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# ALMOST TANGENT STRUCTURES

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**0.** Let $M$ be a differentiable manifold of class $C^\infty$ and of dimension $2n$. A (1, 1) tensor field $J$ of rank $n$ on $M$ such that $J^2=0$ defines a class of conjugate $G$-structures on $M$. A group $G$ for a representative structure consists of all matrices of the form

$$\begin{bmatrix} A & 0 \\ B & A \end{bmatrix} \tag{0.1}$$

where $A$, $B$ are matrices of order $n \times n$ and $A$ is non-singular. This structure is called an almost tangent structure [4]. Suppose that such a structure is defined on $M$ then $M$ is called an almost tangent manifold. A (1, 1) tensor field $J$ on $M$ can be defined by specifying its components to be

$$J_0=\begin{bmatrix} 0 & 0 \\ I_n & 0 \end{bmatrix} \tag{0.2}$$

relative to any adapted frame. If $\sigma=X_1, \cdots, X_{2n}$ is any adapted moving frame defined at a given point $m \in M$, then $JX_a=X_{a+n}$, $JX_{a+n}=0$ $(a=1, \cdots, n)$. The tensor field $J$ has constant rank $n$ and it satisfies the equation $J^2=0$. Conversely any such tensor field $J$ determines an almost tangent structure on $M$ [5]. The (1, 1) tensor field $J$ on an almost tangent manifold $M$ determines a linear mapping $J_m: v \rightarrow (Jm)v$ on each tangent vector space $T_mM$. The function $\operatorname{Ker} J: m \rightarrow$ kernel $J_m$ is an $n$-dimensional distribution on $M$. If $\sigma$ is an adapted moving frame at any given point $m \in M$, then the vector fields $X_{n+1}, \cdots, X_{2n}$ form a local basis for the distribution $\operatorname{Ker} J$ at $m$.

In this paper we shall study the conditions under which an almost tangent structure is integrable, and show that the group of automorphisms of such a structure is not necessarily a Lie group even on a compact manifold.

**1.** Suppose that we have any $G$-structure on a manifold $M$ of dimension $n$ with adapted fram bundle $P(M, G)$. Let $\theta$ be the canonical 1-form on $P(M, G)$ with values in $R^n$ and $\omega$ the connection form of a given linear connection on $P$. If $\Theta=D\theta$ is the torsion form then the torsion tensor $T(\Theta)$ has values in $V= =R^n \otimes \overset{2}{\wedge} R_n$ and is of type $R=\mu \otimes \overset{2}{\wedge} \mu^*$ where $\mu$ is a representation of $G$ in $R^n$ defined by the matrix multiplication. We denote $W=L(G) \otimes R_n$, where $L(G)$ is

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the Lie algebra of $G$. If the linear mapping $\partial: W \to V$ is defined by $\partial S: u, v \to (Su)v-(Sv)u$, where $u, v \in R^n$ and $S \in W$, then the subspace $\partial W$ of $V$ is invariant under $RG$. Consider the natural surjection $\nu: V \to V/\partial W$. Since $\partial W$ is invariant under $RG$ we can define a linear representation $e$ of $G$ in $V/\partial W$ by $(eg)\circ\nu = \nu\circ(Rg)$. The function $B=\nu(T(\Theta))$ is a linear function on $P$ with values in $V/\partial W$ and is of type $e$. It is independent of the choice of the connection on $P$ and is the Bernard tensor, or the structure tensor for the $G$-structure [1]. S. S. Chern [3] originally defined this in a different way as follows. Let $Z$ be a subspace of $V$ complementary to $\partial W$. The natural projection $\lambda: V \to Z$ determines a mapping $\underset{\sim}{\lambda}: V/\partial W \to Z$ such that $\underset{\sim}{\lambda}\circ\nu=\lambda$, $\nu\circ\lambda=\nu$ and $\nu\circ\underset{\sim}{\lambda}$ is the identity function on $V/\partial W$. The function $C=\lambda(T(\Theta))$ is a linear function on $P$ with values in $Z$ and is of type $\lambda\circ R$. It is called the Chern tensor for the $G$-structure. It is independent of the choice of the connection on $P$, but does depend on the choice of subspace $Z$. It is easy to show that the vanishing of the Chern tensor is equivalent to the vanishing of the Bernard tensor.

The following result for an integrable $G$-structure is known.

LEMMA 1.1. [1] *The Bernard tensor of an integrable G-structure is zero.*

**2.** In this section we shall give some conditions under which an almost tangent structure is integrable.

THEOREM 2.1. *An almost tangent structure is integrable if and only if its Chern tensor is zero.*

*Proof.* Let

$$\theta^1, \cdots, \theta^{2n} \tag{2.1}$$

be an adapted moving coframe defined at a given point $m \in M$. The codistribution $\operatorname{Ker} J$ is spanned by $\theta^1, \cdots, \theta^n$. If

$$d\theta^i = \frac{1}{2}\gamma^i_{jk}\theta^j \wedge \theta^k \tag{2.2}$$

$(i, j, k=1, \cdots, 2n)$ we define

$$\gamma = \frac{1}{2}\gamma^i_{jk} e_i \otimes e^j \wedge e^k . \tag{2.3}$$

A complementary subspace $Z$ of $V$ to $\partial W$ is spanned by $e_i \otimes e^{b+n} \wedge e^{c+n}$ $(b, c=1, \cdots, n)$ and the projection $\lambda: V \to Z$ is given by $\gamma^i_{jk} e_i \otimes e^j \wedge e^k \to C^i_{jk} e_i \otimes e^j \wedge e^k$ where

$$C^i_{bk}=0 \qquad C^a_{b+n}=\gamma^a_{b+n\ c+n} \tag{2.4}$$

$$C^{a+n}_{b+n\ c+n}=\gamma^{a+n}_{b+n\ c+n}+\gamma^a_{c\ b+n}-\gamma^a_{b\ c+n} \tag{2.5}$$

The Chern tensor $C$ is determined on $\pi^{-1}U$ by the function $\underset{\sim}{C}=\lambda\circ\gamma$ on $U$ with values in $Z$ calculated above, where $\pi$ is the natural projection of $P(M, G)$ on $M$ and $U$ is a neighbourhood of the point $m \in M$, on which the coframe (2.1) is defined. If the Chern tensor is zero we have from (2.4)

$$\gamma^{a}_{b+n\ c+n}=0\,. \tag{2.6}$$

Hence from the Frobenius theorem it follows that the codistribution Ker $J$ is integrable. Consequently there exists a chart $x$ at the point $m$ such that

$$\begin{bmatrix} dx^1 \\ \vdots \\ dx^{2n} \end{bmatrix}=\begin{bmatrix} A & 0 \\ B & D \end{bmatrix}\begin{bmatrix} \theta^1 \\ \\ \theta^{2n} \end{bmatrix}.$$

Therefore the moving coframe

$$\bar{\theta}^1, \cdots, \bar{\theta}^{2n} \tag{2.7}$$

at $m$ given by

$$\begin{bmatrix} \bar{\theta}^1 \\ \cdots \\ \bar{\theta}^{2n} \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & E \end{bmatrix}\begin{bmatrix} dx^1 \\ \vdots \\ dx^{2n} \end{bmatrix} \tag{2.8}$$

where $E=AD^{-1}$, is adapted for the almost tangent structure. For the coframe (2.7) the corresponding $\bar{\gamma}$ satisfy $\bar{\gamma}^{a}_{ij}=0$. Hence the vanishing of the Chern tensor implies from (2.5)

$$\bar{\gamma}^{a+n}_{b+n\ c+n}=0\,. \tag{2.9}$$

From (2.8) we get $\bar{\theta}^{a+n}=E^a_b\,dx^{b+n}$, and hence

$$d\bar{\theta}^{a+n}=dE^a_b\wedge dx^{b+n}\,. \tag{2.10}$$

Using (2.10) we have from (2.9)

$$\left(\frac{\partial E^a_d}{\partial x^{e+n}}-\frac{\partial E^a_e}{\partial x^{d+n}}\right)E^d_b\,E^e_c=0\,.$$

Since the matrix $E$ is non-singular we get

$$\frac{\partial E^a_b}{\partial x^{c+n}}=\frac{\partial E^a_c}{\partial x^{b+n}} \tag{2.11}$$

Condition (2.11) implies that the system of differential equations

$$\frac{\partial H^{a+n}}{\partial x^{b+n}}=E^a_b \tag{2.12}$$

has a solution $H^{a+n}$ at the point $m$. We define a chart $y$ at $m$ as follows

$$y^a=x^a, \qquad y^{a+n}=H^{a+n}(x^1, \cdots, x^{2n})\,. \tag{2.13}$$

It is easy to verify that $y$ does define a chart at $m$. From (2.8), (2.12) and (2.13) we get

$$\begin{bmatrix} dy^1 \\ \vdots \\ dy^{2n} \end{bmatrix}=\begin{bmatrix} I & 0 \\ * & I \end{bmatrix}\begin{bmatrix} \bar{\theta}^2 \\ \vdots \\ \bar{\theta}^{2n} \end{bmatrix}$$

Therefore the chart $y$ at $m$ is adapted for the almost tangent structure. We can find such charts whose domains cover $M$. Hence the almost tangent structure is integrable.

Conversely it follows from Lemma 1.1 that the Chern tensor of an integrable almost tangent structure is zero.

Associated with any (1, 1) tensor field $J$ on a manifold $M$ we have a (1, 2) tensor field $N$, the Nijenhuis tensor. If $J$ defines a $G$-structure on $M$ which is integrable then the Nijenhuis tensor is zero. The converse is true sometimes. But in general the vanishing of the Nijenhuis tensor is not a sufficient condition for the integrability of the $G$-structure [7].

For an almost tangent structure the following theorem is known [5].

THEOREM 2.2. *For an almost tangent structure the Nijenhuis tensor vanishes if and only if its Chern tensor vanishes.*

A different concept of integrability was introduced by Chern [3] which is now called almost transitivity. A $G$-structure is said to almost transitive if its Bernard tensor is constant. An integrable $G$-structure is almost transitive but the converse is not necessarily true. For example, a Lie group carries an $I$-structure, the structure constants of which determine the Bernard tensor which is always a constant but not necessarily zero. The $I$-structure on a non-abelian Lie group is almost transitive but not integrable.

THEOREM 2.3. *If a group $G$ contains the element $-I$ then the Bernard tensor of a $G$-structure is zero if it is constant.*

*Proof.* If the value of the Bernard tensor is $k$ at some point $p \in P$, then its value at $p(-I)$ is

$$\begin{aligned}\nu(T(\Theta)(p(-I))) &= \nu(R(-I))(T(\Theta)p) \\ &= \nu(-T(\Theta)\,p) \\ &= -\nu(T(\Theta)p) \quad \text{(since the mapping } \nu \text{ is linear)} \\ &= -k\,.\end{aligned}$$

Since the Bernard tensor is constant $k=-k$, and so $k=0$.

Combining the above results we have

THEOREM 2.4. *For an almost tangent structure the following conditions are equivalent.*

1. *It is integrable.*
2. *Its Nijenhuis tensor is zero.*
3. *Its Chern tensor is zero.*
4. *It is almost transitive.*

**3.** Suppose we have a $G$-structure on a manifold $M$ of dimension $n$ with adapted frame bundle $P(M, G)$. A local diffeomorphism $f$ of $M$ into itself induces a local automorphism $f_*$ of the frame bundle $H(M, GL(R^n))$. $f$ is a local automorphism of the $G$-structure if $f_*$ maps adapted frames into adápted frames.

A vector field $X$ in $M$ is a $G$-vector field if the local diffeomorphisms generated by $X$ are local automorphisms of the $G$-structure. For a given $G$-structure the problem is to determine whether the group of global automorphism is a Lie group. A solution may sometimes be obtained using a following particular case of Palais's theorem [9].

THEOREM 3.1. *Let $Q$ be the group of automorphisms of a $G$-structure. A necessary and sufficient condition that $Q$ is a Lie group is that the set $S$ of all complete $G$-vector fields generates a finite dimensional Lie algebra $s$ and in this case the Lie algebra of $Q$ is $s$.*

The following result of which Bochner's [2] result is a particular case is known [10].

THEOREM 3.2. *Let $S$ be a space of vector fields $X$ on a compact manifold $M$ such that for every point $m \in M$ there is a system of elliptic differential equations defined on a neighbourhood of that point and satisfied by all $X^i$ given locally by $X = X^i \partial/\partial x^i$. Then the dimension of $S$ is finite.*

A $G$-structure is said to be elliptic if the $G$-vector fields satisfy an elliptic system of differential equations in a neighbourhood of each point $m \in M$.

From Theorems 3.1 and 3.2 we get

THEOREM 3.3. *On a compact manifold the group of automorphisms of an elliptic $G$-structure is a Lie group.*

The ellipticity of $G$-structure can also be expressed as follows.

THEOREM 3.4. [6] *A $G$-structure is elliptic if and only if the Lie algebra $L(G)$ of the group $G$ contains no element of rank one.*

The almost tangent group is not elliptic for, if $B$ is an $n \times n$ matrix of rank one, then the matrix

$$\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}$$

belongs to $L(G)$. Hence by Theorem 3.9 it is not elliptic. In order to show that the group of automorphisms of an almost tangent structure is not necessarily a Lie group we consider two almost tangent manifolds of which one is compact. It can be shown that a diffeomorphism $f: M \to M$ is an automorphism for an integrable $G$-structure on $M$ if for each point $m \in M$, there exist adapted charts $x$, $\tilde{x}$ at $m$ and $f(m)$ such that the matrix

$$\left[\frac{\partial(\bar{x}^i\circ f)}{\partial x^j}\right]$$

has values in the group $G$. A vector field $X$ is a $G$-vector field if for each point $m\in M$, there exists an adapted chart

$$\left[\frac{\partial X^i}{\partial x^j}\right]$$

has values in the Lie algebra $L(G)$ where $X=X^i\,\partial/\partial x^i$.

THEOREM 3.5. *The group of automorphisms of the almost tangent structure on the tangent manifold $TM$ of any manifold $M$ is not a Lie group.*

*Proof.* The tangent vectors of any differentiable manifold $M$ of dimension $n$ form a differentiable manifold $TM$ of dimension $2n$. Let $\pi: TM\to M$ be the natural projection. Corresponding to any chart $x$ defined on a neighbourhood $U$ of a point $m\in M$ we can define a standard chart on $\pi^{-1}U$ which we denote by $(x, y)$. If $U\cap\bar{U}=\phi$, then the charts $(x, y)$ and $(\bar{x}, \bar{y})$ on $\pi^{-1}U$, $\pi^{-1}\bar{U}$ are related by a change of coordinates whose Jacobian matrix is of the form (1, 1) where

$$A=\left[\frac{\partial x^a}{\partial\bar{x}^b}\right],\qquad B=\left[\frac{\partial^2 x^a}{\partial\bar{x}^b\partial\bar{x}^c}\bar{y}^c\right].$$

The natural moving frames associated with these charts therefore define an integrable almost tangent structure on $TM$.

A diffeomorphism $f$ of $M$ induces a diffeomorphism $f_*$ of $TM$. If $v$ is any point in $TM$, $(x, y)$ and $(\bar{x}, \bar{y})$ charts at $v$ and $f_*v$ then

$$\frac{\partial(\bar{x}\circ f_*, \bar{y}\circ f_*)}{\partial(x, y)}=\begin{bmatrix}\dfrac{\partial(\bar{x}^i\circ f)}{\partial x^j} & 0\\ * & \dfrac{\partial(\bar{x}^i\circ f)}{\partial x^j}\end{bmatrix}$$

which has values in the almost tangent group. Hence $f_*$ is an automorphism of the almost tangent structure on $TM$. The set $\tilde{Q}$ of all diffeomorphisms $f_*$ of $TM$ is a group isomorphic to group $Q$ of diffeomorphisms of $f$ of $M$, for, if $f_1$ and $f_2$ are diffeomorphisms of $M$, then $(f_1\circ f_2)_*=f_{1*}\circ f_{2*}$ and $f_{1*}\neq f_{2*}$ if and only if $f_1\neq f_2$. As the group $Q$ is not a Lie group, $\tilde{Q}$ is not a Lie group.

As the manifold $TM$ considered above is not compact we now study a compact manifold with a similar property.

THEOREM 3.6. *The group of automorphisms of an almost tangent structure on the torus $T=S^1\times S^1$ is not a Lie group.*

*Proof.* The torus $T$ can be covered by coordinates charts $(x_i^1, x_i^2)$ such that the change of coordinates on $U_i\cap U_j$ is of the form

$$x_i^1=x_j^2+n_1,\qquad x_i^2=x_j^2+n_2$$

where $n_1$, $n_2$ are integers. These charts therefore define a parallelisation on $T$, and this can be extended to an integrable almost tangent structure. For any integer $p$, the local vector fields

$$\frac{\partial}{\partial x_i^1}+\sin(2p\pi x_i^1)\frac{\partial}{\partial x_i^2}$$

agree on the intersection of their domains, therefore they define a global vector field $X$ on $T$. At any given point

$$\left[\frac{\partial X^a}{\partial x_i^b}\right]=\begin{bmatrix}0 & 0\\ 2p\pi\cos(2p\pi x_i^1) & 0\end{bmatrix}$$

$(a, b=1, 2)$ for each of these charts so $X$ is a $G$-vector field. As $p$ varies we get a set of complete $G$-vector fields on the torus $T$ which are linearly independent, hence they form an infinite dimensional space. Therefore the group of automorphisms is not a Lie group.

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