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BOUNDED BIHARMONIC FUNCTIONS ON THE POINCARÉ N-BALL

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An important role in the harmonic and biharmonic classification theory of Riemannian manifolds is played by the Poincaré N-ball B_α^N, that is, the manifold $\{x=(x^1, \cdots, x^N) \mid r=|x|<1\}$ endowed with the metric $ds=\lambda(r)|dx|$, $\lambda(r)=(1-r^2)^\alpha$, α a real constant, and $|dx|$ the Euclidean metric. The existence of harmonic and quasiharmonic functions with various boundedness properties on B_α^N has been completely characterized in terms of α, and so has the existence of biharmonic functions which are positive or have a finite Dirichlet integral (Sario, Wang [22], [24], [25], Hada, Sario, Wang [1], [2]). In contrast, the existence of bounded biharmonic functions has remained an open problem. The difficulty lies in the fact that the space of these functions is not a Hilbert space. The purpose of the present paper is to give a complete solution to this problem.

It will be necessary to divide the investigation into the following eight cases, which require a variety of different methods.

Case I: $\alpha<-1$.
Case II: $\alpha>3/(N-4)$.
Case III: $-1<\alpha<1/(N-2)$.
Case IV: $1/(N-2)<\alpha<3/(N-4)$, and α is not an integral multiple of $1/(N-2)$.
Case V: $1/(N-2)<\alpha<3/(N-4)$, and α is an integral multiple of $1/(N-2)$.
Case VI: $\alpha=1/(N-2)$.
Case VII: $\alpha=3/(N-4)$.
Case VIII: $\alpha=-1$.

The solutions in Cases I and II will be based on the use of testing functions and on the self-adjointness of the Laplace-Beltrami operator $\Delta=\delta d$.

Case III is a consequence of what is already known about the existence of bounded quasiharmonic functions on B_α^N.

In Cases IV and V we expand the solutions of a differential equation at the boundary in order to determine their boundedness. In Case V the roots of the

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indicial equation differ by an integer, and the convergence proof requires more delicate estimates than in Case IV.

Case VI is solved by using the reasoning developed in Cases IV and V.

The most intriguing cases are VII and especially VIII. The absence of Hilbert space methods necessitates the construction of all biharmonic functions and an estimation of their orders of growth.

The outcome of our study is as follows. Let O_{H^2B} and $\tilde{O}_{H^2B}$ be the classes of Riemannian manifolds for which the class H^2B of bounded nonharmonic biharmonic functions is void or nonvoid, respectively. Then

$$B_\alpha^N \in \tilde{O}_{H^2B} \Leftrightarrow \begin{cases} \alpha > -1 & \text{for } N=2, 3, 4, \\ -1<\alpha<\dfrac{3}{N-4} & \text{for } N>4. \end{cases}$$

This result will have consequences in several directions of biharmonic classification theory. These applications, an elaborate problem in its own right, will be discussed in later studies.

For a convenient survey of recent work on biharmonic classification theory we append a Bibliography.

1. We start by recalling some properties of harmonic functions on B_α^N. Let (r, θ), $\theta=(\theta^1, \cdots, \theta^{N-1})$, be the polar coordinates. A function $S_n(\theta)$ is, by definition, a spherical harmonic of degree n if $\Delta_0(r^n S_n(\theta))=0$, where Δ_0 is the Laplace-Beltrami operator relative to the Euclidean metric. Denote by H the class of harmonic functions. Then $f_n(r)S_n(\theta)\in H$, $n\geqq 0$, $S_n\not\equiv 0$, if

$$f_n(r)=r^n+\sum_{i=1}^{\infty} c_{n,2i} r^{n+2i},$$

$$c_{n,2i}=\prod_{j=1}^{i} \frac{(n-2+2j)(n+N-4+2(N-2)\alpha+2j)-n(n+N-2)}{(n+2j)(n+N-2+2j)-n(n+N-2)}.$$

In fact,

$$\begin{aligned}\Delta(f_n S_n)&=(\Delta f_n)S_n+f_n\Delta S_n\\ &=-\lambda^{-2}\Big[f_n''+\Big(\frac{N-1}{r}+\frac{(N-2)\lambda'}{\lambda}\Big)f_n'-n(n+N-2)r^{-2}f_n\Big]S_n=0,\end{aligned}$$

which gives

$$\begin{aligned} r^2(1-r^2)f_n''(r)+r[(N-1)-(N-1+2(N-2)\alpha)r^2]f_n'(r)&\\ -n(n+N-2)(1-r^2)f_n(r)&=0.\end{aligned}$$

Since $r=0$ is a regular singular point, there is a power series solution of the form $f_n=r^p\sum_{i=0}^{\infty} c_{ni}r^i$, $c_{n0}=1$. On substituting this into our equation and equating to 0 the coefficient of the lowest power we obtain the indicial equation with roots $p=n$, $-(n+N-2)$. Since $f_n S_n$ must be harmonic at the origin, only $p=n$

qualifies, and we have the asserted expansion for f_n, with the c_{ni} obtained by annihilating the coefficients of the other powers of r.

It is clear that if $n\geqq 0$, then $f_n(r)>0$ for every $0<r<1$, and $\lim_{r\to 1}f_n(r)>0$. In fact, if $f_n(r)=0$ for some such r, or $\lim_{r\to 1}f_n(r)=0$, then by $f_nS_n\in H$ we would have $f_nS_n\equiv 0$ for $S_n\not\equiv 0$, hence $f_n\equiv 0$, in violation of $f_n(r)/r^n\to 1$ as $r\to 0$. This also shows that f_n is actually positive for $0<r<1$. Observe that, in particular, f_n is bounded away from 0 in some neighborhood of 1.

Every $h\in H(B_\alpha^N)$ has an expansion $h=\sum_{n=0}^\infty f_n(r)S_n(\theta)$. This follows from such an expansion on $\{|x|=r_0<1\}$, its harmonic extension to $\{|x|\leqq r_0\}$, which exists since $f_n(r)\neq 0$, $0<r<1$, and the invariance of the S_n's in the expansion as r_0 varies.

2. Note that for $N=2$ or $\alpha=0$, we have $c_{ni}=0$ for all $i\geqq 1$, and therefore $f_n(r)=r^n$. To study the order of growth of f_n as $r\to 1$, $N\geqq 3$, we change the variable to $\rho=1-r$. For convenience we take the liberty of writing $f_n(r)=f_n(1-\rho)$ as $f_n(\rho)$. The differential equation then transforms into

$$L(f)=\rho^2f_n''(\rho)+\rho a(\rho)f_n'(\rho)+b(\rho)f_n(\rho)=0\,,$$

where

$$\begin{cases} a(\rho)=\dfrac{2(N-2)\alpha(1-\rho)}{2-\rho}-\dfrac{(N-1)\rho}{1-\rho}\,, \\[2ex] b(\rho)=-n(n+N-2)\left(\dfrac{\rho}{1-\rho}\right)^2. \end{cases}$$

This is again a linear equation, with $\rho=0$ a regular singular point. The roots of the indicial equation

$$q(p)=p(p-1)+a(0)p+b(0)=p(p-1)+(N-2)\alpha p=0$$

are $p=0,\ 1-(N-2)\alpha$.

LEMMA 1. *For $N\geqq 3$ and $n>0$,*

$$f_n(\rho)\sim\begin{cases} K\rho^{1-(N-2)\alpha}\,, & \alpha>\dfrac{1}{N-2}\,, \\[2ex] -K\log\rho\,, & \alpha=\dfrac{1}{N-2}\,, \\[2ex] K\,, & \alpha<\dfrac{1}{N-2}\,, \end{cases}$$

as $\rho\to 0$, with K some positive constant, not always the same.

Proof. For $1-(N-2)\alpha<0$, two linearly independent solutions $\not\equiv 0$ are of the form

$$\begin{cases} f_{n1}=\sigma_1\,, \\ f_{n2}=\rho^{1-(N-2)\alpha}\sigma_2+c(\log\rho)\sigma_1\,, \end{cases}$$

where σ_1, σ_2 are power series in ρ with $\sigma_1(0)\neq 0$, $\sigma_2(0)\neq 0$. Since linear combinations of these two solutions span the solution space, there exist real constants A, B such that $f_n=Af_{n1}+Bf_{n2}$. We recall that B_α^N belongs to the class O_G of parabolic manifolds (i.e., manifolds without Green's functions) if and only if $\alpha\geqq 1/(N-2)$ Sario and Wang [22]), and that O_G is contained in the class O_{HB} of manifolds which do not carry nonconstant bounded harmonic functions (see, e.g., Sario and Nakai [7]). The function f_{n1} cannot be bounded, for otherwise $f_{n1}S_n\in HB$, in violation of $B_\alpha^N\in O_G$. Since f_{n1} is bounded near $\rho=0$, it must have a singularity at $\rho=1$, that is, $r=0$. Therefore $B\neq 0$, for otherwise $f_n=Af_{n1}$, contrary to the fact that f_n does not have a singularity at $r=0$. Hence $f_n\sim K\rho^{1-(N-2)\alpha}$.

For $1-(N-2)\alpha=0$,

$$\begin{cases} f_{n1}=\sigma_1\,, \\ f_{n2}=\rho\sigma_2+(\log\rho)\sigma_1 \end{cases}$$

are linearly independent solutions, and the reasoning is the same as above.

For $1-(N-2)\alpha>0$, we have

$$\begin{cases} f_{n1}=\rho^{1-(N-2)\alpha}\sigma_1\,, \\ f_{n2}=\sigma_2+c(\log\rho)\rho^{1-(N-2)\alpha}\sigma_1\,, \end{cases}$$

and therefore $f_n=Af_{n1}+Bf_{n2}\sim K$. That $K>0$ follows from the fact that $f_n\geqq 0$ and is bounded away from 0 in a neighborhood of 1.

We shall now embark upon a discussion of the various cases in the order described in the introduction.

3. Case I: $\alpha<-1$.

LEMMA 2. *$B_\alpha^N\in O_{H^2B}$ for $\alpha<-1$, $N\geqq 2$.*

Proof. Suppose there exists a $u\in H^2B(B_\alpha^N)$, with $\Delta u=h$. For fixed numbers $0<\beta<\gamma<1$, take a function $s_0\in C_0[0,1)$, $s_0\geqq 0$, $s_0\not\equiv 0$, supp $s_0\subset(\beta,\gamma)$, and set $s_t(r)=s_0((1-r)/t)$, $t>0$. We know that $h=\sum_{n=0}^\infty f_nS_n$, where $S_n\not\equiv 0$ for some $n\geqq 0$. Set $\varphi_t=s_tS_n$. Since $\lambda^N\sim c(1-r)^{N\alpha}$ as $r\to 1$,

$$|(h,\varphi_t)|>c\int_{1-\gamma t}^{1-\beta t}f_ns_t\lambda^N dr>c\int_{1-\gamma t}^{1-\beta t}s_t(1-r)^{N\alpha}dr$$

$$\sim ct^{N\alpha}\int_{1-\gamma t}^{1-\beta t}s_t dr\sim ct^{N\alpha+1}\,.$$

Here and later c is a positive constant, not always the same.

On the other hand,

$$\Delta\varphi_t=-\lambda^{-2}\Big[s_t''+\Big(\frac{N-1}{r}+\frac{(N-2)\lambda'}{\lambda}\Big)s_t'-n(n+N-2)r^{-2}s_t\Big]S_n\,.$$

It follows that

$$(1, |\Delta\varphi_t|)<t^{(N-2)\alpha}\Big(c_1\int_{1-\gamma t}^{1-\beta t}s_t''dr+c_2t^{-1}\int_{1-\gamma t}^{1-\beta t}s_t'dr+c_3\int_{1-\gamma t}^{1-\beta t}s_tdr\Big)$$
$$=t^{(N-2)\alpha}(O(t^{-1})+t^{-1}O(1)+O(t))$$
$$=O(t^{(N-2)\alpha-1}).$$

For $\alpha<-1$, $t^{N\alpha+1}$ grows more rapidly than $t^{(N-2)\alpha-1}$ as $t\to 0$. This contradicts $|(h, \varphi_t)|=|(u, \Delta\varphi_t)|\leqq K(1, |\Delta\varphi_t|)$.

4. Case II: $\alpha>3/(N-4)$.

LEMMA 3. *$B_\alpha^N\in O_{H^2B}$ for $\alpha>3/(N-4)$, $N>4$.*

Proof. For $\alpha>1/(N-2)$, $n>0$, $f_n\sim(1-r)^{1-(N-2)\alpha}$, and

$$|(h, \varphi_t)|>c\int_{1-\gamma t}^{1-\beta t}f_n s_t\lambda^N dr>c\int_{1-\gamma t}^{1-\beta t}s_t(1-r)^{2\alpha+1}dr$$
$$=ct^{2\alpha+1}\int_{1-\gamma t}^{1-\beta t}s_t dr\sim ct^{2\alpha+2}.$$

We have a contradiction for $2\alpha+2<(N-2)\alpha-1$, that is, $\alpha>3/(N-4)$, and infer that $h=c$. If $c\neq 0$, then $c^{-1}u$ belongs to the class QB of bounded quasiharmonic functions v, characterized by $\Delta v=1$. But we know that $B_\alpha^N\in\tilde{O}_{QB}$ if and only if $-1<\alpha<1/(N-2)$ (Sario and Wang [22]). Therefore $c=0$, and $u\in H$, in violation of $u\in H^2B$.

5. Case III: $-1<\alpha<1/(N-2)$.

LEMMA 4. *$B_\alpha^N\in\tilde{O}_{H^2B}$ for $\alpha>-1$, $N=2$; $-1<\alpha<1/(N-2)$, $N\geqq 3$.*

This is a direct consequence of $B_\alpha^N\in\tilde{O}_{QB}$ for the above values of α (loc. cit.).

6. Case IV: $1/(N-2)<\alpha<3/(N-4)$, $\alpha\neq m/(N-2)$, m an integer.

LEMMA 5. *If α is not an integral multiple of $1/(N-2)$, then $B_\alpha^N\in\tilde{O}_{H^2B}$ for $1/(N-2)<\alpha$, $N=3, 4$; $1/(N-2)<\alpha<3/(N-4)$, $N>4$.*

Proof. We seek functions $g_n(r)$ such that $\Delta(g_n(r)S_n(\theta))=f_n(r)S_n(\theta)$. On writing the left-hand side explicitly, we see that g_n satisfies

$$r^2g_n''(r)+r\frac{N-1-(N-1+2(N-2)\alpha)r^2}{1-r^2}g_n'(r)$$
$$-n(n+N-2)g_n(r)=-r^2(1-r^2)^{2\alpha}f_n(r).$$

Since the right-hand side is of the form $r^{n+2}\sigma(r)$, where $\sigma(r)$ is a power series with radius of convergence 1, we are guaranted a solution $\tilde{g}_n(r)=r^{n+2}\tilde{\sigma}(r)$; hence

$\tilde{\sigma}(r)$ is a power series whose radius of convergence is also 1. However, a priori there is no assurance that $\tilde{g}_n(r)$ is bounded.

In search of a bounded solution, we set $\rho=1-r$, suppress the subindex n in our notation and obtain

$$L(g)=\rho^2 g''(\rho)+\rho a(\rho)g'(\rho)+b(\rho)g(\rho)=-\rho^2\lambda^2(\rho)f(\rho),$$

where $a(\rho)$, $b(\rho)$ and L are the same as in No. 2. The roots of the indicial equation $q(p)=p(p-1)+(N-2)\alpha p=0$ are again $p_0=0$ and $p_1=1-(N-2)\alpha$. Since α is not an integral multiple of $1/(N-2)$, the roots do not differ by an integer. Therefore f is of the form

$$f=A\sum_{i=0}^{\infty} c_i\rho^i+B\sum_{i=0}^{\infty}\gamma_i\rho^{1-(N-2)\alpha+i}.$$

Thus the right-hand side of our differential equation takes the form

$$-\rho^{2\alpha+2}(2-\rho)^{2\alpha}f(\rho)=A\sum_{i=0}^{\infty}\tilde{c}_i\rho^{2\alpha+2+i}+B\sum_{i=0}^{\infty}\tilde{\gamma}_i\rho^{3-(N-4)\alpha+i}.$$

If

$$L(g_{n1})=\sum_{i=0}^{\infty}\tilde{c}_i\rho^{2\alpha+2+i}, \qquad L(g_{n2})=\sum_{i=0}^{\infty}\tilde{\gamma}_i\rho^{3-(N-4)\alpha+i},$$

then $g_n=Ag_{n1}+Bg_{n2}$ will be a solution. Let $a(\rho)=\sum_{i=0}^{\infty}\alpha_i\rho^i$, $b(\rho)=\sum_{i=0}^{\infty}\beta_i\rho^i$. The function

$$g_{n1}=\sum_{i=0}^{\infty}\alpha_i\rho^{p_2+i}, \qquad p_2=2\alpha+2,$$

gives

$$L(g_{n1})=q(p_2)d_0\rho^{p_2}+\sum_{i=1}^{\infty}\{q(p_2+i)d_i+\sum_{j=0}^{i-1}[(p_2+j)\alpha_{i-j}+\beta_{i-j}]d_j\}\rho^{p_2+i}$$

$$=\sum_{i=0}^{\infty}\tilde{c}_i\rho^{p_2+i}.$$

Therefore

$$\begin{cases} d_0=\dfrac{\tilde{c}_0}{q(p_2)}, \\[2ex] d_i=\dfrac{\tilde{c}_i-\sum_{j=0}^{i-1}[(p_2+j)\alpha_{i-j}+\beta_{i-j}]d_j}{q(p_2+i)}, \end{cases}$$

$i=1, 2, \cdots$. That the denominator is never zero is clear since the roots of q are nonpositive whereas $p_2+i=2\alpha+2+i>0$ for all $i\geqq 0$.

In the same fashion we find a solution $g_{n2}=\sum_{i=0}^{\infty}\delta_i\rho^{p_3+i}$, $p_3=3-(N-4)\alpha$. In the cases $N=3, 4$, the condition $\alpha<3/(N-4)$ is not needed to assure that the δ_i's have a nonvanishing denominator $q(3-(N-4)\alpha+i)$ and that g_{n2} is bounded near $\rho=0$. Thus for these dimensions we have obtained g_n for all $\alpha>1/(N-2)$, $\alpha\neq m/(N-2)$.

7. To show that g_n is well defined, we must establish the convergence of

$\sum_{i=0}^{\infty} d_i \rho^i$ and $\sum_{i=0}^{\infty} \delta_i \rho^i$.

Again we shall only consider $\sum_{i=0}^{\infty} d_i \rho^i$ since the convergence of $\sum_{i=0}^{\infty} \delta_i \rho^i$ follows in the same manner. Choose a fixed $0<\rho_0<1$. By virtue of the analyticity of $a(\rho)$, $b(\rho)$, and $\sum_{i=0}^{\infty} \tilde{c}_i \rho^i$ for $0 \leqq \rho < 1$, there exists an $M>0$ such that

$$|\alpha_i| \leqq M\rho_0^{-i}, \qquad |\beta_i| \leqq M\rho_0^{-i}, \qquad \text{and} \qquad |\tilde{c}_i| \leqq M\rho_0^{-i},$$

$i=0, 1, \cdots$. Define $D_0=|d_0|$, and

$$D_i = \frac{M[\rho_0^{-i} + \sum_{j=0}^{i-j} (p_2+j+1)\rho_0^{j-i} D_j]}{q(p_2+i)}.$$

Since

$$|d_i| \leqq \frac{M[\rho_0^{-i} + \sum_{j=0}^{i-j} (p_2+j+1)\rho_0^{i-j} |d_j|}{q(p_2+i)},$$

we have by a trivial induction $|d_i| \leqq D_i$ for all i. We shall show by the ratio test that $\sum_{i=0}^{\infty} D_i \rho^i$ converges for $\rho<\rho_0$. Clearly

$$\begin{aligned} q(p_2+i+1)D_{i+1} &= M[\rho_0^{-i-1} + \sum_{j=0}^{i} (p_2+j+1)\rho_0^{j-i-1} D_j] \\ &= \rho_0^{-1} M[\rho_0^{-i} + \sum_{j=0}^{i-1} (p_2+j+1)\rho_0^{j-i} D_j] + \rho_0^{-1} M(p_2+i+1)D_i \\ &= \rho_0^{-1}[q(p_2+i) + M(p_2+i+1)]D_i . \end{aligned}$$

Hence

$$\frac{D_{i+1}\rho^{i+1}}{D_i \rho^i} = \frac{(i+p_2-p_0)(i+p_2-p_1) + M(i+p_2+1)}{(i+p_2+1-p_0)(i+p_2+1-p_1)} \cdot \frac{\rho}{\rho_0},$$

which approaches ρ/ρ_0 as $i \to \infty$.

We would like to say that $g_n S_n \in H^2B(B_\alpha^N)$, but $g_n S_n$ may fail to be biharmonic at the center of B_α^N. However, $\tilde{g}_n S_n$ is biharmonic at $r=0$. Since g_n and $\tilde{g}_n$ are particular solutions of the same linear differential equation, they differ by a solution of the homogeneous equation. Therefore, in the notation of Lemma 1, $\tilde{g}_n = g_n + Cf_{n1} + Df_{n2}$ for appropriate constants C and D. The function $f_n S_n$ with $f_n = Af_{n1} + Bf_{n2}$ is harmonic at $r=0$, and a fortiori $\hat{g}_n S_n$ with $\hat{g}_n = \tilde{g}_n - Df_n/B$ is biharmonic at $r=0$. Also $\hat{g}_n = g_n + (C - AD/B)f_{n1}$ is bounded since both g_n and f_{n1} are bounded. Thus $\hat{g}_n S_n \in H^2B$.

To simplify the notation we shall henceforth assume that g_n has been normalized so that $g_n S_n$ is biharmonic on all of B_α^N. Furthermore, we note for later use that $r^{-n} g_n$ is real analytic at $r=0$.

8. **Case V:** $1/(N-2)<\alpha<3/(N-4)$, $\alpha=m/(N-2)$.

LEMMA 6. *If α is an integral multiple of $1/(N-2)$, then $B_\alpha^N \in \tilde{O}_{H^2B}$ for $1/(N-2)<\alpha$, $N=3, 4$, $1/(N-2)<\alpha<3/(N-4)$, $N>4$.*

Proof. In the notation of Lemma 5,

$$L(g)=\rho^2 g''+\rho a(\rho)g'(\rho)+b(\rho)g(\rho)=-\rho^2\lambda^2(\rho)f(\rho),$$

where this time, by virtue of the proof of Lemma 1, $f(\rho)$ is of the form

$$f(\rho)=A\sum_{i=0}^{\infty}c_i\rho^i+B(\sum_{i=0}^{\infty}\gamma_i\rho^{1-(N-2)\alpha+i}+c(\log\rho)\sum_{i=0}^{\infty}c_i\rho^i).$$

Hence,

$$-\rho^2\lambda^2(\rho)f(\rho)=A\sum_{i=0}^{\infty}\tilde{c}_i\rho^{2\alpha+2+i}$$

$$+B(\sum_{i=0}^{\infty}\tilde{\gamma}_i\rho^{3-(N-4)\alpha+i}+c(\log\rho)\sum_{i=0}^{\infty}\tilde{c}_i\rho^{2\alpha+2+i}).$$

By the proof of Lemma 5, there exist g_{n1}, g_{n2} such that

$$L(g_{n1})=\sum_{i=0}^{\infty}\tilde{c}_i\rho^{2\alpha+2+i},\qquad L(g_{n2})=\sum_{i=0}^{\infty}\tilde{\gamma}_i\rho^{3-(N-4)\alpha+i}.$$

Therefore, if we can find a g_{n3} such that

$$L(g_{n3})=(\log\rho)\sum_{i=0}^{\infty}\tilde{c}_i\rho^{2\alpha+2+i},$$

then

$$g_n=Ag_{n1}+B(g_{n2}+cg_{n3})$$

will be a solution. We shall show that such a g_{n3}, of the form

$$g_{n3}=(\log\rho)\sum_{i=0}^{\infty}d_i\rho^{2\alpha+2+i}+\sum_{i=0}^{\infty}\delta_i\rho^{2\alpha+2+i},$$

exists. On substituting this into our equation we obtain

$$\begin{cases} d_0=\dfrac{\tilde{c}_0}{q(2\alpha+2)}. \qquad \delta_0=-\dfrac{4\alpha+3+\alpha_0}{q(2\alpha+2)}d_0, \\ d_i=\dfrac{\tilde{c}_i-\sum_{j=0}^{i-1}[(2\alpha+2+j)\alpha_{i-j}+\beta_{i-j}]d_j}{q(2\alpha+2+i)}, \\ \delta_i=-\dfrac{(4\alpha+3+2i+\alpha_0)d_i+\sum_{j=0}^{i-1}\alpha_{i-j}d_j+\sum_{j=0}^{i-1}[(2\alpha+2+j)\alpha_{i-j}+\beta_{i-j}]\delta_j}{q(2\alpha+2+i)}, \end{cases}$$

since

$$L((\log\rho)\sum_{i=0}^{\infty}d_i\rho^{2\alpha+2+i})$$

$$=(\log\rho)\sum_{i=0}^{\infty}[q(2\alpha+2+i)d_i+\sum_{j=0}^{i-1}((2\alpha+2+j)\alpha_{i-j}+\beta_{i-j})d_j]\rho^{2\alpha+2+i}$$

$$+\sum_{i=0}^{\infty}[(4\alpha+3+2i+\alpha_0)d_i+\sum_{j=0}^{i-1}\alpha_{i-j}d_j]\rho^{2\alpha+2+i}$$

and

$$L\left(\sum_{i=0}^{\infty}\delta_i\rho^{2\alpha+2+i}\right)$$

$$=\sum_{i=0}^{\infty}[q(2\alpha+2+i)\delta_i+\sum_{j=0}^{i-1}((2\alpha+2+j)\alpha_{i-j}+\beta_{i-j})\delta_j]\rho^{2\alpha+2+i}.$$

9. Again as in the proof of Lemma 5, $\sum_{i=0}^{\infty}d_i\rho^i$ converges, and it suffices to show the convergence of $\sum_{i=0}^{\infty}\delta_i\rho^i$ for $\rho<\rho_0$. Let $M>0$ be such that $|\alpha_i|\leqq M\rho_0^{-i}$, $|\beta_i|\leqq M\rho_0^{-i}$, and $|d_i|\leqq M\rho_0^{-i}$. Define D_i by $D_0=|\delta_0|$ and

$$q(2\alpha+2+i)D_i=M[(4\alpha+3+\alpha_0+(M+2)i)\rho_0^{-i}+\sum_{j=0}^{i-1}(2\alpha+3+j)\rho_0^{j-i}D_j].$$

We obtain in the same manner as in No. 7 that $|\delta_i|\leqq D_i$. Moreover,

$$q(2\alpha+3+i)D_{i+1}=\rho_0^{-1}M[(4\alpha+3+\alpha_0+(M+2)i)\rho_0^{-i}+\sum_{j=0}^{i-1}(2\alpha+3+j)\rho_0^{j-i}D_j]$$

$$+\rho_0^{-1}M[(M+2)\rho_0^{-i}+(2\alpha+3+i)D_i]$$

$$=\rho_0^{-1}[q(2\alpha+2+i)D_i+M(2\alpha+3+i)D_i+M(M+2)\rho_0^{-i}].$$

Therefore, for $i=0, 1, 2, \cdots$,

$$D_{i+1}=\rho_0^{-1}(A_{i+1}D_i+B_{i+1}\rho_0^{-i}),$$

where

$$A_{i+1}=\frac{q(2\alpha+2+i)+M(2\alpha+3+i)}{q(2\alpha+3+i)},\qquad B_{i+1}=\frac{M(M+2)}{q(2\alpha+3+i)}.$$

From this, we see that

$$D_{i+1}=M_{i+1}\rho_0^{-(i+1)},$$

where

$$M_{i+1}=D_0A_1A_2\cdots A_{i+1}+B_1A_2\cdots A_{i+1}+\cdots+B_iA_{i+1}+B_{i+1}.$$

Hence

$$\frac{D_{i+1}\rho^{i+1}}{D_i\rho^i}=\frac{M_{i+1}}{M_i}\cdot\frac{\rho}{\rho_0}=\left(A_{i+1}+\frac{B_{i+1}}{M_i}\right)\frac{\rho}{\rho_0}.$$

But

$$A_{i+1}=\frac{q(2\alpha+2+i)+M(2\alpha+3+i)}{q(2\alpha+3+i)},$$

which converges to 1 as $i\to\infty$. It remains to show that $B_{i+1}/M_i\to0$ as $i\to\infty$. We have

$$A_{i+1}=\frac{q(2\alpha+2+i)+M(2\alpha+3+i)}{q(2\alpha+3+i)}>\frac{q(2\alpha+2+i)}{q(2\alpha+3+i)},$$

so that

$$B_jA_{j+1}A_{j+2}\cdots A_i\geqq\frac{M(M+2)}{q(2\alpha+2+i)},\qquad 1\leqq j\leqq i.$$

Also,

$$D_0=K\frac{M(M+2)}{q(2\alpha+2)}$$

for some constant $K>0$. Consequently

$$\frac{B_{\imath+1}}{M_\imath}=\frac{B_{\imath+1}}{D_0A_1\cdots A_\imath+B_1A_2\cdots A_\imath+\cdots+B_{\imath-1}A_\imath+B_\imath}$$
$$<\frac{1}{\imath+K}\frac{q(2\alpha+2+i)}{q(2\alpha+3+i)},$$

which approaches 0 as $\imath\to\infty$.

10. Case VI: $\alpha=1/(N-2)$.

LEMMA 7. $B^N_{1/(N-2)}\in\tilde{O}_{H^2B}$, $N\geqq 3$.

Proof. For $\alpha=1/(N-2)$, the indicial equation has the repeated root 0. Therefore, $f(\rho)$ has the form

$$f(\rho)=A\sum_{\imath=0}^{\infty}c_i\rho^\imath+B\Big(\sum_{\imath=0}^{\infty}\gamma_i\rho^{1+\imath}+(\log\rho)\sum_{\imath=0}^{\infty}c_i\rho^i\Big),$$

and

$$-\rho^2\lambda^2(\rho)f(\rho)=A\sum_{\imath=0}^{\infty}\tilde{c}_i\rho^{2\alpha+2+\imath}+B\Big(\sum_{\imath=0}^{\infty}\tilde{\gamma}_i\rho^{2\alpha+3+\imath}+(\log\rho)\sum_{\imath=0}^{\infty}\tilde{c}_i\rho^{2\alpha+2+i}\Big).$$

The existence of a $g(\rho)$ satisfying $L(\rho)=-\rho^2\lambda^2(\rho)f(\rho)$ follows by taking $g=Ag_{n1}+B(g_{n2}+g_{n3})$ with

$$L(g_{n1})=\sum_{\imath=0}^{\infty}\tilde{c}_i\rho^{2\alpha+2+\imath},\quad L(g_{n2})=\sum_{\imath=0}^{\infty}\tilde{\gamma}_i\rho^{2\alpha+3+\imath},\quad L(g_{n3})=(\log\rho)\sum_{\imath=0}^{\infty}\tilde{c}_i\rho^{2\alpha+2+\imath}$$

as can be done by virtue of the proofs of Lemmas 5 and 6.

11. We insert here an expansion lemmas for the general biharmonic function on B^N_α and all α. It will be utilized in the remaining cases $\alpha=3/(N-4)$ and $\alpha=-1$. We recall that any spherical harmonic $S_n(\theta)$ of degree n can be written as a finite linear combination of fundamental spherical harmonics $S_{nm}(\theta)$, $m=1,\cdots,m_n$.

LEMMA 8. *For all α, every biharmonic function $u(r,\theta)$ on B^N_α has an expansion*

$$u(r,\theta)=\sum_{n=0}^{\infty}\sum_{m=1}^{m_n}(a_{nm}f_n(r)+b_{nm}g_n(r))S_m(\theta),$$

where $g_n(r)$ satisfies

$$\Delta(g_n(r)S_n(\theta))=f_n(r)S_n(\theta)$$

and $g_n(r)\neq 0$, $0<r<1$.

Remark. That for all α there exists at least one $g_n(r)$ satisfying the hypothesis is clear. For let $\tilde{g}(r)$ be as at the beginning of the proof of Lemma 5.

(Note that $\tilde{g}(r)$ is well defined not only for the α's considered in Lemma 5 but for all α). The function $r^{-n}\tilde{g}(r)$ is bounded near $r=0$, and $\lim_{r\to 1}\tilde{g}_n(r)\leqq\infty$ exists. Consequently, since $r^{-n}f_n(r)$ is bounded away from 0, there exists a constant C such that $r^{-n}\tilde{g}_n(r)+Cr^{-n}f_n(r)\neq 0$, $0<r<1$. A fortiori $g_n(r)=\tilde{g}_n(r)+Cf_n(r)\neq 0$, $0<r<1$.

Proof. For $u\in H^2(B_\alpha^N)$, Δu has by No. 1 an expansion

$$\Delta u(r, \theta)=\sum_{n=0}^{\infty}\sum_{m=1}^{m_n} b_{nm}f_n(r)S_{nm}(\theta)\,,$$

where the b_{nm}'s are constants. For a fixed $0<r_0<1$, since $g_n(r_0)\neq 0$, there exist constants c_{nm} such that

$$\sum_{n=0}^{\infty}\sum_{m=1}^{m_n} c_{nm}g_n(r)S_{nm}(\theta)$$

converges absolutely and uniformly to u on the sphere $S(r_0)$ of radius r_0. Let $B(r_0)$ be the ball bounded by $S(r_0)$, and denote by $g(x, y)$ the Green's function on $B(r_0)$. The Riesz decomposition of g_nS_{nm} reads

$$g_nS_{nm}=h_{nm}+G(f_nS_{nm})\,,$$

where h_{nm} is the harmonic part, and the potential part

$$G(f_nS_{nm})(x)=\int_{B(r_0)}f_n(y)S_{nm}(y)g(x, y)dy$$

vanishes identically in $S(r_0)$. By taking inner products with S_{nm} over $S(r_0)$ on both sides of the decomposition, we obtain for some constant d_{nm},

$$h_{nm}=d_{nm}f_nS_{nm}$$

on $S(r_0)$ and hence on $B(r_0)$. Substituting the decomposition into the expansion of u on $S(r_0)$ we see that

$$\sum_n\sum_m c_{nm}h_{nm}$$

is harmonic, and converges absolutely and uniformly to u on $S(r_0)$. It follows that

$$\begin{aligned}\sum_n\sum_m (c_{nm}h_{nm}&+b_{nm}G(f_nS_{nm}))\\ &=\sum_n\sum_m [(c_{nm}-b_{nm})h_{nm}+b_{nm}(h_{nm}+G)f_nS_{nm}))]\\ &=\sum_n\sum_m (a_{nm}f_n+b_{nm}g_n)S_{nm}\,,\end{aligned}$$

where $a_{nm}=(c_{nm}-b_{nm})d_{nm}$. Since $\sum_n\sum_m b_{nm}f_nS_{nm}$ converges absolutely and uniformly on $B(r_0)$, so does $\sum_n\sum_m b_{nm}G(f_nS_{nm})$. As a consequence, the expansion we have deduced is absolutely and uniformly convergent on $B(r_0)$ and converges to

u on $S(r_0)$. On applying Δ to the expansion, we obtain Δu and conclude that the expansion is indeed that of u on $B(r_0)$. That the a_{nm}'s and b_{nm}'s are independent of r_0 follows easily by the uniqueness of the coefficients of an expansion in the S_{nm}'s.

12. Case VII: $\alpha=3/(N-4)$.

LEMMA 9. $B^N_{3/(N-4)}\in O_{H^2B}$, $N>4$.

Proof. Again we seek a function $g(\rho)$ such that

$$L(g(\rho))=\rho^2 g''(\rho)+\rho a(\rho)g'(\rho)+b(\rho)g(\rho)=-\rho^2\lambda^2(\rho)f(\rho)\,.$$

The roots 0 and $1-(N-2)\alpha$ of the indicial equation are now distinct and nonpositive; the function $f(\rho)$ has the form

$$f(\rho)=A\sum_{i=0}^{\infty} c_i\rho^i+B\Big(\sum_{i=0}^{\infty}\gamma_i\rho^{1-(N-2)\alpha+i}+c(\log\rho)\sum_{i=0}^{\infty}c_i\rho^i\Big)\,,\qquad B\neq 0\,,$$

and

$$-\rho^2\lambda^2(\rho)f(\rho)=A\sum_{i=0}^{\infty}\tilde{c}_i\rho^{2\alpha+2+i}+B\Big(\sum_{i=0}^{\infty}\tilde{\gamma}_i\rho^i+c(\log\rho)\sum_{i=0}^{\infty}\tilde{c}_i\rho^{2\alpha+2+i}\Big)\,.$$

Let g_{n1} and g_{n3} be as in the proof of Lemma 6. We must assure the existence of a g_{n2} such that

$$L(g_{n2})=\sum_{i=0}^{\infty}\tilde{\gamma}_i\rho^i\,.$$

We try

$$g_{n2}(\rho)=\sum_{i=0}^{\infty}d_i\rho^i+d(\log\rho)\sum c_i\rho^i\,,\qquad d_0=1\,.$$

On substituting we obtain

$$d=\frac{\tilde{\gamma}_0}{(\alpha_0-1)c_0}\,,$$

$$d_i=\frac{\tilde{\gamma}_i-\sum_{j=0}^{i-1}(j\alpha_{i-j}+\beta_{i-j})d_j-d[(2i+\alpha_0-1)c_i+\sum_{j=0}^{i-1}\alpha_{i-j}c_j]}{q(i)}\,,$$

$i=1, 2, \cdots$. That $\sum_{i=0}^{\infty}d_i\rho^i$ converges is seen in the same manner as before. It follows that

$$g_n(\rho)=Ag_{n1}+B(g_{n2}+cg_{n3})$$

satisfies

$$\Delta(g_nS_n)=f_nS_n\,.$$

Moreover, $g_n(\rho)\sim C\log\rho$ as $\rho\to 0$. Using the remark at the end of No. 7, and arguing as in the remark prior to the proof of Lemma 8, we can assume $g_n\neq 0$, $0<r<1$. Hence by Lemma 8, if $u\in H^2$, then u has an expansion

$$u(r,\theta)=\sum_{n=0}^{\infty}\sum_{m=1}^{m_n}(a_{nm}f_n+b_{nm}g_n)S_{nm}\,,$$

where $b_{nm} \neq 0$ for some $n \geqq 0$. Now suppose u is bounded. Then $\int_\theta u S_{nm} d\theta$ is bounded as a function of r. But

$$\int_\theta u S_{nm} d\theta = (S_{nm}, S_{nm})(a_{nm} f_n(r) + b_{nm} g_n(r))$$

is not bounded since $f_n(\rho) \sim \rho^{1-(N-2)\alpha}$ and $g_n(\rho) \sim \log \rho$ are not bounded as $\rho \to 0$. Thus we have a contradiction.

13. Case VIII: $\alpha = -1$.

Solving for g_n was simplest in Case IV, for the hypothesis assured that there was no difficulty with the indicial roots. In Cases V and VI the indicial roots differed by an integer or were repeated; this complicated the form of f_n. However, the difficulty encountered in Case VII was more critical in that the indical root 0 prevented us from solving for d_0, and thereby required the addition of the term $d(\log \rho)\sum_{i=0}^\infty c_i \rho^i$ in the expression for g_{n2}. In the remaining case to be now discussed, the indicial roots cause the greatest complication and necessitate a quite involved expression for g_n.

LEMMA 10. $B_{-1}^N \in O_{H^2B}$, $N \geqq 2$.

Proof. Once more we look for a g satisfying

$$L(g(\rho)) = -\rho^2 \lambda^2(\rho) f(\rho).$$

Since $\alpha = -1$,

$$f(\rho) = A \sum_{i=0}^\infty c_i \rho^{N-1+i} + B\Big(\sum_{i=0}^\infty \gamma_i \rho^i + c(\log \rho) \sum_{i=0}^\infty c_i \rho^{N-1+i}\Big), \qquad B \neq 0,$$

and

$$-\rho^2 \lambda^2(\rho) f(\rho) = A \sum_{i=0}^\infty \tilde{c}_i \rho^{N-1+i} + B\Big(\sum_{i=0}^\infty \tilde{\gamma}_i \rho^i + c(\log \rho) \sum_{i=0}^\infty \tilde{c}_i \rho^{N-1+i}\Big).$$

That there exists a g_{n1} with $L(g_{n1}) = \sum_{i=0}^\infty \tilde{c}_i \rho^{N-1+i}$ follows from the proof of Lemma 9 and the fact that the indicial roots are 0 and $N-1$. Thus, the present task is to find a g_{n2} such that

$$L(g_{n2}) = \sum_{i=0}^\infty \tilde{\gamma}_i \rho^i + c(\log \rho) \sum_{i=0}^\infty \tilde{c}_i \rho^{N-1+i}.$$

We shall express g_{n2} as the sum of three functions,

$$g_{n2} = \Phi_1 + \Phi_2 + \Phi_3.$$

Let

$$\Phi_1 = e_1 (\log \rho)\Big(\sum_{i=0}^\infty \gamma_i \rho^i + c(\log \rho) \sum_{i=0}^\infty c_i \rho^{N-1+i}\Big) + \sum_{i=0}^{N-2} d_i \rho^i,$$

where

$$e_1 = \frac{\tilde{\gamma}_0}{(\alpha_0 - 1)\gamma_0}, \qquad d_0 = 1,$$

$$d_i=\frac{\tilde{\gamma}_i-\sum_{j=0}^{i-1}(j\alpha_{i-j}+\beta_{i-j})d_j-e_1[(2i+\alpha_0-1)\gamma_i+\sum_{j=0}^{i-1}\alpha_{i-j}\gamma_j]}{q(i)},$$

$i=1, \cdots, N-2$. In view of

$$L((\log\rho)(\sum_{i=0}^{\infty}\gamma_i\rho^i+c(\log\rho)\sum_{i=0}^{\infty}c_i\rho^{N-1+i}))$$

$$=c(\log\rho)\sum_{i=0}^{\infty}[(2N-3+2i)c_i+\sum_{j=0}^{i}\alpha_{i-j}c_j]\rho^{N-1+i}+2c\sum_{i=0}^{\infty}c_i\rho^{N-1+i}$$

$$+(\alpha_0-1)\gamma_0+\sum_{i=1}^{\infty}[(2i+\alpha_0-1)\gamma_i+\sum_{j=0}^{i-1}\alpha_{i-j}\gamma_j]\rho^i,$$

and

$$L(\sum_{i=0}^{N-2}d_i\rho^i)=\sum_{i=1}^{N-2}[q(i)d_i+\sum_{j=0}^{i-1}(j\alpha_{i-j}+\beta_{i-j})d_j]\rho^i+\sum_{i=N-1}^{\infty}\sum_{j=0}^{N-2}(j\alpha_{i-j}+\beta_{i-j})d_j\rho^i,$$

it follows that

$$L(\Phi_1)=\sum_{i=0}^{N-2}\tilde{\gamma}_i\rho^i+\sum_{i=0}^{\infty}s_i\rho^{N-1+i}+c(\log\rho)\sum_{i=0}^{\infty}\sigma_i\rho^{N-1+i},$$

with the s_i's and σ_i's constants. Next choose

$$\Phi_2=e_2(\log\rho)^2\sum_{i=0}^{\infty}c_i\rho^{N-1+i}+c(\log\rho)\sum_{i=0}^{\infty}\delta_i\rho^{N-1+i}$$

with

$$e_2=\frac{\tilde{c}_0-c\sigma_0}{2(2N-3+\alpha_0)c_0},\qquad \delta_0=1,$$

$$\delta_i=\frac{\tilde{c}_i-\sigma_i-\sum_{j=0}^{i-1}((N-1+j)\alpha_{i-j}+\beta_{i-j})\delta_j-2e_2[(2N-3+2i+\alpha_0)c_i+\sum_{j=0}^{i-1}\alpha d_{i-j}c_j]}{q(N-1+i)}$$

Since

$$L((\log\rho)^2\sum_{i=0}^{\infty}c_i\rho^{N-1+i})$$

$$=2\sum_{i=0}^{\infty}c_i\rho^{N-1+i}+2(\log\rho)\sum_{i=0}^{\infty}[(2N-3+2i+\alpha_0)c_i+\sum_{j=0}^{i-1}\alpha_{i-j}c_j]\rho^{N-1+i},$$

and

$$L((\log\rho)\sum_{i=0}^{\infty}\delta_i\rho^{N-1+i})=\sum_{i=0}^{\infty}[(2N-3+2i+\alpha_0)\delta_i+\sum_{j=0}^{i-1}\alpha_{i-j}\delta_j]\rho^{N-1+i}$$

$$+(\log\rho)\sum_{i=0}^{\infty}[q(N-1+i)\delta_i+\sum_{j=0}^{i-1}((N-1+j)\alpha_{i-j}+\beta_{i-j})\delta_j]\rho^{N-1+i},$$

we have

$$L(\Phi_1+\Phi_2)=\sum_{i=0}^{N-2}\tilde{\gamma}_i\rho^i+\sum_{i=0}^{\infty}\tilde{s}_i\rho^{N-1+i}+c(\log\rho)\sum_{i=0}^{\infty}\tilde{c}_i\rho^{N-1+i}.$$

Finally, set

$$\Phi_3=e_3(\log\rho)\sum_{i=0}^{\infty}c_i\rho^{N-1+i}+\sum_{i=0}^{\infty}\tilde{d}_i\rho^{N-1+i},$$

where

$$e_3=\frac{\tilde{\gamma}_{N-1}-\tilde{s}_0}{(2N-3+\alpha_0)c_0}, \qquad \tilde{d}_0=1,$$

$$\tilde{d}_i=\frac{\tilde{\gamma}_{N-1+i}-\tilde{s}_i-\sum_{j=0}^{i-1}((N-1+j)\alpha_{i-j}+\beta_{i-j})\tilde{d}_j-e_3[(2N-3+2i+\alpha_0)c_i+\sum_{j=0}^{i-1}\alpha_{i-j}c_j]}{q(N-1+i)},$$

$i=1, 2, \cdots$. We infer from

$$L((\log\rho)\sum_{i=0}^{\infty}c_i\rho^{N-1+i})=\sum_{i=0}^{\infty}[(2N-3+2i+\alpha_0)c_i+\sum_{j=0}^{i-1}\alpha_{i-j}c_j]\rho^{N-1+i}$$

and

$$L(\sum_{i=0}^{\infty}\tilde{d}_i\rho^{N-1+i})=\sum_{i=0}^{\infty}[q(N-1+i)\tilde{d}_i+\sum_{j=0}^{i-1}((N-1+j)\alpha_{i-j}+\beta_{i-j})\tilde{d}_j]\rho^{N-1+i}$$

that g_{n2} satisfies our equation. Clearly Φ_1 is well defined. The convergence of Φ_2 and Φ_3 is seen by arguing as before.

As in Lemma 9, $g_n \sim C\log\rho$, and the proof is herewith complete.

14. We have thus obtained the following complete solution of our problem:

THEOREM. *$B_\alpha^N \in \tilde{O}_{H^2B}$ if and only if*

$$\begin{cases} \alpha>-1 & \textit{for}\quad N=2, 3, 4, \\ -1<\alpha<\dfrac{3}{N-4} & \textit{for}\quad N>4, \end{cases}$$

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