KŌDAI MATH. SEM. REP.
23 (1971), 238—247

NOTES ON ALMOST ISOMETRIES

BY KENTARO YANO AND MARIKO KONISHI*

§ 0. Introduction.

Chern and Hsiung [1] proved a theorem stating that a volume-preserving almost isometry between two compact submanifolds in euclidean space satisfying certain conditions is an isometry.

The purpose of the present paper is to give a different approach to this subject, which seems to be somewhat related to a paper of Gardner [3].

§ 1. Almost isometries.

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ where here and throughout the paper the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, 3, \cdots, n\}$. We denote by g_{ji}, $\{{}_j{}^h{}_i\}$, ∇_i, $K_{kji}{}^h$ and K_{ji} the components of the metric tensor, the Christoffel symbols formed with g_{ji}, the operator of covariant differentiation with respect to the Christoffel symbols, the curvature tensor and the Ricci tensor respectively.

A transformation is said to be affine when it does not change the Levi-Civita connection defined by the Christoffel symbols. In order that an infinitesimal transformation v^h be an affine transformation, it is necessary and sufficient that

$$\mathcal{L}_v\left\{ {h \atop j\ i} \right\} = \nabla_j\nabla_i v^h + K_{kji}{}^h v^k = 0, \tag{1.1}$$

where $\mathcal{L}_v$ denotes the Lie derivative with respect to the infinitesimal transformation v^h, [7].

An infinitesimal transformation v^h satisfying

$$g^{ji}\left(\mathcal{L}_v\left\{ {h \atop j\ i} \right\}\right) = g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i = 0 \tag{1.2}$$

is called an infinitesimal almost isometry, where $K_i{}^h = K_{it}g^{th}$. Thus an affine transformation is an almost isometry.

A transformation is said to be isometric when it does not change the Riemannian metric. In order that an infinitesimal transformation v^h be an isometry, it is necessary and sufficient that

Received September 24, 1970.

* Formerly MARIKO TANI.

(1. 3) $$\mathcal{L}_v g_{ji}=\nabla_j v_i+\nabla_i v_j=0,$$

where $v_i=g_{ih}v^h$. Since (1. 3) implies (1. 1) and consequently (1. 2), an isometry is an almost isometry.

One of the present authors proved in [6, 8] that a necessary and sufficient condition for an infinitesimal transformation v^h in a compact Riemannian manifold M to be an isometry is that

(1. 4) $$g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i=0 \qquad \text{and} \qquad \nabla_i v^i=0,$$

or that the infinitesimal transformation v^h defines an almost isometry and preserves the volume.

Since (1. 1) implies (1. 4), we see that an infinitesimal affine transformation in a compact Riemannian manifold is an isometry.

Now consider a transformation in the Riemannian manifold M and denote by $\bar{g}_{ji}$ the transformed metric and by $\overline{\{_j{}^h{}_i\}}$ the transformed Christoffel symbols. We put

(1. 5) $$\overline{\left\{ {h \atop j\ i} \right\}}=\left\{ {h \atop j\ i} \right\}+U_{ji}{}^h,$$

$U_{ji}{}^h$ being components of a tensor field of type (1, 2).

If

(1. 6) $$U_{ji}{}^h=0,$$

then the transformation is affine, if

(1. 7) $$g^{ji}U_{ji}{}^h=0$$

then the transformation is an almost isometry with respect to g_{ji}, if

(1. 8) $$\bar{g}^{ji}U_{ji}{}^h=0,$$

then the transformation is an almost isometry with respect to $\bar{g}_{ji}$ and if

(1. 9) $$\det(\bar{g}_{ji})=\det(g_{ji})$$

det( ) denoting the determinant formed with elements between braces, the transformation is volume-preserving. In this case we have

(1. 10) $$U_{jt}{}^t=0.$$

If

(1. 11) $$\bar{g}_{ji}=g_{ji},$$

the transformation is an isometry.

Denoting by $\bar{\nabla}_i$ the operator of covariant differentiation with respect to $\overline{\{{}_j{}^h{}_i\}}$, we have

$$0=\bar{\nabla}_k\bar{g}_{ji}=\partial_k\bar{g}_{ji}-\overline{\left\{ {t \atop k\ j} \right\}}\bar{g}_{ti}-\overline{\left\{ {t \atop k\ i} \right\}}\bar{g}_{jt}$$

$$=\partial_k\bar{g}_{ji}-\left(\left\{ {t \atop k\ j} \right\}+U_{kj}{}^t\right)\bar{g}_{ti}-\left(\left\{ {t \atop k\ i} \right\}+U_{ki}{}^t\right)\bar{g}_{jt},$$

from which

$$\nabla_k\bar{g}_{ji}=U_{kj}{}^t\bar{g}_{ti}+U_{ki}{}^t\bar{g}_{jt}. \tag{1.12}$$

Thus, for an affine transformation, we have

$$\nabla_k\bar{g}_{ji}=0, \tag{1.13}$$

and consequently for an affine transformation in an irreducible Riemannian manifold, we have, by a theorem of Schur,

$$\bar{g}_{ji}=c^2g_{ji}, \tag{1.14}$$

c being a non-zero constant, that is, the transformation is homothetic. If the affine transformation preserves the volume, then we have

$$\bar{g}_{ji}=g_{ji}, \tag{1.15}$$

that is, the transformation is isometry. See [4], [5].

It is not yet known whether an almost isometry preserving the volume is an isometry or not.

§ 2. Integral formulas.

Let E be euclidean space of dimension $m\,(>n)$ and consider an immersion X: $M\to E$, that is, a differentiable mapping X of M into E such that the induced linear mapping on the tangent space is univalent everywhere. We regard $X(p)$, $p\in M$ as a position vector in E. As the Riemannian manifold M is covered by a system of coordinate neighborhoods $\{U;\ x^h\}$, we can consider the position vector X as function of $x^1, x^2, \cdots, x^n$.

We put

$$X_i=\partial_iX, \qquad \partial_i=\partial/\partial x^i, \tag{2.1}$$

then X_i are n linearly independent vectors tangent to $X(M)$. Assuming that M is oriented and the immersion X: $M\to E$ is orientation-preserving, we choose $m-n$

mutually orthogonal unit normals C_x to $X(M)$ in such a way that $X_1, \cdots, X_n, C_{n+1}, \cdots, C_m$ give the positive orientation of E, where here and throughout the paper the indices x, y, z run over the range $\{n+1, \cdots, m\}$.

Now the components of the metric tensor are given by

$$g_{ji}=X_j\cdot X_i, \tag{2.2}$$

the dot denoting the inner product of vectors in E, and the equations of Gauss are written as

$$\nabla_j X_i=\partial_j X_i-\left\{ {h \atop j\ i} \right\} X_h=h_{jix}C_x, \tag{2.3}$$

where h_{jix} are components of the second fundamental tensors with respect to the normals C_x. The equations of Weingarten are written as

$$\nabla_j C_x=\partial_j C_x=-h_j{}^i{}_x X_i+l_{jxy}C_y, \tag{2.4}$$

where $h_j{}^i{}_x=h_{jtx}g^{ti}$ and l_{jxy} are components of the so-called third fundamental tensor.

We now put

$$X=X_i z^i+C_x\alpha_x, \tag{2.5}$$

where z^i are components of a vector field of M and α_x are $m-n$ functions of M and compute the covariant derivative of X. Then we obtain

$$X_j=h_{jix}z^i C_x+X_i\nabla_j z^i+(-h_j{}^i{}_x X_i+l_{jxy}C_y)\alpha_x+C_x\nabla_j\alpha_x,$$

from which

$$\nabla_j z^i=\delta_j^i+h_j{}^i{}_x\alpha_x \tag{2.6}$$

and

$$\nabla_j\alpha_x=-h_{jix}z^i-l_{jyx}\alpha_y. \tag{2.7}$$

From (2.6), we obtain

$$\nabla_j z_i=g_{ji}+h_{jix}\alpha_x, \tag{2.8}$$

where $z_i=g_{ih}z^h$ and

$$\nabla_i z^i=n+g^{ji}h_{jix}\alpha_x. \tag{2.9}$$

Thus, supposing that M is compact, we obtain the integral formula

$$\int_M (n+g^{ji}h_{jix}\alpha_x)dV=0, \tag{2.10}$$

dV denoting the volume element.

We now consider the second immersion $\bar{X}$: $M \to E$ and proceed as above, we then obtain similarly

$$\bar{\nabla}_j \bar{z}_i = \bar{g}_{ji} + \bar{h}_{jix} \bar{\alpha}_x \tag{2.11}$$

and

$$\bar{\nabla}_i \bar{z}^i = n + \bar{g}^{ji} \bar{h}_{jix} \bar{\alpha}_x \tag{2.12}$$

and consequently

$$\int_M (n + \bar{g}^{ji} \bar{h}_{jix} \bar{\alpha}_x) d\bar{V} = 0. \tag{2.13}$$

Now, two immersions X: $M \to E$ and $\bar{X}$: $M \to E$ define a diffeomorphism f: $X(M) \to \bar{X}(M)$. We assume that this mapping is volume-preserving, that is,

$$\det(\bar{g}_{ji}) = \det(g_{ji}). \tag{2.14}$$

Now the inequality of Gårding [2] says that

$$\bar{g}^{ji} g_{ji} \geqq n \{\det(\bar{g}_{ji}) / \det(g_{ji})\}^{1/n}, \tag{2.15}$$

equality holding if and only if $\bar{g}_{ji}$ and g_{ji} are proportional. Thus we have, from (2.14) and (2.15),

$$\bar{g}^{ji} g_{ji} \geqq n, \tag{2.16}$$

equality holding if and only if $\bar{g}_{ji} = g_{ji}$.

Similarly we have

$$g^{ji} \bar{g}_{ji} \geqq n \tag{2.17}$$

equality holding if and only if $\bar{g}_{ji} = g_{ji}$.

Now, since

$$\begin{aligned} \bar{\nabla}_j z_i &= \partial_j z_i - \overline{\left\{ {h \atop j\ i} \right\}} z_h \\ &= \partial_j z_i - \left(\left\{ {h \atop j\ i} \right\} + U_{ji}{}^h \right) z_h \\ &= \nabla_j z_i - U_{ji}{}^h z_h, \end{aligned}$$

if the transformation is an almost isometry with respect to $\bar{g}_{ji}$, then we have

$$\begin{aligned} \bar{g}^{ji} \bar{\nabla}_j z_i &= \bar{g}^{ji} \nabla_j z_i \\ &= \bar{g}^{ji} (g_{ji} + h_{jix} \alpha_x) \end{aligned}$$

$$\geqq n+\bar{g}^{ji}h_{jix}\alpha_x$$

by virtue of (2. 8) and (2. 16). Thus integrating, we have

$$\int_M (n+\bar{g}^{ji}h_{jix}\alpha_x)dV \leqq 0, \tag{2. 18}$$

since $d\bar{V}=dV$, equality holding if and only if $\bar{g}_{ji}=g_{ji}$.

Combining (2. 10) and (2. 18), we obtain

$$\int_M (\bar{g}^{ji}h_{jix}\alpha_x - g^{ji}h_{jix}\alpha_x)dV \leqq 0, \tag{2. 19}$$

equality holding if and only if $\bar{g}_{ji}=g_{ji}$.

Similarly if the transformation is an almost isometry with respect to g_{ji}, then we have

$$\int_M (g^{ji}\bar{h}_{jix}\bar{\alpha}_x - \bar{g}^{ji}\bar{h}_{jix}\bar{\alpha}_x)dV \leqq 0, \tag{2. 20}$$

equality holding if and only if $\bar{g}_{ji}=g_{ji}$.

§ 3. Theorem of Chern and Hsiung.

Chern and Hsiung [1] proved

THEOREM. *Let $\bar{X}$, X: $M \to E$ be two immersed compact submanifolds and let f: $X(M) \to \bar{X}(M)$ be a volume-preserving almost isometry with respect to the metric of $\bar{X}(M)$. If*

$$\bar{g}^{ji}h_{jix}\alpha_x \geqq g^{ji}h_{jix}\alpha_x, \tag{3. 1}$$

then f is an isometry.

Under these assumptions, we have, from (2. 19),

$$\int_M (\bar{g}^{ji}h_{jix}\alpha_x - g^{ji}h_{jix}\alpha_x)dV = 0,$$

and consequently

$$\bar{g}_{ji}=g_{ji}. \tag{3. 2}$$

If f is volume-preserving almost isometry with respect to the metric of $X(M)$ and

$$g^{ji}\bar{h}_{jix}\bar{\alpha}_x \geqq \bar{g}^{ji}\bar{h}_{jix}\bar{\alpha}_x \tag{3. 3}$$

then we have, from (2. 20),

$$\int_M (g^{ji}\bar{h}_{jix}\bar{\alpha}_x - \bar{g}^{ji}\bar{h}_{jix}\bar{\alpha}_x)dV = 0,$$

and consequently we can conclude (3. 2).

§4. A theorem.

We have put

$$X = X_i z^i + C_x \alpha_x, \tag{4. 1}$$

from which

$$X_i \cdot X = z_i \tag{4. 2}$$

and consequently

$$\frac{1}{2}\nabla_i(X\cdot X) = z_i. \tag{4. 3}$$

Thus

$$\frac{1}{2}\nabla_j\nabla_i(X\cdot X) = g_{ji} + h_{jix}\alpha_x \tag{4. 4}$$

by virtue of (2. 8).

We have similarly

$$\frac{1}{2}\bar{\nabla}_j\bar{\nabla}_i(\bar{X}\cdot\bar{X}) = \bar{g}_{ji} + \bar{h}_{jix}\bar{\alpha}_x. \tag{4. 5}$$

Since

$$\begin{aligned}\frac{1}{2}\bar{\nabla}_j\bar{\nabla}_i(\bar{X}\cdot\bar{X}) &= \frac{1}{2}\left\{\partial_j\partial_i(\bar{X}\cdot\bar{X}) - \overline{\left\{ {h \atop j\ i} \right\}}\partial_h(\bar{X}\cdot\bar{X})\right\} \\ &= \frac{1}{2}\left\{\partial_j\partial_i(\bar{X}\cdot\bar{X}) - \left(\left\{ {h \atop j\ i} \right\} + U_{ji}{}^h\right)\partial_h(\bar{X}\cdot\bar{X})\right\} \\ &= \frac{1}{2}\left\{\nabla_j\nabla_i(\bar{X}\cdot\bar{X}) - U_{ji}{}^h\nabla_h(\bar{X}\cdot\bar{X})\right\},\end{aligned}$$

we have, from (4. 4) and (4. 5),

$$\frac{1}{2}\nabla_j\nabla_i(\bar{X}\cdot\bar{X} - X\cdot X) - \frac{1}{2}U_{ji}{}^h\nabla_h(\bar{X}\cdot\bar{X})$$
$$= \bar{g}_{ji} - g_{ji} + \bar{h}_{jix}\bar{\alpha}_x - h_{jix}\alpha_x.$$

Thus, if the transformation is a volume-preserving almost isometry with respect to g_{ji}, we have

$$\frac{1}{2} g^{ji}\nabla_j\nabla_i(\bar{X}\cdot\bar{X}-X\cdot X)$$

$$=g^{ji}(\bar{g}_{ji}-g_{ji})+g^{ji}\bar{h}_{jix}\bar{\alpha}_x-g^{ji}h_{jix}\alpha_x.$$

$$\geqq g^{ji}\bar{h}_{jix}\bar{\alpha}_x-g^{ji}h_{jix}\alpha_x,$$

equality holding if and only if $\bar{g}_{ji}=g_{ji}$.

Thus, if

$$g^{ji}\bar{h}_{jix}\bar{\alpha}_x\geqq g^{ji}h_{jix}\alpha_x,$$

we have

$$\frac{1}{2} g^{ji}\nabla_j\nabla_i(\bar{X}\cdot\bar{X}-X\cdot X)\geqq 0,$$

from which, M being assumed to be compact,

$$\frac{1}{2} g^{ji}\nabla_j\nabla_i(\bar{X}\cdot\bar{X}-X\cdot X)=0,$$

by virtue of Bochner's lemma [5].

Thus

$$g^{ji}(\bar{g}_{ji}-g_{ji})=0$$

and consequently

$$\bar{g}_{ji}=g_{ji}.$$

Thus we have

THEOREM. *Let X, $\bar{X}$: $M\to E$ be two immersed compact submanifolds and let f: $X(M)\to\bar{X}(M)$ be a volume-preserving almost isometry with respect to the metric of $\bar{X}(M)$.*

If

$$g^{ji}\bar{h}_{jix}\bar{\alpha}_x\geqq g^{ji}h_{jix}\alpha_x,$$

then f is an isometry.

§ 5. Almost parallel displacement.

We know that

$$\nabla_j X_i=h_{jix}C_x \tag{5. 1}$$

and

$$\bar{\nabla}_j \bar{X}_i = \bar{h}_{jix} \bar{C}_x. \tag{5.2}$$

We call an almost parallel displacement with respect to g_{ji} a transformation for which

$$g^{ji}(\bar{\nabla}_j \bar{X}_i - \nabla_j X_i) = 0. \tag{5.3}$$

Now, putting

$$A = \bar{X} - X, \qquad A_i = \bar{X}_i - X_i$$

we have

$$\begin{aligned} \nabla_j A_i &= \partial_j(\bar{X}_i - X_i) - \left\{ {h \atop j\ i} \right\} (\bar{X}_h - X_h) \\ &= \partial_j \bar{X}_i - \left(\overline{\left\{ {h \atop j\ i} \right\}} - U_{ji}{}^h \right) \bar{X}_h - \left(\partial_j X_i - \left\{ {h \atop j\ i} \right\} X_h \right) \\ &= \bar{\nabla}_j \bar{X}_i - \nabla_j X_i + U_{ji}{}^h \bar{X}_h. \end{aligned}$$

Thus, for an almost isometry with respect to g_{ji}, we have

$$g^{ji} \nabla_j A_i = g^{ji}(\bar{\nabla}_j \bar{X}_i - \nabla_j X_i).$$

Thus, if an almost isometry is an almost parallel displacement, we have

$$g^{ji} \nabla_j A_i = 0.$$

Thus, if M is compact, all the components of the vector A are constant and A is a constant vector. Thus we have

THEOREM. *If M is compact and f: $X(M) \to \bar{X}(M)$ is an almost isometry and at the same time an almost parallel displacement, then f is a parallel displacement.*

REMARK. Gardner [3] has employed profitably a fixed vector in E to get a general rigidity theorem.

BIBLIOGRAPHY

[1] CHERN, S. S., AND C. C. HSIUNG, On the isometry of compact submanifolds in euclidean space. Math. Ann. **149** (1963), 278–285.
[2] GÅRDING, L., An inequality for hyperbolic polynomials. J. Math. Mech. **8** (1959), 957–965.
[3] GARDNER, R. B., An integral formula for immersions in euclidean space. J. of Diff. Geom. **3** (1969), 242–252.
[4] HANO, J., On affine transformations of a Riemannian manifold. Nagoya Math. J. **9** (1955), 99–109.

[5] KOBAYASHI, S., A theorem on the affine transformation group of a Riemannian manifold. Nagoya Math. J. **9** (1955), 39-41.

[6] YANO, K., On harmonic and Killing vector fields. Ann. of Math. **55** (1952), 328-341.

[7] YANO, K., The theory of Lie derivatives and its applications. North-Holland Publ. Co., Amsterdam (1957).

[8] YANO, K., AND S. BOCHNER, Curvature and Betti numbers. Ann. of Math. Studies **32** (1953).

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.