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QUASI-NORMAL ANALYTIC SPACES, II

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In [2], we have discussed relations between three kinds of analytic sheaves on an analytic space, that is, $\mathcal{O}$, $\mathcal{O}'$ and $\tilde{\mathcal{O}}$ which are, respectively, the sheaf of germs of holomorphic functions, the sheaf of germs of continuous and weakly holomorphic functions, and the sheaf of germs of weakly holomorphic functions.

The present paper is a continuation of [2]. § 1 is devoted to a result concerning the product of analytic spaces, which will be used for dealing with an example in § 3. In § 2, we discuss a quasi-normality condition in the case of subvarieties which consist of certain submanifolds; § 3 contains some examples.

The notations and terminology of [2] will be used without any specific mention.

§ 1. Quasi-normality of product spaces.

We prove the following

THEOREM 1. *Let* $X=X_1\times\cdots\times X_m$ *be the Cartesian product of analytic spaces* X_i, $i=1,\cdots,m$. *Let* $p=(p_1,\cdots,p_m)\in X$. *Then,* X *is quasi-normal at* p *if and only if* X_i *are quasi-normal at* p_i *for all* i.

Proof. It is sufficient to treat the case in which $m=2$. Let $V=V_1\times V_2$ where V_i are neighborhoods of p_i which are subvarieties of open subsets D_i of $\boldsymbol{C}^{n_i}$, p_i being origins of $\boldsymbol{C}^{n_i}$, $i=1,2$. Assume that V is quasi-normal at $p=(p_1,p_2)$, and let $\boldsymbol{f}\in{}_{V_1}\mathcal{O}'_{p_1}$. Choose a representative f of $\boldsymbol{f}$ which is continuous and weakly holomorphic on $V_1\cap\Delta_1$ where Δ_1 is a suitable neighborhood of p_1 in D_1. Let $\Delta=\Delta_1\times\Delta_2$ where Δ_2 is a neighborhood of p_2 in D_2, and let π denote the projection of $V\cap\Delta$ onto $V_1\cap\Delta_1$. Since

$$\mathcal{R}(V\cap\Delta)=(\mathcal{R}(V_1)\cap\Delta_1)\times(\mathcal{R}(V_2)\cap\Delta_2),$$

we see that $f\circ\pi$ is continuous and weakly holomorphic on $V\cap\Delta$, hence holomorphic on $V\cap\Delta'$, where $\Delta'=\Delta'_1\times\Delta'_2$ with $\Delta'_i\subset\Delta_i$. From this follows that f is holomorphic on $V_1\cap\Delta'_1$, which implies that $\boldsymbol{f}\in{}_{V_1}\mathcal{O}_{p_1}$.

Conversely, let V_i be quasi-normal at p_i, $i=1,2$. We can choose neighborhoods Δ_i of p_i in D_i so that $V_i\cap\Delta_i$ are quasi-normal spaces; this is possible, because the set of points where a space is quasi-normal is open ([2], p. 182). Let $\boldsymbol{F}\in{}_V\mathcal{O}'_p$. There exist neighborhoods Δ'_i of p_i, $\Delta'_i\subset\Delta_i$, and a representative F of $\boldsymbol{F}$ such that F is

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bounded, continuous and weakly holomorphic on $V\cap(\Delta'_1\times\Delta'_2)$. For an arbitrarily fixed point $u\in V_1\cap\Delta'_1$, we define F_u by $F_u(v)=F(u,v)$, $v\in V_2\cap\Delta'_2$; F_v is similarly defined on $V_1\cap\Delta'_1$ for a fixed $v\in V_2\cap\Delta'_2$. If $u\in\mathfrak{R}(V_1)\cap\Delta'_1$, the assumption implies that F_u is holomorphic on $V_2\cap\Delta'_2$. Now, let $u\in V_1\cap\Delta'_1$; choose $u_n\in\mathfrak{R}(V_1)\cap\Delta'_1$ such that $u_n\to u$, $n\to\infty$. Since $\{F_{u_n}\}$ is a uniformly bounded sequence of holomorphic functions on $V_2\cap\Delta'_2$, a subsequence $\{F_{u_{n(i)}}\}$ converges to a function G uniformly on compact subsets, which is holomorphic on $V_2\cap\Delta'_2$. The continuity of F implies that $G(v)=F_u(v)$ for $v\in V_2\cap\Delta'_2$. Similarly, F_v is holomorphic, and a generalized theorem of Hartogs ([1], p. 292) assures that F is holomorphic on $V\cap(\Delta'_1\cap\Delta'_2)$; hence we have $\boldsymbol{F}\in{}_V\mathcal{O}_p$. This completes the proof.

§ 2. Quasi-normality at certain reducible points.

Let $W=M_1\cup M_2$, where M_i are connected complex submanifolds of an open neighborhood of 0 in $\boldsymbol{C}^n$. We are interested in the problem: What condition is necessary and sufficient for W to be quasi-normal at 0? This is equivalent to the following: What is a necessary and sufficient condition for a continuous function on a neighborhood U of 0 in W which is holomorphic on $M_i\cap U$, $i=1,2$, to be holomorphic on U? In this section, we are concerned with the following restricted case. Let D be a connected open neighborhood of 0 in $\boldsymbol{C}^n$ and $f_1,\cdots,f_m$ be holomorphic functions on D such that $f_i(0)=0$, $i=1,\cdots,m$. Let

$$M_1=D\times\{0\}\subset D\times\boldsymbol{C}^m,\ M_2=\{(z,f_1(z),\cdots,f_m(z))\,|\,z=(z_1,\cdots,z_n)\in D\}.$$

M_2 is the graph of the holomorphic map $f=(f_1,\cdots,f_m)\colon D\to\boldsymbol{C}^m$. Now, $W=M_1\cup M_2$ is an analytic subvariety of $D\times\boldsymbol{C}^m$; M_i are irreducible branches. Let $V=\{z\in D\,|\,f_i(z)=0, i=1,\cdots,m\}$. The set of singular points of W is then $\mathfrak{S}(W)=M_1\cap M_2=V\times\{0\}$.

We denote by ${}_n\mathcal{O}_0$ the ring of germs of holomorphic functions at 0 in $\boldsymbol{C}^n$. The germ of a variety defined by the germs $\boldsymbol{f}_i$ will be denoted by $\boldsymbol{V}=\boldsymbol{V}(\boldsymbol{f}_1,\cdots,\boldsymbol{f}_m)$; the ideal of $\boldsymbol{V}$ will be denoted by id $\boldsymbol{V}$, that is, id $\boldsymbol{V}=\{\boldsymbol{g}\in{}_n\mathcal{O}_0\,|\,\boldsymbol{g}$ vanishes on $\boldsymbol{V}\}$.

THEOREM 2. *Let $f_1,\cdots,f_m$ be holomorphic functions on D such that $f_i(0)=0$, $i=1,\cdots,m$. Let $W=M_1\cup M_2$ be as above. Let $\mathcal{J}=(\boldsymbol{f}_1,\cdots,\boldsymbol{f}_m)$, the ideal of ${}_n\mathcal{O}_0$ generated by the germs $\boldsymbol{f}_i$. Then W is quasi-normal at 0 if and only if*

$$\text{id}\ \boldsymbol{V}(\boldsymbol{f}_1,\cdots,\boldsymbol{f}_m)=\mathcal{J}$$

or, equivalently, $\sqrt{\mathcal{J}}=\mathcal{J}$ where $\sqrt{\mathcal{J}}$ is the radical of $\mathcal{J}$.

Proof. Let W be quasi-normal at 0; let $\boldsymbol{g}\in$id $\boldsymbol{V}(\boldsymbol{f}_1,\cdots,\boldsymbol{f}_m)$. $\boldsymbol{g}$ is represented by a holomorphic function g on a polydisk $\Delta_1\subset D$; $g=0$ on the variety $\{z\in\Delta_1\,|\,f_i(z)=0, i=1,\cdots,m\}$. By means of the projection $\pi\colon D\times\boldsymbol{C}^m\to D$, we define a function G as follows:

$$G=\begin{cases}0, & \text{on}\quad M_1\cap(\Delta_1\times\{0\}),\\ g\circ\pi, & \text{on}\quad M_2\cap(\Delta_1\times\boldsymbol{C}^m).\end{cases}$$

G is continuous and weakly holomorphic on $W\cap(\Delta_1\times \boldsymbol{C}^m)$, hence holomorphic on a neighborhood of 0 in W. Consequently, there exist suitable polydisks $\Delta'_1\subset\Delta_1$, $\Delta_2\subset \boldsymbol{C}^m$ such that G extends to a holomorphic function $\tilde{G}$ on $\Delta'_1\times\Delta_2$. The Taylor series expansion of $\tilde{G}$ is so arranged that

$$\tilde{G}(z, w)=g_0(z)+\sum_{i=1}^{m} g_i(z, w_i, \cdots, w_m)w_i,$$

where g_i are holomorphic; $z=(z_1, \cdots, z_n)$, $w=(w_1, \cdots, w_m)$. From the definition of G we see that $g_0(z)=0$. Let Δ''_1 be so small that $f(\Delta''_1)\subset\Delta_2$. Putting $w_i=f_i(z)$, $i=1, \cdots, m$, we obtain

$$g(z)=\sum_{i=1}^{m} a_i(z)f_i(z), \qquad z\in\Delta''_1,$$

where a_i are holomorphic functions on Δ''_1. We have thus

$$\boldsymbol{g}=\sum_{i=1}^{m} \boldsymbol{a}_i\boldsymbol{f}_i.$$

Conversely, let id $\boldsymbol{V}(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_m)=\mathcal{J}$ and let $\boldsymbol{G}\in {}_W\mathcal{O}'_0$, i.e., a germ of a continuous, weakly holomorphic function at 0. Let G be a representative of $\boldsymbol{G}$ on $W\cap(\Delta_1\times\Delta_2)$, and let $G_i=G|M_i\cap(\Delta_1\times\Delta_2)$, $i=1, 2$. We may suppose that $f(\Delta_1)\subset\Delta_2$. Let φ_i: $\Delta_1\to M_i\cap(\Delta_1\times\Delta_2)$ be such that $\varphi_1(z)=(z, 0)$, $\varphi_2(z)=(z, f(z))$, $z\in\Delta_1$. G_i are holomorphic by the Riemann extension theorem, hence $G_i\circ\varphi_i$ are holomorphic functions on Δ_1. Since $\boldsymbol{G}_1\circ\varphi_1-\boldsymbol{G}_2\circ\varphi_2\in\mathcal{J}$ by assumption, there exist a polydisk $\Delta'_1\subset\Delta_1$ and holomorphic functions a_i on Δ'_1, $i=1, \cdots, m$, such that

$$G_2\circ\varphi_2=G_1\circ\varphi_1+\sum_{i=1}^{m} a_if_i \quad \text{on} \quad \Delta'_1.$$

We define $\tilde{G}$ by

$$\tilde{G}(z, w)=G(z, 0)+\sum_{i=1}^{m} a_i(z)w_i, \qquad (z, w)\in\Delta'_1\times\Delta_2.$$

$\tilde{G}$ is a holomorphic function on $\Delta'_1\times\Delta_2$ and it is easily seen that $\tilde{G}=G$ on $W\cap(\Delta'_1\times\Delta_2)$. We have thus $\boldsymbol{G}\in {}_W\mathcal{O}_0$. The second statement is clear from the Nullstellensatz. This completes the proof.

COROLLARY 3. *If* $(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_m)$ *is a prime ideal, then* $M_1\cup M_2$ *is quasi-normal at* 0.

COROLLARY 4. *If the map* $f=(f_1, \cdots, f_m)$ *is nonsingular at* 0, *then* $M_1\cup M_2$ *is quasi-normal at* 0.

Proof. First, let $m<n$. We denote by $J_f(0)$ the Jacobian matrix of f at 0. Since $\operatorname{rank} J_f(0)=m$, there exists a local coordinate set $w_1, \cdots, w_n$ at 0 in which $w_i=f_i$, $i=1, \cdots, m$. For $\boldsymbol{g}\in$ id $\boldsymbol{V}(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_m)$, we have as in Theorem 2,

$$g=\sum_{i=1}^{n} a_i(w_i, \cdots, w_n)w_i.$$

Putting $w_1=\cdots=w_m=0$, we obtain

$$g=\sum_{i=1}^{m} a_i(w_i, \cdots, w_n)w_i,$$

which implies that $\boldsymbol{g}\in(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_m)$. In case where $n\leqq m$, $\{f_1, \cdots, f_m\}$ contains a local coordinate set at 0; therefore, $(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_m)$ is the maximal ideal of ${}_n\mathcal{O}_0$. It follows that id $\boldsymbol{V}(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_m)=(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_m)$, completing the proof.

It should be noted that the converse of Corollary 4 does not hold in general. In fact, let $n\geqq 2$; let $\boldsymbol{f}_1\in{}_n\mathcal{O}_0$ for which $f_1(0)=0$, $(\partial f_1/\partial z_1)(0)\neq 0$. Let $\mathcal{J}=(\boldsymbol{f}_1)$, then $\sqrt{\mathcal{J}}=\mathcal{J}$ from the above. Take $\boldsymbol{f}_2, \cdots, \boldsymbol{f}_m\in\mathcal{J}$, then clearly $\sqrt{(\boldsymbol{f}_1, \boldsymbol{f}_2, \cdots, \boldsymbol{f}_m)}=(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_m)$. But rank $J_f(0)=1$, which is not maximal.

The following shows, however, the converse is also true in a special case.

PROPOSITION 5. *Let $f_1, \cdots, f_m$ be holomorphic on D with $f_i(0)=0$, $i=1, \cdots, m$. Let* $\dim_0 \boldsymbol{V}(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_m)=0$. *Then $M_1\cup M_2$ is quasi-normal at 0 if and only if $f=(f_1, \cdots, f_m)$ is non-singular at 0.*

Proof. Let $\mathcal{J}=(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_m)$. If $M_1\cup M_2$ is quasi-normal at 0, then $\boldsymbol{z}_i\in$ id $\boldsymbol{V}(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_m)=\mathcal{J}$, $i=1, \cdots, n$, so we have

$$\boldsymbol{z}_i=\sum_{k=1}^{m} \boldsymbol{a}_{ik}\boldsymbol{f}_k, \qquad \boldsymbol{a}_{ik}\in{}_n\mathcal{O}_0, \qquad i=1, \cdots, n;$$

hence

$$\sum_{k=1}^{m} a_{ik}(0)\frac{\partial f_k}{\partial z_j}(0)=\delta_{ij}, \qquad i, j=1, \cdots, n.$$

It follows that rank $J_f(0)=n$.

COROLLARY 6. *Let $D\subset \boldsymbol{C}^1$ and f_i, $i=1, \cdots, m$, be holomorphic functions on D; $f_i(0)=0$. Then $M_1\cup M_2$ is quasi-normal at 0 if and only if $f_i'(0)\neq 0$ for some i.*

§ 3. Examples.

EXAMPLE 1. We construct a space which is irreducible at p and $\mathcal{O}_p\subsetneqq\mathcal{O}'_p\subsetneqq\tilde{\mathcal{O}}_p$. Let

$$V=\{(z, w, u)\in\boldsymbol{C}^3\,|\,z^2-w^2u=0\},\quad W=\{(x, y)\in\boldsymbol{C}^2\,|\,x^2-y^3=0\},$$

and let $X=V\times W$. X is a subvariety of $\boldsymbol{C}^5$ and irreducible at the origin $p\in\boldsymbol{C}^5$, but not locally irreducible there because of the same property of V ([3], p. 93). Consequently, we have $\mathcal{O}'_p\subsetneqq\tilde{\mathcal{O}}_p$ by Theorem 1 in [2]. On the other hand, X is not quasi-normal at p by Theorem 1 in § 1 because of that property of W; thus we have $\mathcal{O}_p \subsetneqq \mathcal{O}'_p$.

EXAMPLE 2. There exists a space X such that X is irreducible at p and $\mathcal{O}_p = \mathcal{O}'_p \subsetneqq \tilde{\mathcal{O}}_p$.

Proof. First, we note the following fact: Let φ be a holomorphic homeomorphism of an analytic space X onto an analytic space Y. Then, (1) X is irreducible at p if and only if Y is irreducible at $\varphi(p)$; (2) X is locally irreducible at p if and only if Y is locally irreducible at $\varphi(p)$. In fact, (2) is immediate from (1). (1) is easily seen from the following fact:

There exist a neighborhood V of p and a neighborhood W of $\varphi(p)$ such that V is a subvariety of an open subset D_1 of $\boldsymbol{C}^n$ and W is a subvariety of an open subset D_2 of $\boldsymbol{C}^m$, and φ is a holomorphic homeomorphism of V onto W. $\{V\}$ and $\{W\}$ can be chosen so that they are bases of neighborhoods at p and $\varphi(p)$.

Now, let X be an analytic space which is irreducible but not locally irreducible at $p \in X$, e.g., the variety V in Example 1 with $p=0 \in \boldsymbol{C}^3$. We denote by Y the quasi-normalization of X with the projection π'. π' is a holomorphic homeomorphism of Y onto X. Let $\pi'(q)=p$, $q \in Y$. Then Y is irreducible, but not locally irreducible at q. Y is clearly quasi-normal. We have thus ${}_Y\mathcal{O}_q = {}_Y\mathcal{O}'_q \subsetneqq {}_Y\tilde{\mathcal{O}}_q$.

Last of all, we make a list of examples for all the possible cases:

	irreducible at p	reducible at p
$\mathcal{O}_p = \mathcal{O}'_p = \tilde{\mathcal{O}}_p$	normal at p	
$\mathcal{O}_p \subsetneqq \mathcal{O}'_p = \tilde{\mathcal{O}}_p$	$z^2 - w^3 = 0$	
$\mathcal{O}_p = \mathcal{O}'_p \subsetneqq \tilde{\mathcal{O}}_p$	Example 2	Theorem 6, [2]
$\mathcal{O}_p \subsetneqq \mathcal{O}'_p \subsetneqq \tilde{\mathcal{O}}_p$	Example 1	Corollary 6

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