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QUASIHARMONIC DEGENERACY OF RIEMANNIAN N-MANIFOLDS

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The harmonic classification of Riemannian manifolds of higher dimension was recently brought to completion by the construction of manifolds which carry bounded but no Dirichlet finite nonconstant harmonic functions (Kwon [2], Hada-Sario-Wang [1]). In contrast, almost nothing is known about the quasiharmonic classification of higher dimensional manifolds. The purpose of the present study is to establish inclusion relations constituting such a classification.

Let Q be the class of quasiharmonic functions [6], i. e., solutions of $\Delta u=1$, where $\Delta=d\delta+\delta d$ is the Laplace-Beltrami operator. Denote by QP, QB, QD, and QC the classes of quasiharmonic functions which are positive, bounded, Dirichlet finite, and bounded Dirichlet finite, respectively. The notation $\tilde{O}_{QX}^N$, O_{QX}^N, with $X=P$, B, D, or C, will be used for the classes of Riemannian manifolds M of dimension $N>2$ for which $QX(M)=\emptyset$ or $\neq\emptyset$, respectively. The class of parabolic N-manifolds, characterized by the nonexistence of Green's functions, is designated by O_G^N. We shall prove that the complete classification scheme

$$O_G^N<O_{QP}^N<O_{QB}^N\cap O_{QD}^N<O_{QB}^N\,,\ O_{QD}^N<O_{QB}^N\cup O_{QD}^N=O_{QC}^N$$

holds for every N. Here "$<$" signifies strict inclusion and "O_{QB}^N, O_{QD}^N" means that the inequalities are valid for both classes.

1. We shall cover the relations in the order given above. Regarding O_G^N and O_{QP}^N, every $u\in QP$ is superharmonic by $\Delta u=1$ and its existence therefore implies that of the Green's functions (see e. g. [7]). To prove $O_G^N<O_{QP}^N$, it thus suffices to find a hyperbolic N-manifold with $QP=\emptyset$. The trivial example of the Euclidean N-space does not qualify for $N=2$, whereas the following example is valid for all N. It is an N-torus cut open along a pair of opposite faces and equipped with a suitable metric.

LEMMA 1. *Let T be the space*

$$0<x<1\,,\qquad |y_i|\leqq 1\,,\qquad i=1,\cdots,N-1\,,$$

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with each face $y_i=1$ *identified with the opposite face* $y_i=-1$ *by parallel translation. Endow* T *with the metric*

$$ds^2=(x^{-2}+((1-x)^{-2})dx^2+(x^{-2}+(1-x)^{-2})^{1/(N-1)}\sum_{i=1}^{N-1}dy_i^2 .$$

Then $T\in\tilde{O}_G^N\cap O_{QP}^N$.

Proof. The metric tensor has the determinant $g=(x^{-2}+(1-x)^{-2})^2$, and the first entry of the conjugate tensor is $g^{xx}=(x^{-2}+(1-x)^{-2})^{-1}$. Characterization of an $h(x)$ in the class H of harmonic functions,

$$\Delta h=-g^{-\frac{1}{2}}(g^{\frac{1}{2}}g^{xx}h')'=0 ,$$

gives $h''=0$, that is, $h=ax+b$. Therefore, the harmonic measure of the ideal boundary of T is positive, which entails $T\in\tilde{O}_G^N$ (see e.g. [7]).

We recall that a manifold M belongs to $\tilde{O}_{QP}$, $\tilde{O}_{QB}$, or $\tilde{O}_{QD}$ according as the potential

$$G1(p)=\int_M g(p, q)dq$$

exists or is bounded at some, and hence at every, point $p\in M$, or has a finite Dirichlet integral

$$D(G1)=G(1, 1)=\iint_{M\times M} g(p, q)dp\, dq ,$$

respectively (Nakai-Sario [3]). Here g is the Green's function with pole q, and dp, dq are the volume elements at p and q.

For the function

$$u(x)=\log x+\log(1-x)$$

we obtain by direct computation $\Delta u=1$. If $G1$ exists, then by the symmetry of T with respect to all y_i-coördinates, $G1$ is a function of x only. Since $\Delta G1=1$, we have

$$G1(x)=u(x)+h(x)$$

for some $h=ax+b$. In view of $u(x)+h(x)\to-\infty$ as $x\to 0$ and $x\to 1$, this contradicts $G1>0$. Thus $T\in O_{QP}^N$.

2. Next we show that $O_{QP}^N<O_{QB}^N\cap O_{QD}^N$. Since the boundedness as well as Dirichlet finiteness of $G1$ trivially implies its existence, we only have to find an N-manifold with $QP\neq\emptyset$, $QB=QD=\emptyset$. We shall henceforth tacitly understand that $|y_i|\leqq 1$ implies identification of opposite faces by pairs.

LEMMA 2. *The N-manifold*

$$T:\ 0<x<1 , \qquad |y_i|\leqq 1 , \qquad i=1, \cdots, N-1 ,$$

with the metric

$$ds^2=x^{-3}dx^2+x^{3/(N-1)}\sum_{i=1}^{N-1}dy_i^2$$

belongs to $\tilde{O}_{QP}^N\cap O_{QB}^N\cap O_{QD}^N$ *for every* N.

Proof. Clearly

$$u=x^{-1}\in QP,$$

and the most general $h(x)\in H$ is

$$h(x)=ax^{-2}+b.$$

Since $G1$ exists,

$$G1=u+h=ax^{-2}+x^{-1}+c$$

for some a, c. This function is unbounded, hence $T\in O_{QB}^N$.

Let D_x stand for the Dirichlet integral over the subregion of T from x to 1, and denoted by c a constant, not always the same. Then

$$D_x(u)=c\int_x^1 x^{-4}x^3dx=-c\log x,$$

$$D_x(h)=c\int_x^1 x^{-6}x^3dx=c(1-x^{-2}),$$

and $D_x(G1)>|D_x(u)-D_x(h)|$, which is unbounded as $x\to 0$. Therefore $T\in O_{QD}^N$.

3. We proceed to show that $O_{QB}^N\cap O_{QD}^N<O_{QB}^N$.

LEMMA 3. *The N-manifold*

$$T:\ 0<x<1,\qquad |y_i|\leqq 1,\qquad i=1,\cdots,N-1,$$

with the metric

$$ds^2=x^{-2}dx^2+x^{2/(N-1)}\sum_{i=1}^{N-1}dy_i^2$$

belongs to $O_{QB}^N\cap\tilde{O}_{QD}^N$ *for every* N.

Proof. We have

$$u=\log x^{-1}\in Q$$

with

$$D(u)=c\int_0^1 x^{-2}x^2dx<\infty.$$

Therefore $T\in\tilde{O}_{QD}^N$.

The general $h(x)\in H$ is

$$h=ax^{-1}+b.$$

For some a, b,

$$G1=\log x^{-1}+ax^{-1}+b,$$

which is unbounded. A fortiori $T\in O_{QB}^N$.

4. The relation $O_{QB}^N \cap O_{QD}^N < O_{QD}^N$ is known for $N>2$. In fact, by Sario-Wang [8], the Poincaré N-ball $B_\alpha^N : r<1$, $ds=(1-r^2)^\alpha |dx|$, belongs to O_{QB}^N if and only if $\alpha \notin (-1, (N-2)^{-1})$, and to O_{QD}^N if and only if $\alpha \notin (-3(N+2)^{-1}, (N-2)^{-1})$. Therefore $B_\alpha^N \in \tilde{O}_{QB}^N \cap O_{QD}^N$ is characterized by $\alpha \in (-1, -3(N+2)^{-1}]$. Here we give two other examples, valid for all N.

LEMMA 4. *The N-manifold*

$$T:\ |x|<\infty, \quad |y|\leqq 1, \quad |z_i|\leqq 1, \quad i=1, \cdots, N-2,$$

with the metric

$$ds^2=(1+x^4)^{-1}dx^2+(1+x^4)(1+x^6)^2 dy^2+\sum_{i=1}^{N-2} dz_i^2$$

belongs to $\tilde{O}_{QB}^N \cap O_{QD}^N$ *for every N.*

Proof. We readily verify that the function

$$u(x)=-\int_0^x (1+s^4)^{-1}(1+s^6)^{-1}\int_0^s (1+r^6)dr\,ds$$

belongs to Q. As $|x|\to\infty$, $u'(x)\sim cx^{-3}$ and

$$u(x)=O(1).$$

Consequently $T\in\tilde{O}_{QB}^N$.

Suppose there exists a $u\in QD$. Take a function $\varphi_0\in C_0^2((-\infty,\infty))$, $\varphi_0\geqq 0$, supp $\varphi_0\subset(0,1)$. For $t>0$, set $\varphi_t(x)=\varphi_0(x-t)$. Then supp $\varphi_t\subset(t, t+1)$,

$$(1,\varphi_t)=c\int_t^{t+1}(1+x^6)\varphi_t dx>ct^6,$$

and

$$D(\varphi_t)=c\int_t^{t+1}\varphi_t'^2(1+x^4)(1+x^6)dx=O(t^{10}).$$

Therefore $(1,\varphi_t)/\sqrt{D(\varphi_t)}\to\infty$ as $t\to\infty$. This violates Stokes' formula, which gives $(\Delta u, \varphi_t)=(1,\varphi_t)<\sqrt{D(u)}\sqrt{D(\varphi_t)}$. We conclude that $T\in O_{QD}^N$.

5. Our second example for $\tilde{O}_{QB}^N \cap O_{QD}^N=\emptyset$ is as follows.

LEMMA 5. *The N-manifold*

$$T:\ 0<x<1, \quad |y_i|\leqq 1, \quad i=1,\cdots,N-1,$$

with the metric

$$ds^2=x^{-1}dx^2+x^{-5/(N-1)}\sum_{i=1}^{N-1}dy_i^2$$

belongs to $\tilde{O}_{QB}^N \cap O_{QD}^N$ *for every N.*

Proof. The general solution of $\Delta u=1$ is

$$u=ax^3+\frac{1}{2}x+b,$$

which is bounded. Since $G1=u$ for some (a, b), $T\in\tilde{O}_{QB}^N$. By virtue of

$$D(G1)=D(u)=c\int_0^1\left(3ax^2+\frac{1}{2}\right)^2 xx^{-3}dx=\infty,$$

we have $T\in O_{QD}^N$.

6. Trivially $O_{QB}^N\cup O_{QD}^N\subset O_{QC}^N$. If there exist $u\in QB$ and $v\in QD$, then $G1$ is bounded and Dirichlet finite, hence in QC. This gives our last relation $O_{QB}^N\cup O_{QD}^N=O_{QC}^N$.

In the case $N=2$ the relations $O_{QP}<O_{QB}$; $O_{QP}<O_{QD}$; $O_{QB}\not\subset O_{QD}$; $O_{QD}\not\subset O_{QB}$; $O_{QB}<O_{QC}$; and $O_{QD}<O_{QC}$ can also be established (Nakai-Sario [3]) by means of the identity $\Delta u=\lambda^{-2}\Delta_e u$ between the Riemannian Δ, Euclidean Δ_e, and the conformal metric $ds=\lambda(z)|dz|$.

7. We collect our results:

THEOREM. *The strict inclusion relations*

$$O_G^N<O_{QP}^N<O_{QB}^N\cap O_{QD}^N<O_{QB}^N,\quad O_{QD}^N<O_{QB}^N\cup O_{QD}^N=O_{QC}^N$$

between quasiharmonic null classes of Riemannian N-manifolds are valid for every dimension $N\geqq 2$.

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