Φ -BOUNDED SOLUTIONS OF $\Delta u = Pu$ ON A RIEMANN SURFACE

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1. Let R be an open Riemann surface, $P \not\equiv 0$ a C^1 nonnegative density function. A P -harmonic function u is a solution of the equation $\Delta u = Pu$. Let $\Phi(t)$ be a non-negative finite real-valued function defined on $[0, \infty)$. A P -harmonic function u is called Φ -bounded if $\Phi(|u|)$ admits a P -harmonic majorant. The family of Φ -bounded P -harmonic functions on R is denoted by $P\Phi(R)$, and $O_{P\Phi}$ is the class of pairs (R, P) for which $P\Phi(R)$ contains no functions other than constants.

The study of Φ -bounded harmonic functions was initiated by Parreau [9] for a convex increasing Φ . His work was extended by Nakai [6, 7] for the case of the arbitrary Φ defined above. Recently Chow-Glasner [2] have given analogues of Parreau's results for P -harmonic functions. In this paper we show that the essential part of Nakai's results on Φ -bounded harmonic functions can be generalized to P -harmonic functions.

2. We shall make important use of the two basic quantities

$$\bar{d}(\Phi) = \overline{\lim}_{t \rightarrow \infty} \frac{\Phi(t)}{t},$$

$$\underline{d}(\Phi) = \underline{\lim}_{t \rightarrow \infty} \frac{\Phi(t)}{t}.$$

A subregion G of R is called *normal* if the relative boundary ∂G is regular for the harmonic and hence P -harmonic Dirichlet problem. Denote by SO_{PB} the class of normal subregions G of R such that every PB -function which vanishes continuously on ∂G reduces to identically zero. If O_{PB} represents the class of pairs (R, P) for which $PB(R)$ consists only of constants, we maintain:

LEMMA 1. *If there exists a normal subregion $G \notin SO_{PB}$, then $(R, P) \notin O_{PB}$.*

Proof. Let $\{R_n\}$ be an exhaustion of R by regular subregions and $\{\omega_n\}$ a sequence of continuous functions on R such that $\omega_n \in P(R_n)$, $\omega_n|_{R-R_n} = 1$. Then clearly $0 < \omega_{n+p} \leq \omega_n \leq 1$ on R and hence $\{\omega_n\}$ converges uniformly on compact subsets to a P -harmonic function ω . It is known (Ozawa [8]) that $(R, P) \in O_{PB}$ if and only if $\omega \equiv 0$ on R .

Since $G \notin SO_{PB}$ there exists a function $u \in PB(G)$ such that $0 \leq u \leq 1$ on G , $u|_{\partial G} = 0$. Extend u to R by defining $u = 0$ on $R - G$. Then u is P -subharmonic on R , that is u satisfies the sub-mean value property in some parametric disc about each point

Received May 4, 1971.

of R . Moreover

$$\omega_n \geq u$$

on $\partial(R_n \cap G)$. Therefore $\omega_n \geq u$ on $R_n \cap G$ and thus on R_n . Letting $n \rightarrow \infty$

$$\omega \geq u \not\equiv 0$$

on R , and $(R, P) \notin O_{PB}$.

We are ready to prove:

THEOREM 1. *If $\bar{d}(\Phi) = \infty$ then $O_{PB} \subset O_{P\Phi}$.*

Proof. Suppose $(R, P) \notin O_{P\Phi}$. Then there exists a function $u \not\equiv 0$ in $P\Phi(R)$ and a P -harmonic function v such that

$$\Phi(|u|) \leq v$$

on R . We may assume that u is unbounded. Since $\bar{d}(\Phi) = \infty$ there exists a sequence $\{r_n\}$ of positive real numbers, $\lim_{n \rightarrow \infty} r_n = \infty$, and $\Phi(r_n) > 0$. Setting $a_n = r_n / \Phi(r_n)$, $\lim_{n \rightarrow \infty} a_n = 0$. Let

$$G_n = \{z \in R : |u(z)| < r_n\}.$$

Then $G_1 \subset G_2 \subset \dots$ and $R = \bigcup_{n=1}^{\infty} G_n$.

Assume $G_n \in SO_{PB}$, $n=1, 2, \dots$. Since $|u|$ is P -subharmonic, $a_n v - |u|$ is P -superharmonic and also bounded from below. Furthermore

$$a_n v - |u| \geq a_n \Phi(|u|) - |u| = a_n \Phi(r_n) - r_n = 0$$

on ∂G_n for each n . Therefore

$$a_n v - |u| \geq 0$$

on G_n , i.e. $|u| \leq a_n v$ on G_n , $n=1, 2, \dots$. Let $z \in R$. Then $z \in G_n$ for all sufficiently large n , and

$$|u(z)| \leq a_n v(z).$$

Letting $n \rightarrow \infty$ we infer $u \equiv 0$ on R , a contradiction. Hence for some integer n , $G_n \notin SO_{PB}$. By Lemma 1 $R \notin O_{PB}$ as desired.

Next, for Φ with $\Phi(0)=0$ we define $SO_{P\Phi}$ to be the class of normal subregions of R such that every $P\Phi$ -function which vanishes continuously on ∂G reduces to identically zero. Note that this definition assumes a priori that the zero function belongs to $P\Phi(R)$ and hence the restriction on Φ .

COROLLARY. *If $\Phi(0)=0$ and $\bar{d}(\Phi) = \infty$ then $SO_{PB} \subset SO_{P\Phi}$.*

The proof is analogous to that of the theorem with a few slight modifications.

3. Let $PN(R)$ be the space of nonnegative P -harmonic functions on R . Myrberg [4] has shown that there always exists a positive nonconstant solution of $\Delta u = Pu$, and thus the null class $O_{PN} = \phi$. To preserve the analogy with the harmonic case, define

$$PI(R) = \{u \in P(R) : \inf_R u > 0\},$$

and O_{PI} the null class for which $PI(R) = \phi$.

THEOREM 2. If $\bar{d}(\Phi) < \infty$ then $O_{P\Phi} \subset O_{PI}$.

Proof. Since $\bar{d}(\Phi) < \infty$, there exists $t_0 \in [0, \infty)$ and a constant c such that

$$\Phi(t) \leq ct \quad \text{for all } t \geq t_0.$$

Suppose $(R, P) \notin O_{PI}$, i.e. there exists a function $u \in PI(R)$. Then for some constant $\alpha > 0$,

$$u \geq \alpha$$

on R . We can choose a constant M sufficiently large such that

$$v = Mu \geq M\alpha \geq t_0,$$

and $v \in PI(R) \subset PN(R)$. It follows that

$$\Phi(|v|) = \Phi(v) \leq cv,$$

that is $v \in P\Phi(R)$ and a fortiori $(R, P) \notin O_{P\Phi}$.

REMARK. It is an open question whether or not the null class O_{PI} is nonvoid. If $\bar{d}(\Phi) < \infty$ and O_{PI} is void, then every open Riemann surface carries a nonconstant $P\Phi$ -function. It can be demonstrated that there always exists a smooth density $P \not\equiv 0$ such that $\Delta u = Pu$ has an Evans solution e on R (Nakai's generalization of BreLOT's theorem), and hence $e \in PI(R)$. Consequently for this choice of P , R carries a nonconstant $P\Phi$ -function whenever $\bar{d}(\Phi) < \infty$. Moreover, it has been conjectured by Ozawa (cf. Nakai [5]) that for pairs $(R, P) \in O_{PB}$ there exists an Evans solution of $\Delta u = Pu$ (the converse being trivial). Therefore the validity of the conjecture would imply the existence of a $P\Phi$ -function on R for $\bar{d}(\Phi) < \infty$ and $(R, P) \in O_{PB}$.

4. Consider the class

$$PP'(R) = \{u \in P(R) : u = u_1 - u_2, u_1, u_2 \in PN(R)\}.$$

It is easily seen that $u \in PP'(R)$ if and only if $|u|$ has a P -harmonic majorant. A function $u \in PP'(R)$ is called *essentially positive*.

THEOREM 3. If $\underline{d}(\Phi) > 0$ then $P\Phi(R) \subset PP'(R)$.

Proof. Let $u \in P\Phi(R)$. Then there exists a function $v \in PN(R)$ such that

$$\Phi(|u|) \leq v$$

on R . Since $\underline{d}(\Phi) > 0$ there is a $t_0 > 0$ and a constant $c > 0$ with

$$\Phi(t) > ct \quad \text{for all } t \geq t_0.$$

As a consequence $\Phi(t) + ct_0 \geq ct$ for all $t \in [0, \infty)$. Hence

$$c|u| \leq \Phi(|u|) + ct_0 \leq v + ct_0,$$

and we infer

$$|u| \leq \frac{v + ct_0}{c}.$$

In view of the fact that $(v + ct_0)/c$ is a P -superharmonic function on R , the family $\mathcal{F}$ of all P -superharmonic functions which dominate $|u|$ on R is a non-void Perron family. The function

$$w = \inf \mathcal{F}$$

is P -harmonic, $|u| \leq w$ and we conclude that $u \in PP'(R)$.

Next we introduce the operator B' by defining

$$(B'u)(z) = \sup \{v(z) : v \in PB(R), v \leq u \text{ on } R\}$$

for $u \in PN(R)$. If $u \in PP'(R)$, $u = u_1 - u_2$, $u_1, u_2 \in PN(R)$, then

$$B'u = B'u_1 - B'u_2.$$

The operator B' is positive, linear, and involutive.

The class $PB'(R)$ of *quasibounded* P -harmonic functions is defined by

$$PB'(R) = \{u \in PP'(R) : B'u = u\}$$

(cf. e.g. Kwon-Sario-Wang [3]). In accordance with the harmonic case we prove:

THEOREM 4. If $\bar{d}(\Phi) = \infty$ then $P\Phi(R) \cap PP'(R) \subset PB'(R)$.

The proof is a suitable modification of the one given by Nakai [7]. In fact, let $u \in P\Phi(R) \cap PP'(R)$. Then $|u|$ has a P -harmonic majorant, and $Mu = u \vee 0 + (-u) \vee 0$ is the least P -harmonic majorant of $|u|$ on R . By the commutativity of M , it suffices to prove $B'Mu = Mu$.

Since $\bar{d}(\Phi) = \infty$, there exists as before a sequence $\{r_n\}$ of positive real numbers such that $\lim_{n \rightarrow \infty} r_n = \infty$, $\Phi(r_n) > 0$, and $\lim_{n \rightarrow \infty} a_n = 0$, where $a_n = r_n / \Phi(r_n)$. Let

$$\Omega_n = \{z \in R: |u(z)| < r_n\}.$$

Then $\Omega_1 \subset \Omega_2 \subset \dots$ and $R = \bigcup_{n=1}^{\infty} \Omega_n$. Take an exhaustion $\{R_n\}$ of R by regular subregions, and let $w_m \in P(R_m \cap \Omega_n)$,

$$w_m|(\partial R_m) \cap \Omega_n = \min(Mu - B'Mu, r_n),$$

$$w_m|(\partial \Omega_n) \cap R_m = 0$$

for each $n=1, 2, \dots$. By setting $w_m=0$ on $R_m - \Omega_n$, w_m becomes P -subharmonic on R_m . Since r_n is P -superharmonic on R , $\min(Mu - B'Mu, r_n)$ is P -superharmonic on R . Thus

$$w_{m+1} \leq \min(Mu - B'Mu, r_n)$$

on $R_{m+1} \supset R_m$, and therefore

$$w_{m+1} \leq w_m$$

on R_m .

Now consider $w'_m \in P(R_m)$ such that

$$w'_m|(\partial R_m) \cap \Omega_n = \min(Mu - B'Mu, r_n),$$

$$w'_m| \partial R_m - \bar{\Omega}_n = 0.$$

Then $0 \leq w'_m \leq Mu - B'Mu$ on R_m , and hence we can choose a convergent subsequence, say again $\{w'_m\}$, and set

$$w' = \lim_{m \rightarrow \infty} w'_m.$$

Clearly $0 \leq w' \leq Mu - B'Mu$. It follows that

$$0 \leq B'w' \leq B'(Mu - B'Mu) = B'Mu - B'^2Mu = 0.$$

Since w' is bounded,

$$w' = B'w' = 0$$

on R . Furthermore, $w'_m \geq w_m$ on $\partial(R_m \cap \Omega_n)$ and thus $w'_m \geq w_m$ on R_m . As a consequence $w \equiv 0$ on R .

On $(\partial R_m) \cap \Omega_n$ we have $|u| < r_n$ and

$$|u| - B'Mu \leq Mu - B'Mu.$$

Therefore

$$|u| - B'Mu \leq \min(Mu - B'Mu, r_n)$$

on $(\partial R_m) \cap \Omega_n$, i.e. $|u| \leq B'Mu + w_m$. On the other hand

$$|u| = r_n = a_n \Phi(|u|) \leq a_n v$$

for some $v \in PN(R)$. Thus

$$|u| \leq a_n v + B' M u + w_m$$

on $\partial(R_m \cap \Omega_n)$. In view of the P -subharmonicity of $|u|$,

$$|u| \leq a_n v + B' M u + w_m$$

on $R_m \cap \Omega_n$. Letting $m \rightarrow \infty$ and then $n \rightarrow \infty$ we obtain

$$|u| \leq B' M u$$

on R , and hence $M u \leq B' M u$. By the definition of B' , $B' M u \leq M u$ and the theorem is proved.

COROLLARY. If $\bar{d}(\Phi) = \infty$ and $\underline{d}(\Phi) > 0$ then $P\Phi(R) \subset PB'(R)$.

5. In this section we redefine the class $P\Phi(R)$ as the space of P -harmonic functions u on R such that $\Phi(|u|)$ admits a P -superharmonic majorant. This enlarges the space $P\Phi(R)$ somewhat. In contrast to Theorem 2 we maintain:

THEOREM 5. If $\bar{d}(\Phi) < \infty$ then there always exists a nonconstant $P\Phi$ -function on R .

Proof. Since $\bar{d}(\Phi) < \infty$ there exists a $t_0 \in [0, \infty)$ such that

$$\Phi(t) < ct \quad \text{for all } t \geq t_0$$

and some constant c . Let u be a nonconstant positive P -harmonic function on R . Then the function

$$v = u + t_0$$

is P -superharmonic, $v \geq t_0$, and consequently

$$\Phi(|v|) = \Phi(v) \leq cv.$$

Then v is nonconstant, $v \in P\Phi(R)$.

At this point we impose the further restriction that Φ is bounded in a neighborhood of $t=0$, for otherwise $SO_{P\Phi}$ (with $\Phi(0)=0$) would consist of all bordered surfaces.

THEOREM 6. If $\Phi(0)=0$ and $\bar{d}(\Phi)=\infty$ then $SO_{PB}=SO_{P\Phi}$.

Proof. It remains to be shown that $SO_{PB} \supset SO_{P\Phi}$. Let $\Omega \in SO_{P\Phi}$ and $u \in PB(\Omega)$, $u|_{\partial\Omega}=0$. Since Φ is bounded in a neighborhood of $t=0$, there is a real number b such that for all $t \in [0, b)$

$$\Phi(t) \leq c$$

for some constant c . The boundedness of u implies that for some constant M the function

$$v = \frac{u}{M}$$

belongs to $PB(\Omega)$, $v|_{\partial\Omega} = 0$ and $|v| < b$. Therefore $\Phi(|v|) \leq c$ and $v \in PB(\Omega)$. We infer that $v \equiv 0$ on Ω and thus $u \equiv 0$ on Ω , i.e. $\Omega \in SO_{PB}$.

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