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ON PRIME ENTIRE FUNCTIONS

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§1. An entire function $F(z)=f\circ g(z)$ is said to be prime if every factorization of the above form implies that one of the functions $f(z)$ or $g(z)$ is linear.

Ozawa [5] has recently proved the following.

THEOREM A. *Let $F(z)$ be an entire function of order ρ, $1/2<\rho<1$ and with only negative zeros. Assume that $n(r)\sim\lambda r^{\rho}$, $\lambda>0$ where $n(r)$ indicates the number of zeros of $F(z)$ in $|z|<r$. Further assume that there are two indices j and k such that a_j, a_k are zeros of $F(z)$ whose multiplicities p_j, p_k satisfy $(p_j, p_k)=1$. Then $F(z)$ is prime.*

The purpose of this note is to extend Theorem A to higher orders and to prove the following.

THEOREM. *Let $F(z)$ be an entire function of non-integral order ρ $(>1/2)$, with only negative zeros. Assume that $n(r)\sim\lambda r^{\rho}$, $\lambda>0$. Further assume that there are two indices j and k such that a_j, a_k are zeros of $F(z)$ whose multiplicities p_j, p_k satisfy $(p_j, p_k)=1$. Then $F(z)$ is prime.*

In order to prove this we quote several known results.

LEMMA 1. (Edrei [1]). *Let $f(z)$ be an entire function. Assume that there exists an unbounded sequence $\{h_\nu\}_{\nu=1}^{\infty}$ such that all the roots of the equations $f(z)=h_\nu$, $\nu=1, 2, \cdots$, be real. Then $f(z)$ is a polynomial of degree at most two.*

LEMMA 2. (Pólya [6]). *Suppose that $f(z)$, $g(z)$ are entire functions and that $\phi(z)=f\circ g(z)$ is of finite order. Then either $g(z)$ is a polynomial or $f(z)$ is of order zero.*

LEMMA 3. (Hardy-Littlewood [2]). *If $F(z)$ is a positive integrable function such that, when $t\to 0$,*

$$\int_0^{\infty} F(x)e^{-xt}dx\sim t^{-\beta} \qquad (\beta>0),$$

then, when $x\to\infty$,

$$\int_0^{x} F(u)du\sim\frac{x^{\beta}}{\Gamma(\beta+1)}.$$

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LEMMA 4. (Titchmarsh [7]). *If $\phi(x)$ and $\phi'(x)$ are integrable over any finite interval and when $x\to\infty$,*

$$\phi(x)\sim x^{-\beta}, \quad \phi'(x)=O(x^{-\beta-1}) \qquad (0<\beta<1),$$

then, when $t\to 0$,

$$\int_0^\infty \phi(x) \frac{\cos}{\sin} xtdx \sim \frac{\sin}{\cos} \frac{1}{2}\pi\beta\Gamma(1-\beta)t^{\beta-1}.$$

§ 2. The main lemmas.

LEMMA 5. (Hellerstein-Shea [3]). *Let $f(z)$ be an entire function of order ρ, $q<\rho<q+1$ where q is a nonnegative integer, with real negative zeros and $n(r)\sim\lambda r^\rho(\lambda>0)$ as $r\to\infty$, then*

$$\log f(re^{i\theta}) \sim e^{i\rho\theta}\pi\lambda\frac{r^\rho}{\sin \pi\rho}, \qquad r\to\infty$$

for each fixed θ in $-\pi<\theta<\pi$.

Our proofs depend upon the following lemma which is the extension of the theorem of [8].

LEMMA 6. *Let $f(z)$ be an entire function of order $q<\rho<q+1$ where q is a nonnegative integer, with real negative zeros and if*

$$\log f(x) \sim \pi\lambda \operatorname{cosec} \pi\rho\cdot x^\rho \qquad (\lambda>0)$$

then $n(x)\sim\lambda x^\rho$.

Proof. We can write

$$f(z)=e^{P(z)}\prod_{n=1}^\infty \left(1+\frac{z}{a_n}\right)\exp\left(-\frac{z}{a_n}+\cdots+\frac{1}{q}\left(-\frac{z}{a_n}\right)^q\right), \qquad a_n>0$$

where $P(z)$ is a polynomial of degree $d\leqq q$. If we write $|f(x)/e^{P(x)}|=g(x)$, then we have

$$\log g(x) \sim \pi\lambda \operatorname{cosec} \pi\rho\cdot x^\rho.$$

From the representation

$$\log\left|\frac{f(xe^{i\theta})}{e^{P(xe^{i\theta})}}\right|=(-1)^q x^{q+1}\int_0^\infty \frac{n(r)}{r^{q+1}}\cdot\frac{r\cos(q+1)\theta+x\cos q\theta}{r^2+2rx\cos\theta+x^2}dr,$$

we have

$$\log g(x)=(-1)^q x^{q+1}\int_0^\infty \frac{n(r)(r+x)}{r^{q+1}(r+x)^2}dr.$$

Hence

$$|\log g(x)| \geqq x^{q+1}\int_x^\infty \frac{n(r)dr}{r^{q+1}(x+r)} \geqq x^{q+1}\cdot n(x)\int_x^\infty \frac{dr}{(x+r)r^{q+1}} \geqq \frac{n(x)}{2(q+1)}.$$

Therefore we have

$$n(x)=O(x^\rho).$$

Also

$$\frac{g'(x)}{g(x)} = (-1)^q\int_0^\infty \frac{(q+1)x^q r+qx^{q+1}}{r^{q+1}(r+x)^2}\, n(r)dr=O(x^{\rho-1}).$$

Now

$$(-1)^q\cdot\frac{\log g(x^2)}{x^{2q+2}} = 2\int_0^\infty \frac{n(r^2)}{r^{2q+1}(r^2+x^2)}dr.$$

Multiply each side by

$$x^{2q+\alpha}\cos\left\{xt-\frac{1}{2}(2q+\alpha)\pi\right\} \qquad (t>0,\ 1-2\rho<\alpha<2-2\rho)$$

and integrate from 0 to ∞. We may, as in [8], invert the order of integration on the right, and we obtain for a suitable α $(1-2\rho<\alpha<2-2\rho)$

$$\int_0^\infty \log g(x^2)\cos\left(xt-\frac{1}{2}\alpha\pi\right)x^{\alpha-2}dx=\pi\int_0^\infty n(r^2)r^{\alpha-2}e^{-rt}dr.$$

In the Lemma 4, put

$$\phi(x)=\frac{\sin\pi\rho}{\pi\lambda}\, x^{\alpha-2}\log g(x^2), \qquad \beta=2-2\rho-\alpha.$$

Then

$$\int_0^\infty \log g(x^2)\cdot\cos\left(xt-\frac{1}{2}\alpha\pi\right)x^{\alpha-2}dx$$

$$=\frac{\pi\lambda}{\sin\pi\rho}\left\{\cos\frac{1}{2}\alpha\pi\int_0^\infty\phi(x)\cos xtdt+\sin\frac{1}{2}\alpha\pi\int_0^\infty\phi(x)\sin xtdt\right\}$$

$$\sim\frac{\pi\lambda}{\sin\pi\rho}\left\{\cos\frac{1}{2}\alpha\pi\cdot\sin\left(1-\rho-\frac{1}{2}\alpha\right)\pi+\sin\frac{1}{2}\alpha\pi\cdot\cos\left(1-\rho-\frac{1}{2}\alpha\right)\pi\right\}\Gamma(2\rho+\alpha-1)t^{1-2\rho-\alpha}.$$

Hence

$$\int_0^\infty n(r^2)r^{\alpha-2}e^{-rt}dr\sim\lambda\Gamma(2\rho+\alpha-1)t^{1-2\rho-\alpha}.$$

Hence, by Lemma 3, we have

$$\int_0^x n(r^2)r^{\alpha-2}dr \sim \frac{\lambda}{2\rho+\alpha-1}x^{2\rho+\alpha-1}.$$

Thus, for $x > x_0(\varepsilon)$

$$\frac{\lambda(1-\varepsilon)}{2\rho+\alpha-1}x^{2\rho+\alpha-1} < \int_0^x n(r^2)r^{\alpha-2}dr < \frac{\lambda(1+\varepsilon)}{2\rho+\alpha-1}x^{2\rho+\alpha-1}.$$

Hence

$$\int_x^{x+x\delta} n(r^2)r^{\alpha-2}dr < \frac{\lambda(1+\varepsilon)(1+\delta)^{2\rho+\alpha-1}-\lambda(1-\varepsilon)}{2\rho+\alpha-1}x^{2\rho+\alpha-1}$$

$$= \frac{\lambda}{2\rho+\alpha-1}\{(2\rho+\alpha-1)\delta+O(\delta^2)+O(\varepsilon)\}x^{2\rho+\alpha-1}.$$

On the other hand,

$$\int_x^{x+x\delta} n(r^2)r^{\alpha-2}dr \geqq n(x^2)\int_x^{x+x\delta} r^{\alpha-2}dr > n(x^2)\frac{x\delta}{x^{2-\alpha}(1+\delta)^{2-\alpha}}.$$

Hence

$$n(x^2) < \lambda(1+\delta)^{2-\alpha}\left\{1+O(\delta)+O\left(\frac{\varepsilon}{\delta}\right)\right\}x^{2\rho}.$$

and the required upper bound is obtained on taking, e. g., $\delta=\sqrt{\varepsilon}$. The lower bound may be obtained in a similar way. This proves Lemma 6.

§ 3. Proof of Theorem.

Let $F(z)$ be $f\circ g(z)$. Assume that $f(w)$ is transcendental. If $f(w)=0$ has only a finite number of roots, then we can write

$$f(w)=P(w)e^{H(w)}$$

where $P(w)$ is a polynomial and $H(w)$ is also a polynomial, in view of $\rho<+\infty$. Since, by Lemma 2, $g(z)$ is a polynomial, ρ is an integer. This is a contradiction. Hence $f(w)=0$ has an infinite number of roots $\{w_n\}$, $w_n\to\infty$ as $n\to\infty$. Consider the equations $g(z)=w_n$, $n=1, 2, \cdots$. All their roots lie on the real negative axis. Then by Lemma 1 $g(z)$ is a polynomial of degree at most two. Therefore $g(z)$ must be linear.

Suppose, next, that $F(z)=f\circ g(z)$ with a polynomial $f(w)$. In this case, we have

$$F(z)=Ag_1(z)^{l_1}\cdots g_p(z)^{l_p}, \qquad g_j(z)=g(z)-w_j.$$

From the representation

$$F(z)=e^{p(z)}\prod_{n=1}^{\infty}E\left(\frac{z}{a_n},\ q\right),$$

we may put

$$g_j(z)=e^{p_j(z)}\prod_{n=1}^{\infty}E\left(\frac{z}{a_{jn}},\ q\right).$$

And it is clear that

$$|g_j(r)|\sim|g_k(r)|, \qquad r\to\infty,$$

for any j and k. Thus for each s, $1\leqq s\leqq p$

$$|F(r)|\sim|A|\prod_{j=1}^{p}|g_s(r)|^{l_j}=|A|\cdot|g_s(r)|^{\alpha} \qquad \left(\alpha=\sum_{j=1}^{p}l_j\right), \qquad r\to\infty$$

By Lemma 5 we have

$$\log|F(r)|\sim\frac{\pi\lambda}{\sin\pi\rho}r^{p}, \qquad r\to\infty.$$

Hence

$$\log|g_s(r)|\sim\frac{\pi(\lambda/\alpha)}{\sin\pi\rho}r^{\rho} \qquad (1\leqq s\leqq p), \qquad r\to\infty.$$

Then by Lemma 6

$$n(r,\ g_s(z))\sim\frac{\lambda}{\alpha}r^{\rho}, \qquad r\to\infty.$$

Therefore, in view of $\rho>1/2$, we can choose a rectilinear ray issuing from the origin by Lemma 5 such that along the ray,

$$g(z)\to w_1, \quad g(z)\to w_2 \qquad (z\to\infty).$$

This is clearly a contradiction. Therefore we have $F(z)=A(g(z)-w_1)^{l_1}$. By the existence of two zeros whose multiplicities are coprime, l_1 must reduce to 1. Hence we have

$$F(z)=A(g(z)-w_1),$$

which is the desired result.

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