a scalar function $p$ such that the complex conformal connection*

$$\Gamma_{ji}^h=\left\{ {h \atop ji} \right\}+\delta_j^h p_i+\delta_i^h p_j-g_{ji}p^h+F_j{}^h q_i+F_i{}^h q_j-F_{ji}q^h\,,$$

*where* $p_i = \partial_i p$, $p^h = p_t g^{th}$, $q_i = -p_t F_i{}^t$ *and* $q^h = q_t g^{th}$, *is of zero curvature, then the Bochner curvature tensor of the manifold vanishes.*

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