ON PRIME ENTIRE FUNCTIONS

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§ 1. An entire function $F(z)=f\circ g(z)$ is said to be prime (pseudo-prime) if every factorization of the above form implies that one of the functions $f(z)$ or $g(z)$ is linear (a polynomial). It is almost trivial that z^p with a prime p is prime. However it is hard to say that known examples of prime transcendental entire functions are rich in number. It is known that e^z+z is prime, which had been stated in Rosenbloom's pioneering paper [7] without proof and was explicitly proved by Gross [3]. For the pseudo-primeness we had published several papers [5]. Our methods may be classified two types. One depends upon the Picard theorem and the other the following elegant theorem due to Edrei [2]:

LEMMA. *Let $f(z)$ be an entire function. Assume that there exists an unbounded sequence $\{h_\nu\}_{\nu=1}^\infty$ such that all the roots of the equations $f(z)=h_\nu$, $\nu=1,2,\cdots$ be real. Then $f(z)$ is a polynomial of degree at most two.*

However we need another consideration in order to assure the primeness of individual functions.

In this paper we shall give a method, which guarantees the primeness. This method has close connection with the famous Wiman theorem. In the last part we shall give several examples of prime functions, whose proof depends upon their special forms.

§ 2. We shall prove the following criterion of primeness.

THEOREM 1. *Let $F(z)$ be an entire function of order less than one:*

$$\prod_{l=1}^\infty \left(1-\frac{z}{a_l}\right)^{p_l}, \qquad a_l>0, \qquad a_{l+1}>a_l.$$

Suppose that there are two indices j and k such that $(p_j, p_k)=1$. Further suppose that there is a sequence $\{r_n\}$ such that $a_{n-1}<r_n<a_n$ and $\lim_{n\to\infty}F(r_n)=\infty$. Then $F(z)$ is prime.

Proof. Let $F(z)$ be $f\circ g(z)$. Assume that $f(w)$ is transcendental. Then its order must be less than or equal to the order of $F(z)$. Hence $f(w)=0$ has an infinite number of roots $\{w_n\}$, $w_n\to\infty$ as $n\to\infty$. Consider the equations $g(z)=w_n$, $n=1,2,\cdots$. All their roots lie on the real positive axis. Then by Edrei's theorem

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$g(z)$ is a polynomial of degree at most two. If $g(z)$ is a polynomial of degree two, then $g(z)=w_n$ has two roots $z_{1,n}$, $z_{2,n}$ satisfying almost $z_{1,n}\sim-z_{2,n}$ for any sufficiently large n. This is plainly a contradiction. Thus $g(z)$ is linear. Therefore we may put aside this case.

Assume that $f(w)$ is a polynomial. Then

$$F(z)=Ag_1(z)^{l_1}\cdots g_k(z)^{l_k}, \qquad g_j(z)=g(z)-w_j.$$

In this case any zero of $F(z)$ is not divided into two or more different factors of $F(z)$. Thus we may put

$$g_j(z)=c_j\prod_{l=1}^{\infty}\left(1-\frac{z}{a_{j,l}}\right)^{p_{j,l}},$$

where $\{a_{j,l}\}$ is a subset of $\{a_l\}$ and $\{p_{j,l}\}$ is a subset of $\{p_l\}$ such that $a_{j,l}=a_s$ for a certain s and simultaneously $p_{j,l}=p_s$. Of course $\{a_{j,l}\}\cap\{a_{i,m}\}=\phi$ for $j\neq i$ and $\cup_{j=1}^{k}\{a_{j,l}\}=\{a_s\}$. Since $g_j(z)$ has its order less than one,

$$m_j^*(r)=\min_{|z|=r}|g_j(z)|=|g_j(r)|.$$

Further $m_j^*(r)-|w_j-w_i|\leq m_i^*(r)\leq m_j^*(r)+|w_j-w_i|$, $i\neq j$. Thus $m_i^*(r)$, $m_j^*(r)$ tend to ∞ simultaneously. Since $F(r_n)\to\infty$ as $n\to\infty$, $m_j^*(r_n)$ tends to ∞ as $n\to\infty$ for every j. Consider the sequence $\{a_{1,l}\}$, the set of zeros of $g_1(z)$. We denote it $\{a_{l_1}\}$. Consider the sequence of intervals Δ_{l_1}: (r_{l_1}, r_{l_1+1}). Then $g_2(z)$ does not have any zero in the annulus $r_{l_1}<|z|<r_{l_1+1}$. By the maximum modulus principle

$$|g_2(z)|\geq\min(m_2^*(r_{l_1}), m_2^*(r_{l_1+1}))$$

in $r_{l_1}<|z|<r_{l_1+1}$. Especially $|g_2(a_{l_1})|\geq\min(m_2^*(r_{l_1}), m_2^*(r_{l_1+1}))$. Thus $g_2(a_{l_1})\to\infty$ as $l_1\to\infty$. However $g_1(a_{l_1})=0$. This implies that $g(a_{l_1})\to\infty$ but $g(a_{l_1})=w_1$. Clearly this is a contradiction. Thus we have

$$F(z)=A(g(z)-w_1)^{l_1}.$$

In this case $(p_j,p_k)\geq l_1$. Thus $l_1=1$, which implies that $F(z)=A(g(z)-w_1)$. Thus f is linear. q.e.d.

We cannot omit the condition on the existence of $\{r_n\}$ or the condition $(p_j,p_k)=1$. This is shown by the following examples:

$$F=f^2, \qquad f=\prod_{n=1}^{\infty}\left(1+\frac{z}{n^\alpha}\right), \qquad \alpha>2;$$

$$F=f(f-a), \qquad f=\prod_{n=1}^{\infty}\left(1+\frac{z}{n^\alpha}\right), \qquad \alpha>2, \quad a:\text{ real}.$$

To prove this we need Wiman's observation [8]. He found

$$\log |f(re^{i\varphi})| = \frac{\pi r^{\rho}}{\sin \pi \rho} \cos \rho\varphi - \frac{1}{2\rho} \log 2\pi r^{\rho} + O\left(\log \frac{\delta}{r}\right)$$

where $\rho = 1/\alpha$ and δ shows the distance from $z = re^{i\varphi}$ to the nearest zero $-n^{\alpha}$. Another support is given by Besicovitch's work [1].

Now we list two typical examples of prime functions:

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right), \qquad a_n > 0, \qquad a_{n+1} \geqq ka_n, \qquad k > 1;$$

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n^{\alpha}}\right), \qquad \alpha > 2.$$

It is very plausible to state a conjecture that

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n^{\alpha}}\right), \qquad 2 \geqq \alpha > 1$$

and $1/\Gamma(z)$ are prime. Our method does not work.

§ 3. We shall give two methods to construct prime functions.

THEOREM 2. *Let $L(z)$ be an entire function satisfying the condition in Theorem 1 and with the same notations.*

$$\varlimsup_{n \to \infty} \frac{\log \log |L(r_n)|}{\log r_n} = \rho, \qquad 0 < \rho < 1.$$

Let $M(z)$ be an entire function having only negative zeros and being of order less than ρ. Then $F(z) = L(z)M(z)$ is prime.

Proof. Let $F(z)$ be $f \circ g(z)$. Assume that $f(w)$ is transcendental. Then by Edrei's theorem $g(z)$ must be a polynomial of degree at most two. If $g(z)$ is quadratic, then $g(z) = w_n$ for $f(w_n) = 0$ has two roots $z_{n,1}, z_{n,2}$ in general. If n is sufficiently large, then $z_{n,2} \sim -z_{n,1}$. One of them must be a zero of $L(z)$ and the other a zero of $M(z)$. All the zeros of $F(z)$ can be obtained in this manner. Hence $M(z)$ has the same order as the one of $L(z)$, which is at least ρ. This is a contradiction. Thus $g(z)$ is linear, which may be put aside.

Assume that $f(w)$ is a polynomial. Then we have

$$F(z) = Ag_1(z)^{l_1} \cdots g_k(z)^{l_k}, \qquad g_j(z) = g(z) - w_j.$$

Here any zero of $F(z)$ cannot be divided into two or more different factors of $F(z)$. Let $F(z)$ be

$$A \prod_{j=1}^{k} c_j L_j(z) M_j(z),$$

where $L_j(z)$ and $M_j(z)$ are factors of $L(z)$ and $M(z)$, respectively and $c_j L_j(z) M_j(z)$

is $g_j(z)^{l_j}$. Here $g_1(z), \cdots, g_k(z)$ tend to ∞ simultaneously and have the same growth along the same sequence. Since $M(r_n)$ tends to ∞ as $n \to \infty$, $F(r_n)$ tends to ∞. Then

$$\frac{\log \log |F(r_n)|}{\log r_n} \leqq \frac{\log k \max \log |c_j L_j(r_n) M_j(r_n)|}{\log r_n}$$

implies

$$\lim_{n \to \infty} \frac{\log \log |c_j L_j(r_n) M_j(r_n)|}{\log r_n} \geqq \rho.$$

If

$$\lim_{n \to \infty} \frac{\log \log |L_j(r_n)|}{\log r_n} < \rho,$$

then

$$\lim_{n \to \infty} \frac{\log \log |c_j L_j(r_n) M_j(r_n)|}{\log r_n}$$

$$\leqq \lim_{n \to \infty} \frac{\log 2 \max (\log |L_j(r_n)|, \ \log |M_j(r_n)|)}{\log r_n}$$

$$\leqq \lim_{n \to \infty} \max \left(\frac{\log \log |L_j(r_n)|}{\log r_n}, \ \frac{\log \log |M_j(r_n)|}{\log r_n} \right) < \rho,$$

since

$$\overline{\lim_{n \to \infty}} \frac{\log \log |M_j(r_n)|}{\log r_n} < \rho.$$

This is a contradiction.   Thus

$$\lim_{n \to \infty} \frac{\log \log |L_j(r_n)|}{\log r_n} \geqq \rho.$$

This remains true for each j.   Now by the same process as in Theorem 1 we can find a sequence $\{a_{n,1}\}$ such that

$$L_1(a_{n,1}) = 0, \qquad L_2(a_{n,1}) \to \infty$$

as $n \to \infty$, by making use of the maximum modulus principle. Hence $c_1 L_1(a_{n,1}) M_1(a_{n,1}) = 0$ but $c_2 L_2(a_{n,1}) M_2(a_{n,1}) \to \infty$ as $n \to \infty$. Thus $g(z) - w_1 = 0$ and $g(z) - w_2 \to \infty$ along $\{a_{n,1}\}$. This is a contradiction. Thus we have

$$F(z) = A(g(z) - W_1)^{l_1}.$$

The existence of two indices j and k for which $(p_j, p_k) = 1$ implies $l_1 = 1$. Thus $F(z) = A(g(z) - w_1)$, which is the desired result.

THEOREM 3.   *Let $L(z)$ be an entire function satisfying the condition in Theorem 1. Suppose that for an arbitrary $K > 0$*

$$\lim_{n\to\infty}\frac{|L(r_n)|}{r_n{}^K}=\infty$$

and that there is a sequence $\{s\}$ *such that* $a_{s+1}-a_s \neq a_{t+1}-a_t$ *for any* $t \neq s$. *Let* $P(z)$ *be a polynomial having the form*

$$\prod_{l=1}^{N}\left(1-\frac{z}{a_l}\right)^{q_l},\qquad q_l\leqq p_l,\qquad \sum_{l=1}^{N}q_l\geqq 1,\qquad N<s.$$

Let $Q(z)$ *be a polynomial whose zeros are different from those of* $P(z)$. *Further suppose that*

$$F(z)=L(z)L(-z)\frac{Q(z)}{P(z)}$$

has two zeros whose multiplicities are coprime. Then $F(z)$ *is prime.*

Proof. Let $F(z)$ be $f\circ g(z)$. Assume that $f(w)$ is transcendental. Then $g(z)$ is a polynomial of degree at most two. If $g(z)$ is quadratic, then we put Az^2+Bz+C. Let w_n be a zero of $f(w)$, whose modulus is sufficiently large. Then $Az^2+Bz+C=w_n$ has two roots $a_{l(n)}$, $b_{l(n)}$, which satisfy $a_{l(n)}+b_{l(n)}=-B/A$. For $a_{l(n)+1}$ we have $a_{l(n)+1}+b_{l(n)*}=-B/A$. Hence $a_{l(n)+1}-a_{l(n)}=-(b_{l(n)*}-b_{l(n)})$. Between $b_{l(n)*}$ and $b_{l(n)}$ there is no b_x satisfying $b_x+a_t=-B/A$. Hence $l(n)^*=l(n)+1$. By the assumption on the existence of $\{s\}$ we have

$$b_s=-a_s,\qquad b_{s+1}=-a_{s+1}$$

for a certain s. Thus $B=0$. For $1\leqq t\leqq N$ we have

$$A(-a_t)^2+C=Aa_t{}^2+C,$$

which leads us to a fact that a_t is also a zero of $F(z)$ with the same multiplicity as the one of $-a_t$. This is a contradiction. Therefore $g(z)$ is linear, which may be put aside.

Assume that $f(w)$ is a polynomial. Then

$$F(z)=Ag_1(z)^{l_1}\cdots g_k(z)^{l_k},\qquad g_j(z)=g(z)-w_j.$$

Since

$$\left|1-\frac{z^2}{a^2}\right|,\qquad a>0,\qquad z=re^{i\varphi}$$

is monotone increasing for $\varphi\in(0,\pi/2)$ and is symmetric with respect to the real axis and the imaginary axis,

$$|L(re^{i\varphi})L(-re^{i\varphi})|\geqq|L(r)L(-r)|$$

for $-\pi/2\leqq\varphi\leqq\pi/2$ and for $\pi/2\leqq\varphi\leqq 3\pi/2$. Further $|L(-r)|\geqq|L(r)|$. Hence

$$|L(r_ne^{i\varphi})L(-r_ne^{i\varphi})|\geqq|L(r_n)|^2.$$

Thus

$$|F(r_n e^{i\varphi})| \geq |L(r_n)|^2 \frac{|Q(r_n)|}{|P(r_n)|} \to \infty$$

as $n \to \infty$ for $\varphi \in (-\pi/2, \pi/2)$.   Further

$$|F(\pm ri)| = \max_{|z|=r} |L(z)L(-z)| \cdot \frac{|Q(\pm ri)|}{|P(\pm ri)|}$$

$$\geq \frac{M(r, L(z)L(-z))}{|B|r^s \left(1 + O\left(\dfrac{1}{r}\right)\right)} \to \infty$$

as $r \to \infty$.   Let C_n be the boundary curve of the following domain

$$D_n: \quad r_n < |z| < r_{n+1}, \quad -\frac{\pi}{2} < \arg z < \frac{\pi}{2}.$$

Then we have

$$\lim_{\substack{z \to \infty \\ z \in C_n}} F(z) \to \infty.$$

By the factorization of $F(z)$ each $g_j(z) \to \infty$ as $z \to \infty$, $z \in C_n$.   Any zero of $F(z)$ is not divided into two or more different factors. Let $\{a_{n,1}\}$ be the set of zeros of $g_1(z)$.   Applying the minimum modulus principle for $g_2(z)$ in $D_{n,1}$, we have

$$\lim_{\substack{z \to \infty \\ z \in D_{n,1}}} |g_2(z)| \geq \lim_{\substack{z \to \infty \\ z \in C_{n,1}}} |g_2(z)| = \infty.$$

Thus especially

$$g(a_{n,1}) - w_2 = g_2(a_{n,1}) \to \infty, \qquad g(a_{n,1}) - w_1 = g_1(a_{n,1}) = 0.$$

This is a contradiction. Thus $F(z) = A(g(z) - w_1)^{l_1}$.   By the existence of two indices j, k for which $(p_j, p_k) = 1$, $l_1 = 1$.   This is the desired result.

§ 4.   From now on we shall prove the primeness of several functions.

THEOREM 4.   *Suppose that* $\exp H(z)$ *is of hyperorder less than one, where the hyperorder of* $f(z)$ *stands for*

$$\varlimsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r}.$$

Then $ze^{H(z)}$ *is prime.*

Proof.   Let $F(z) = ze^{H(z)}$ be $f \circ g(z)$.   If $f(w)$ is a polynomial of degree greater than two. If $f(w) = a(w - w_1)^{p_1}$, $p_1 \geq 2$, then $f \circ g(z)$ has an infinite number of roots. If $f(w) = a(w - w_1)^{p_1}$, $p_1 \geq 2$, then $g(z) = w_1$ has either an infinite number of roots or only a finite number of roots. Anyway all roots of $f \circ g(z) = 0$ have their multiplicities at least p_1. This is untenable. Assume that $f(w)$ is transcendental. In

this case $f(w)$ has just one zero of multiplicity one. Hence $f(w)=B(w-w_0)\exp L(w)$. Then $g(z)=w_0+Cz\exp M(z)$. If $M(z)$ is a constant, then the result follows. Thus we may assume the nonconstancy of $M(z)$. Then

$$ze^{H(z)}=F(z)=BCze^{M(z)+L(w_0+Cze^{M(z)})}.$$

A slight analysis leads us to a fact that the right hand side term has its hyperorder not less than one. This is untenable.

If the right hand side term has its hyperorder one, then $M(z)$ must be linear and $L(w)$ must be of order at most zero. Hence

$$H(z)=Az+B+L(w_0+Cze^{Az+B})+2p\pi i.$$

If $H(z)=e^z$ and if L is transcendental, then

$$\lim_{r\to\infty}\frac{T(r,L(w_0+Cze^{Az+B}))}{T(r,e^z)}=\infty$$

shows a contradiction. If L is a polynomial, we need a little bit long discussion based upon Borel's impossibility proof of his identity, for which Nevanlinna [4] gave an extension. Then we have the primeness of $z\exp\exp z$.

THEOREM 5. *Let $H(z)$ be transcendental of order less than one. Suppose that there is at least one simple zero and all its zeros lie on the positive real axis. Then $H(z)e^z$ is prime.*

Proof. Let He^z be $f\circ g(z)$. If $f(w)$ and $g(z)$ are transcendental, then by Pólya's theorem [6] $f(w)$ must be of order zero. If $f(w)$ is transcendental and $g(z)$ is a polynomial of degree $n\geq 2$, then $f(w)$ must be of order $1/n$. Hence $f(w)$ has an infinite number of zeros $\{w_m\}$. Consider $g(z)=w_m$, $m=1,2,\cdots$. They must have only zeros on the positive real axis. Hence by Edrei's theorem and by a slight precise observation $g(z)$ must be linear, which is untenable. Assume that $f(w)$ is a polynomial. Then

$$H(z)e^z=A(g(z)-w_1)^{p_1}\cdots(g(z)-w_k)^{p_k}.$$

In this case $g(z)$ must be of order one. Hence at least one of $g(z)-w_j=0$, $j=1,\cdots,k$ has zeros, whose N-function $N(r;w_j,g)$ must be of order 1 if $k\geq 2$. This is untenable, since $N(r;0,H)$ is of order less than one. Hence $f(w)=A(w-w_1)^{p_1}$. If $p_1\geq 2$, then every zero of $H(z)e^z$ has its multiplicity at least p_1. This contradicts the existence of a simple zero. Thus we have the desired result.

Finally we prove the primeness of

$$\int_0^z e^{-t^p}dt, \qquad p(\geq 2):\text{ an integer.}$$

In this case the derived functional equation $F'(z)=g'(z)f'\circ g(z)$ together with the original functional equation $F(z)=f\circ g(z)$ is useful. Assume that f is transcen-

dental. If g is transcendental, then the order of f is zero. Thus $f'(w)=0$ has an infinite number of roots $\{w_n\}$ and hence at least one of $g(z)=w_n$ has an infinite number of roots, which must satisfy $F'(z)=0$. But $F'(z)$ does not have any zero, which is untenable. Hence g must be a polynomial. If its degree is not less than 2, then $g'(z)$ has a zero. This is a contradiction. Thus g is linear, which may be put aside. Assume that $f(w)$ is a polynomial. Then

$$e^{-z^p}=F'(z)=g'(z)A(g(z)-w_1)^{p_1}\cdots(g(z)-w_k)^{p_k}.$$

Here $g(z)$ is of order p. If $k\geq 2$, then there is an infinite number of zeros of a factor $g(z)-w_j$. This is untenable. Hence $F'(z)=Ag'(z)(g(z)-w_1)^{p_1}$. In this case $g'(z)$ and $g(z)-w_1$ do not have any zero. Hence $g'(z)=B\exp L(z)$ and $g(z)-w_1=C\exp M(z)$ with constants B and C. Here $L(z)$ and $M(z)$ must be polynomials. However $g'(z)=CM'(z)\exp M(z)$. Here $M'(z)$ must be a constant. Thus $M(z)$ is linear. Thus $F'(z)=ACD\exp DzC^{p_1}\exp p_1Dz\neq\exp(-z^p)$. This is a contradiction.

By a closer examination we can find several prime functions in a class of functions having the form

$$\int_0^z P(t)e^{-Q(t)}dt, \qquad \deg P\leqq\deg Q-2.$$

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