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SOME PROPERTIES OF EXTREMAL POLYNOMIALS FOR THE ILIEFF CONJECTURE

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Let P_n denote the family of complex polynomials each of degree n, with leading coefficient 1, and having all of its roots in $\bar{D}(0, 1)$, the closed unit disc with center at 0 and radius 1. Let $p\in P_n$ have roots $z_1, \cdots, z_n$ and have roots $w_1, \cdots, w_{n-1}$. For such p we use $I(z_j), I(p)$, and $I(P_n)$ to denote the numbers $\min\{|z_i-w_k|: 1\leqq k\leqq n-1\}$, $\max\{I(z_j): 1\leqq j\leqq n\}$, and $\sup\{I(p): p\in P_n\}$ respectively. Then $p\in P_n$ is called an extremal polynomial for the Ilieff conjecture if $I(p)=I(P_n)$. With this notation the Gauss-Lucas theorem implies that $I(P_n)\leqq 2$ and the conjecture of Ilieff is that $I(p)\leqq 1$ for all $p\in P_n$. We show that there exist extremal polynomials, that an extremal polynomial must have at least one root on each subarc of the unit circle of length $\geqq\pi$, and we find the extremal polynomials for $n=3$ and 4.

We begin with a

LEMMA. *P_n is a compact normal family in the open plane C.*

Proof. Let $R>0$. If $p\in P_n$ then $|p(z)|=|(z-z_1)\cdots(z-z_n)|\leqq(R+1)^n$ when $|z|\leqq R$ so that by the theorem of Stieltjes and Osgood P_n is a normal family in C. Next if $\{p_j\}$ is a sequence in p_n converging almost uniformly, i.e., uniformly on compact sets, to a limit p then the Weierstrass convergence theorem implies that p is a polynomial of degree n with leading coefficient 1 and Hurwitz's theorem implies that all the roots of p lie in $\bar{D}(0, 1)$.

THEOREM 1. *There exists an extremal polynomial.*

Proof. Let $\{p_k\}$ be a sequence in P_n such that $\lim I(p_k)=I(P_n)$. We may assume that $\{p_k\}$ converges almost uniformly to a limit $p\in P_n$. Then $I(p)\leqq I(P_n)$. If equality does not hold then $I(p)+4\varepsilon=I(P_n)$ for some $\varepsilon>0$. Choose δ, $0<\delta<\varepsilon$, so that p has no roots in $0<|z-z_j|<2\delta, j=1, \cdots, n$. If $D(z_j, \delta)$ denotes the disc with center z_j and radius δ then $\cup_1^n D(z_j, \delta)$ contains all the roots of p and each disc $D(z_j, I(p)+\varepsilon)$ contains at least one root of p'. Thus for sufficiently large k all the roots of p_k are contained in $\cup_1^n D(z_j, \delta)$ and each disc $D(z_j, I(p)+\varepsilon)$ contains at least one root of p_k', whence $I(p_k)<I(p)+2\varepsilon=I(P_n)-2\varepsilon$ for these k which is a contradiction.

THEOREM 2. *If $p\in P_n$ and $|z_j|<1$ for the roots $z_1, \cdots, z_n$ of p, then p is not an extremal polynomial.*

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Proof. We may assume that $I(p)=I(z_1)$. By the Gauss-Lucas theorem we have that $|w_j|<1$ for the roots $w_1, \cdots, w_{n-1}$ of p'; also $|z_1-w_j|\geqq I(z_1)$ for $j=1, \cdots, n-1$. Therefore for each w_j there is a sequence $\{s_k^j\}_{k=1}^{\infty}$ of points in $D(0, 1)$ converging to w_j and such that $|z_1-s_k^j|>I(z_1)$. Consider the functions

$$p_k(z)=n\int_{z_1}^{z}\prod_{j=1}^{n-1}(w-s_k^j)dw, \qquad k=1, 2, \cdots.$$

They are polynomials of degree n and leading coefficient 1, and the sequence $\{p_k\}$ converges almost uniformly on C to p. Since the roots of p are in the open disc $D(0, 1)$, Hurwitz's theorem implies there is an integer K such that for $k>K$, p_k has all its roots in $D(0, 1)$. Thus $p_k\in P_n$ for $k>K$ and by construction $I(p_k)>I(z_1)=I(p)$. Hence $I(p)\neq I(P_n)$.

Thus an extremal polynomial in P_n has at least one root on the unit circle $C(0, 1)$. An improvement of this result is given in

THEOREM 3. *If $p\in P_n$ is an extremal polynomial then every closed subarc of $C(0, 1)$ of length greater than or equal to π contains a root of p.*

Proof. Suppose first that p has one distinct root, say z_1, on $C(0, 1)$. We may assume that $z_1=1$. Let $r=\max\{|z_j| : m\leqq j\leqq n\}$ where $z_m, z_{m+1}, \cdots, z_n$ are the roots of p that lie in $D(0, 1)$, and define $s=(1-r)/2$. Then the polynomial $q(z)=p(z+s)$ is in P_n, has all of its roots in $D(0, 1)$ and $I(q)=I(p)$. By theorem 2, however, $I(q)<I(P_n)=I(p)$. Now suppose p has two distinct roots z_1 and z_2 on $C(0, 1)$ and that z_1 and z_2 are separated by a subarc of $C(0, 1)$ of length greater than π and containing no root of p. We may assume that for some θ, $0<\theta<\pi/2$, $z_1=\exp(i\theta)$, $z_2=\exp(-i\theta)$ and p has no roots on $\{\exp(it) : \theta<t<2\pi-\theta\}$. Define $r=\max\{|z_j| : |z_j|<1\}$ if p has a root in $D(0, 1)$ and $r=0$ otherwise. Define $s=\min\{\cos\theta, (1-r)/2\}$. Then as above, the polynomial $q(z)=p(z+s)$ is in P_n, has all of its roots in $D(0, 1)$, and $I(p)=I(q)<I(P_n)=I(p)$. This contradiction establishes the theorem.

As an application of theorem 3 we have

THEOREM 4. *Let $p\in P_n$ be extremal. If z_k is a root of p with $|z_k|<1$ then*

$$I(z_k)\leqq\left[\frac{(1+|z_k|^2)(1+|z_k|)^{n-3}}{n}\right]^{1/(n-1)}$$

Proof. We may assume without loss of generality that $z_1=1$ is a root of p on $C(0, 1)$ nearest to z_k and that $z_k=z_n=r\exp(iy)$ with $0\leqq y\leqq\pi$. By theorem 3, p has a root on the circular arc $\{\exp(it) : 0<t\leqq\pi\}$, say $\exp(iu)=z_2$. Then writing $p(z)=\Pi_{j=1}^{n}(z-z_j)$ and $p'(z)=n\Pi_{j=1}^{n-1}(z-w_j)$ we have

$$p'(z_n)=\prod_{j=1}^{n-1}(z_n-z_j)=n\prod_{j=1}^{n-1}(z_n-w_j)$$

so that

$$n(I(z_n))^{n-1} \leqq n \prod_{j=1}^{n-1} |z_n - w_j| = \prod_{j=1}^{n-1} |z_n - z_j| \leqq |z_n - 1| |z_n - \exp(iu)| (1 + |z_n|)^{n-3}.$$

Since $\varphi(t) = |z_n - \exp(it)| = |r \exp(iy) - \exp(it)|$ is a decreasing function of t for $0 \leqq t \leqq y$, we have that $u \geqq y$ because 1 is a root of p on $C(0, 1)$ nearest to z_n. Since $\varphi(t)$ is increasing for $y \leqq t \leqq \pi$, $\varphi(u) \leqq \varphi(\pi)$. Thus $n(I(z_n))^{n-1} \leqq |z_n - 1| |z_n + 1| (1 + |z_n|)^{n-3} \leqq (1 + |z_n|^2)(1 + |z_n|)^{n-3}$ and the result follows.

We remark here that in [1] it is shown that if $p \in P_n$ has a root z_k on $C(0, 1)$, then $I(z_k) \leqq 1$. Using this and Theorem 4 we can prove a result of Rubinstein [3].

PROPOSITION. *$I(p_n) = 1$ for $n = 3$ and 4.*

Proof. Let $p \in P_n$ be extremal. If z_k is a root of p with $|z_k| < 1$, the bound in theorem 4 gives $I(z_k) < (2/3)^{1/2}$ for $n = 3$ and $I(z_k) < 1$ for $n = 4$. Thus $I(P_n) \leqq 1$ for $n = 3$ or 4, and the polynomials $z^n - 1 \in P_n$ show that $I(P_n) \geqq 1$.

Further it is shown in [3] that if $p \in P_n$ has a root at z_k on $C(0, 1)$ and p is not of the form $z^n - \exp(it)$ for some t, then $I(z_k) < 1$. Using this result and theorem 4 we can prove

THEOREM 5. *If $p \in P_n$ is extremal, then $p(z) = z^n - \exp(it)$ for some t if $n = 2$, 3, and 4.*

Proof. This is immediate for $n = 2$. In the proof of the proposition it was shown that $I(z_k) < (2/3)^{1/2}$ and $I(z_k) < 1$ for $n = 3$ and 4 respectively and $|z_k| < 1$. The result in [3] quoted above completes the proof.

Based on this result we offer the conjecture: If $p \in P_n$ is extremal then $p(z) = z^n - \exp(it)$ for some t.

A result not dependent on the above theorems is given in

THEOREM 6. *If $p \in P_n$ has all of its roots on a line segment that is contained in $\bar{D}(0, 1)$ then $I(p) \leqq 1$.*

Proof. We may assume that p has all its roots on the closed real interval $[-1, 1]$, $p(1) = 0$, and p has at least one root less than 1. Let the roots of p be such that $z_1 = z_2 = \cdots = z_{m-1} < z_m \leqq \cdots \leqq z_n = 1$. Then $I(z_n) \leqq 1$ by the result in [1], and if z_j is a root in the half open interval $[z_m, 1)$ then p has a root on either side of z_j whence by Rolle's theorem $I(z_j) \leqq 1$. If $z_1 = -1$ then by [1] again, $I(z_1) \leqq 1$. If $-1 < z_1$ then $q(z) = p(z + z_1 + 1)$ has roots $z'_j = z_j - (z_1 + 1)$ where $-1 = z'_1 \leqq \cdots z'_n$ so that $I(z_1) = I(z'_1) \leqq 1$. Thus $I(p) \leqq 1$.

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