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ON COMPLETE FLAT SURFACES IN HYPERBOLIC 3-SPACE

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§ 1. Introduction.

"Any 2-dimensional, connected complete and flat Riemannian manifold M isometrically immersed in the Euclidean space E^3 is a plane or a cylinder." This theorem was first proved by Pogorelov [3, 4] in 1956 and an elementary proof was given by Massey [2] in 1962. Correspinding to it, the problems to characterize 2-dimensional connected, complete and flat Riemannian manifolds isometrically immersed in 3-sphere S^3 and in hyperbolic 3-space H^3 arise. The author studied the S^3 case in [7]. In this paper we shall study the H^3 case. Main theorems are Theorem 3 in §3 and Theorem 6 in § 5 which tell us that "any complete flat surface in H^3 is either a horosphere or an equidistant surface of a geodesic line." For the sake of simplicity, all functions are assumed to be smooth, i. e. of class C^∞.

§ 2. Basic considerations.

As the model of the hyperbolic 3-space H^3 we take the upper half space $x^3>0$ in the sense of Poincaré's representation. Without any loss of generality, we may assume that the sectional curvature of H^3 is -1. In this case the metric tensor of H^3 is given by

$$G_{\alpha\beta}=(x^3)^{-2}\delta_{\alpha\beta}. \tag{2.1}$$

Now, let us consider a connected complete surface M (i.e. 2-dimensional Riemannian manifold M immersed) in H^3 and take a coordinate neighborhood U on M. Then, U can be expressed parametrically in the form $x^\alpha=x^\alpha(u^1, u^2)$ $(\alpha, \beta, \gamma, \delta=1, 2, 3)$. If we put $X_i^\alpha\equiv\partial x^\alpha/\partial u^i$ $(i, j, k, l=1, 2)$ and choose the unit normal vector field N^α so that $|X_1, X_2, N|>0$. Then we have

$$g_{ij}=G_{\alpha\beta}X_i^\alpha X_j^\beta, \tag{2.2}$$

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A few months after I completed this paper, I found that the main theorems were also found independently by Yu. A. Volkov and S.M. Vladimirova a little earlier than me (Cf. Isometric immersions of a Euclidean plane in Lobachevskii space, Math. Notices, Acad. Sci. USSR **10** (1972), 619-622, Russian Original (1971)).

$$D_{X_k}X_j^\alpha=\left\{{i \atop j\ k}\right\}X_i^\alpha+h_{jk}N^\alpha, \tag{2.3}$$

$$D_{X_k}N^\alpha=-h_k^iX_i^\alpha, \tag{2.4}$$

where g_{ij}, h_{ij}, $\{{}_j{}^i{}_k\}$ and N^α are the first and the second fundamental tensors of M, the Christoffel's symbol with respect to g_{ij} and the unit normal vector of M, and D_{X_k} means the covariant derivative in H^3 in the direction of X_k^α. (2.3), (2.4) are Gauss' and Weingarten's derived equations.

The integrability conditions of (2.3) and (2.4) are

$$R_{ijkl}-h_{jk}h_{il}+h_{ik}h_{jl}=-g_{jk}g_{il}+g_{ik}g_{jl} \tag{2.5}$$

and

$$\nabla_l h_{jk}-\nabla_k h_{jl}=0 \tag{2.6}$$

known as Gauss' and Codazzi's equations, where R_{ijkl} is the curvature tensor with respect to g_{ij} and ∇_k means the covariant differentiation. When M is flat, (2.5) is equivalent with

$$h_{11}h_{22}-h_{12}^2=g_{11}g_{22}-g_{12}^2. \tag{2.5$'$}$$

Now assume that M is complete. Then, M can be regarded as an isometric immersion of the Euclidean plane E^2 with rectangular coordinates (u^1, u^2) and we have

$$g_{11}=g_{22}=1, \qquad g_{12}=0. \tag{2.7}$$

So (2.5)$'$ and (2.6) reduce to

$$h_{11}h_{22}-h_{12}^2=1, \tag{2.8}$$

$$\partial h_{11}/\partial u^2=\partial h_{12}/\partial u^1,\ \partial h_{22}/\partial u^1=\partial h_{12}/\partial u^2. \tag{2.9}$$

(2.9) tells us that there exists a smooth function $\phi(u^1, u^2)$ defined on the whole plane E^2 such that

$$h_{11}=\phi_{11},\ h_{12}=\phi_{12},\ h_{22}=\phi_{22}, \tag{2.10}$$

where we have put $\phi_{ij}=\partial^2\phi/\partial u^i\partial u^j$. Thus (2.8) reduces to

$$\phi_{11}\phi_{22}-\phi_{12}^2=1. \tag{2.11}$$

Now, by a theorem of Jörgens [1], the differential equation of elliptic type (2.11) admits as solutions only polynomials of the second degree of the variables u^1 and u^2. So h_{ij}'s are constants. If $h_{12}=0$, then

$$h_{11}=\lambda_1,\ h_{22}=\lambda_2 \qquad (h_{12}=0), \tag{2.12}$$

where λ_1, λ_2 are principal curvatures, i.e. eigenvalues of the second fundamental tensor and satisfy

$$\lambda_1\lambda_2=1. \tag{2.13}$$

In the case $h_{12}\neq 0$, we can reduce it to the first case by a suitable orthogonal transformation of rectangular coordinates u^1, u^2 in E^2.

Thus we get the following theorem:

THEOREM 1. *For each complete flat surface M in H^3 regarded as an isometric immersion of the Euclidean plane E^2 with rectangular coordinates (u^1, u^2), the principal curvatures λ_1 and λ_2 are constant and their product is equal to* 1. *Conversely, if we take two constants λ_1 and λ_2 so that their product is equal to* 1, *then there exists a complete flat surface in E^3 such that its principal curvatures coincide with the given λ_1 and λ_2.*

The proof of the latter part follows easily if we define g_{ij} and h_{ij} by (2.7) and (2.12) and apply the first fundamental theorem of surfaces in space forms (Cf. [6]). On M parameter curves are lines of curvature and isothermal.

COROLLARY. *Every complete flat surface in H^3 can not be a minimal surface.*

As both of the first and second fundamental tensors have constant components with respect to a parameter system which covers M, we see, by the fundamental theorem of surfaces in H^3 again, that the following theorem is true.

THEOREM 2. *Every complete flat surface M in H^3 is an orbit space of a* 2-*parametric subgroup of the isometry group $I(H^3)$ of H^3.*

The constants λ_1 and λ_2 have the same sign. If λ_1 and λ_2 are negative, we may change parameters and the unit normal vector so that

$$\bar{u}^1=-u^1,\ \bar{u}^2=u^2,\ \bar{N}^\alpha=-N^\alpha.$$

And the determinant $|\bar{X}_1\bar{X}_2\bar{N}|$ of the new Gaussian frame is positive and $\bar{\lambda}_1=-\lambda_1$, $\bar{\lambda}_2=-\lambda_2$. Hence, we may hereafter assume without any loss of generality that λ_1 and λ_2 are positive.

From the above arguments, we may, without any loss of generality, classify complete flat surfaces into following two types by their principal curvatures:

Umbilical type: $\lambda_1=\lambda_2=1$,

Non-umbilical type: $\lambda_2>1>\lambda_1>0 \quad (\lambda_1\lambda_2=1)$.

§ 3. Complete flat totally umbilical surfaces.

(2.1) shows that the Riemannian metrics of H^3 and E^3 in the upper half space $x^3>0$ are conformal with each other.

In general, for a conformal change of Riemannian metrics $G_{\alpha\beta}=\sigma^2 G^0_{\alpha\beta}$ on a differentiable manifold V^3 we have

(3.1) $$\left\{ \begin{matrix} \alpha \\ \beta \ \ \gamma \end{matrix} \right\} = \left\{ \begin{matrix} \alpha \\ \beta \ \ \gamma \end{matrix} \right\}_0 + \delta^\alpha_\beta \sigma_\gamma + \delta^\alpha_\gamma \sigma_\beta - G_0^{\alpha\delta} \sigma_\delta G^0_{\beta\gamma},$$

where we have put $\sigma_\alpha = \partial \log \sigma / \partial x^\alpha$. We consider a surface F immersed in V^3 and denote its unit normal vector, its first and second fundamental tensors with respect to the Riemannian metric $G^0_{\alpha\beta}$ by N_0^α, g^0_{ij} and h^0_{ij} respectively, and those with respect to the Riemannian metric $G_{\alpha\beta}$ by N^α, g_{ij} and h_{ij} respectively. Then there exist following relations as we can easily verify them:

(3.2) $$N^\alpha = (1/\sigma) N_0^\alpha,$$

(3.3) $$g_{ij} = \sigma^2 g^0_{ij},$$

(3.4) $$h_{ij} = \sigma \{ h^0_{ij} - (N_0^\alpha \sigma_\alpha) g^0_{ij} \}.$$

From (3.4) we see that the following lemma is true. (Cf. [5])

LEEMA. *The totally umbilical property of a surface in a Riemannian manifold* V^3 *is invariant under any conformal change of metrics.*

When F is totally umbilical, then we see easily that

(3.5) $$\Omega = (1/\sigma)(\Omega^0 - N_0^\alpha \sigma_\alpha). \qquad (\Omega, \Omega^0\text{: mean curvatures})$$

By virtue of the Lemma, a complete flat totally umbilical surface M in H^3 is also a totally umbilical surface in E^3. So, it is a piece or the whole of an ordinary sphere or plane in E^3. This tells us that M in consideration is one of proper spheres, horo-spheres, equidistant surfaces or H-planes in H^3 where H-plane means a plane in the sense of hyperbolic geometry. Thus, we have reduced our problem to calculate the function λ for each of these surfaces and to pick up the one for which $\lambda = \pm 1$.

Now, without any loss of generality, we may express any one of surfaces in H^3 described above by an equation of the type

(3.6) $$(x^1)^2 + (x^2)^2 + (x^3 - c)^2 = R^2 (R > 0),$$

the cases $c > R$; $c = R$; $R > c > -R (c \neq 0)$ and $c = 0$ corresponding to a proper sphere, a horo-sphere, an equidistant surface and an H-plane respectively. If we express (3.6) parametrically by

(3.6)′ $$x^1 = u^1,\ x^1 = u^2,\ x^3 = x^3(u^1, u^2),$$

then we see first that

(3.7) $$X^\alpha_j = \begin{cases} \delta^i_j & \text{for } \alpha = i\ (=1 \text{ or } 2), \\ -x^j/(x^3 - c) & \text{for } \alpha = 3. \end{cases}$$

As $\sigma = 1/x^3$ and $G^0_{\alpha\beta} = \delta_{\alpha\beta}$ in our case

(3.8) $$g^0_{ij} = G^0_{\alpha\beta} X^\alpha_i X^\beta_j = \delta_{ij} + x^i x^j / (x^3 - c)^2,$$

(3.9) $$N_0^\alpha=\begin{cases} x^i/R & \text{for } \alpha=i\ (=1 \text{ or } 2), \\ (x^3-c)/R & \text{for } \alpha=3. \end{cases}$$

(We took the normal direction toward outside as the positive direction of the normal.) Then, as

(3.10) $$h_{ij}^0=-(1/R)g_{ij}^0,\ \Omega^0=-1/R,$$

we see by (3.5) that

(3.11) $$\Omega=-c/R,$$

i.e. the mean curvature of the surface (3.6) is $-c/R$. Hence $\lambda=\pm 1$ if and only if the surface in consideration is a horo-sphere. Thus we get the following.

THEOREM 3. *Any complete flat totally umbilical surface in the hyperbolic 3-space is a horo-sphere. It is isometric with the Euclidean plane.*

N. B. A similar theorem holds good for any complete flat totally umbilical hypersurface M^n in H^{n+1} too.

§ 4. Geometrical construction of complete flat surfaces in H^3.

In order to study complete flat non-umbilical surface, we shall study here some geometric properties of complete flat surfaces.

For any curve $x^\alpha=x^\alpha(u^1(s), u^2(s))$ on M defined in some interval of s, we get easily

$$D_T T^\alpha=X_i^\alpha(\nabla_T T^i)+(h_{ij}T^iT^j)N^\alpha,$$

where $T^\alpha=X_i^\alpha T^i$ is the unit tangent vector. Any u^1-curve on M is a geodesic of M as it is the image of a straight line by an isometric immersion of E^2 into H^3. So for a u-curve, we have

(4.1) $$D_T T^\alpha=h_{11}N^\alpha.$$

Now, the Frenet formulas of the u^1-curve are of the form

(4.2) $$\begin{aligned} D_T T^\alpha&=\kappa_1 H, \\ D_T H^\alpha&=-\kappa_1 T^\alpha+\tau_1 B^\alpha, \\ D_T B^\alpha&=-\tau_1 H^\alpha, \end{aligned}$$

Comparing (4.1) with (4.2), we see first $H^\alpha=N^\alpha$ as $\kappa_1>0$ by assumption and $h_{11}=\lambda_1>0$. So we get

(4.3) $$T^\alpha=X_1^\alpha,\ H^\alpha=N^\alpha,\ B^\alpha=-X_2^\alpha,$$

(4.4)$_1$ $$\kappa_1=h_{11}=\lambda_1.$$

On the other hand, we have

$$D_T H^\alpha = D_{X_1} N^\alpha = -h_1^i X_i^\alpha$$
$$= -h_{11} T^\alpha + h_{12} B^\alpha.$$

Comparing this with $(4.2)_2$, we get

$$(4.2)_2 \qquad \tau_1 = h_{12} = 0.$$

In the same way, we see that the Frenet's frame of any u^2-curve on M is given by

$$(4.5) \qquad \bar{T} = X_2, \bar{H} = N, \bar{B} = X_1$$

and the curvature and torsion are given by

$$(4.6) \qquad \kappa_2 = h_{22} = \lambda_2, \tau_2 = -h_{12} = 0$$

respectively. Thus, we get the following

THEOREM 4. *For each complete flat surface M in H^3 with the principal curvatures λ_1 and λ_2, the curvature and torsion of a family of lines of curvature are given by* (4.4) *and those of another family of lines of curvature are given by* (4.6).

Now the above argument suggests us a method how to construct complete flat surfaces in H^3.

THEOREM 5. *Let λ_1 and λ_2 be two positive constants such that their product is equal to* 1. *We first draw a curve Γ_1 with curvature $\kappa_1(u_1, 0) = \lambda_1$ and torsion $\tau_1(u^1, 0) = 0$ in H^3, the parameter being the arc length. Using the moving Frenet's frame (T, H, B) of Γ_1, we draw, for each fixed value u^1, a curve $\Gamma_2(u^1)$ with curvature $\kappa_2(u^1, u^2) = \lambda_2$ and torsion $\tau_2(u^1, u^2) = 0$ with initial Frenet's frame*

$$(4.7) \qquad \bar{T}(u^1, 0) = -B(u^1), \bar{H}(u^1, 0) = H(u^1), \bar{B}(u^1, 0) = T(u^1),$$

the parameter u^2 being arc length. Then, the locus of all $\Gamma_2(u^1)$ $(u^1 \in R)$ is a complete flat surface in H^3.

Proof. By the latter half of Theorem 1, there exists complete flat surfaces in H^3 such that (2.7) and (2.12) hold good and any two of them are congruent under a motion of H^3. We take any one of them and denote it by M. M can be regarded as an isometric immersion of E^2 into H^3 by a map f.

At each point $f(u^1, u^2)$ of M, we define an orthonormal frame (T, H, B) by

$$(4.8) \qquad T(u^1, u^2) = X_1(u^1, u^2), \quad H(u^1, u^2) = N(u^1, u^2), \quad B(u^1, u^2) = -X_2(u^1, u^2).$$

We fix the value u^2, then they constitute the moving Frenet's frame for the u^1-curve and the curvature and torsion are given by

$$(4.9) \qquad \kappa_1(u^1, u^2) = \lambda_1, \ \tau_1(u^1, u^2) = 0.$$

Especially, the moving Frenet frame $(T(u^1), H(u^1), B(u^1))$ of the u^1-curve $u^2=0$ relates to the Gauss' frame of M on the curve by

$$T(u^1)=X_1(u^1, 0),\ H(u^1)=N(u^1, 0),\qquad B(u^1)=-X_2(u^1, 0). \tag{4.10}$$

In the same way, the moving frame

$$\bar{T}(u^1, u^2)=X_2(u^1, u^2),\ \bar{H}(u^1, u^2)=N(u^1, u^2),\qquad \bar{B}(u^1, u^2)=\bar{X}(u^1, u^2) \tag{4.11}$$

gives, for each fixed value of u^1, the moving Frenet's frame of the u^2-curve. The curvature and torsion of the latter curve are given by

$$\kappa_2(u^1, u^2)=\lambda_2,\qquad \tau_2(u^1, u^2)=0. \tag{4.12}$$

By (4.8), (4.10) and (4.11) we get (4.7). This completes the proof.

§ 5. Complete flat non-totally umbilical surfaces.

As a preparation we remark, by (2.1) and (3.1), that

$$\begin{cases} \left\{ {i \atop j\ k} \right\}=0, & \left\{ {3 \atop j\ k} \right\}=-\sigma\delta_{jk}, \\ \left\{ {i \atop 3\ k} \right\}=-\delta_k^i\sigma, & \left\{ {3 \atop 3\ k} \right\}=0, \\ \left\{ {i \atop 3\ 3} \right\}=0, & \left\{ {3 \atop 3\ 3} \right\}=-\sigma \end{cases} \tag{5.1}$$

hold good, where $\sigma=1/x^3$.

First, let us consider a half line Γ_1

$$x^1=t,\ x^2=0,\ x^3=\tan\omega\cdot t\qquad (t>0) \tag{5.2}$$

in the plane $x^2=0$, where ω is the angle such that $\tan\omega=\sqrt{1-\lambda_1^2}/\lambda_1$ and $0<\omega<\pi/2$. Then, the line element du^1 and the unit tangent vector T are given by

$$du^1=\frac{dt}{x^3\cos\omega}, \tag{5.3}$$

$$T=(x^3\cos\omega,\ 0,\ x^3\sin\omega). \tag{5.4}$$

As $x^2=0$ is an H-plane and each H-plane is a totally geodesic surface in H^3, the unit principal normal vector H lies in the H-plane $x^2=0$ and so we see that $H=(-x^3\sin\omega, 0, x^3\cos\omega)$ and the unit binormal vector is given by $B=(0, x^3, 0)$. Putting (5.1) and (5.3) into

$$\frac{\delta T^\alpha}{du^1}=\frac{dt}{du^1}\frac{\delta T^\alpha}{du^1}=\frac{dt}{du^1}\frac{dT^\alpha}{dt}+\left\{ {\alpha \atop \beta\ \gamma} \right\}T^\beta T^\gamma$$

we can easily verify that

$$\frac{\delta T^{\alpha}}{du^1}=\kappa_1 H^{\alpha}, \qquad \kappa_1=\cos\omega=\lambda_1 \tag{5.5}$$

holds good, where κ_1 is the curvature of Γ_1. In the same way, we can easily get

$$\frac{\delta H^{\alpha}}{du^1}=-\kappa_1 T^{\alpha},\ \tau_1=0 \tag{5.6}$$

where τ_1 is the torsion of Γ_1.

Secondly, let us consider a circle Γ_{γ} defined by

$$x^1=\gamma\cos\theta,\ x^2=\gamma\sin\theta,\ x^3=k\ \ (k=\gamma\tan\omega) \tag{5.7}$$

on a plane (a horo-sphere) $x^3=k$, where $\gamma>0$ is a constant. Then, we see that its arc length u^2 and the unit tangent vector $\bar{T}$ are given by

$$\frac{du^2}{d\theta}=\cot\omega \qquad (u^2=\theta\cot\omega), \tag{5.8}$$

$$\bar{T}=(-k\sin\theta,\ k\cos\theta,\ 0). \tag{5.9}$$

We denote the H-plane with center 0 (the origin) and radius γ/λ_1 by π_{γ}, then Γ_{γ} lies on π_{γ}. So, the unit principal normal vector $\bar{H}$ is tangent to π_{γ}, normal to $\bar{T}$ and is given by

$$\bar{H}=\left(\frac{-k^2\cos\theta}{\sqrt{k^2+\gamma^2}},\ \frac{-k^2\sin\theta}{\sqrt{k^2+\gamma^2}},\ \frac{k\gamma}{\sqrt{k^2+\gamma^2}}\right).$$

In the similar way as the case of Γ_1, we can easily verify that

$$\frac{\delta\bar{T}^{\alpha}}{du^2}=\kappa_2\bar{H}^{\alpha}, \qquad \kappa_2=\frac{1}{\lambda_1}=\lambda_2,$$

(5.10)

$$\frac{\delta H^{\alpha}}{du^2}=-\kappa_2\bar{T}^{\alpha}, \qquad \tau_2=0.$$

Thirdly, let us consider a half cone S which is a surface of revolution of the half line Γ_1 around the x^3-axis. Then, it is easy to see that (i) all generating lines have same curvature λ_1 and torsion 0 and are equidistant curves to the x^3-axis and (ii) all circles Γ_{γ} are same curvature λ_2 and torsion 0 and are congruent in H^3. Thus, S has similar properties as a circular cylinder in E^3.

Now, we may regard the curve Γ_1 defined by (5.2) as the curve Γ_1 in Theorem 5. Its arc length u^1 is given by $u^1=\operatorname{cosec}\omega\cdot\log t$. The circle Γ_{γ} corresponds to the curve $\Gamma_2(u^1)$ in Theorem 5 for $u^1=\operatorname{cosec}\omega\cdot\log\gamma$. As $u^2=\theta\cot\omega$, we may easily verify that the Frenet's frame $(\bar{T},\bar{H},\bar{B})$ of Γ_{γ} at the point $\theta=0$, coincides with $(-B,H,T)$ of Γ_1 at the same point. Hence, the cone S is nothing but the complete flat surface corresponding to the given constants λ_1 and λ_2 assured in Theorem 5. (We may easily verify directly that the Gaussian curvature of S is everywhere equal to zero.) S is an orbit space of a 2-parametric subgroup of isometries of the form

$$\bar{x}^1=\rho(x^1\cos\gamma+x^2\sin\gamma),$$

$$\bar{x}^2=\rho(-x^1\sin\gamma+x^2\cos\gamma),$$

$$\bar{x}^3=\rho x^3$$

$\rho>0$ and γ being parameters.

Suppose T be an isometry of H^3, i.e. a composite of some inversions with respect to some H-planes, then $T(S)$ is again a half cone whose axis is orthogonal to the plane $x^3=0$ with vertex on $x^3=0$ or $T(S)$ is one half of a cyclide with two vertices on the plane $x^3=0$. The latter carries a family of congruent equidistant curves corresponding to the family of generating lines of the half cone S and a family of congruent proper circles corresponding to the family of circles Γ_γ $(0<\gamma<\infty)$ on S. There is no distinction between the half cone S and the half cyclide $T(S)$ in hyperbolic geometry. Each of them is an equidistant surface from a geodesic line in H^3 and can be regarded as an analogue of a circular cylinder in the sense of hyperbolic geometry. Thus, we get the following

THEOREM 6. *Any complete flat non-totally umbilical surface in H^3 is an equidistant surface from a geodesic line in H^3.*

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