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POLYNOMIAL STRUCTURES ON MANIFOLDS

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1. Introduction. Let P be a C^∞ manifold. A C^∞ tensor field f of type (1, 1) on P is said to define a *polynomial structure of degree* d on P if d is the smallest integer for which the powers $I, f, \cdots, f^d$ are dependent, and f has constant rank on P. If $\dim P=2n$, an almost complex structure on P is a polynomial structure of degree 2. If $\dim P=2n-1$, an almost contact structure on P is a polynomial structure of degree 3. A (globally framed) f-manifold is a polynomial structure of degree 3 (see [3], [8]). Walker [7] appears to have inaugurated this study since almost product manifolds provide examples of polynomial structures.

Let M be a $(2n+1)$-dimensional almost contact manifold with fundamental affine collineation ϕ, fundamental vector field E and contact form η. In a recent paper [1], the authors considered a $2n$-dimensional manifold P embedded in M, with embedding $i: P\rightarrow M$, and assumed that for each $p\in P$ the tangent vector $E_{i(p)}$ does not belong to the tangent hyperplane of the hypersurface. This means that the fundamental vector field of M can be taken as the "affine normal" to the hypersurface. We therefore had

$$\phi i_* X=i_* JX+\alpha(X)E, \qquad \phi E=0 \tag{1.1}$$

where i_* is the induced tangent map of i. If $\alpha\neq 0$, we called $i(P)$ a noninvariant hypersurface of M. The structure J induced on P by ϕ is almost complex, that is $J^2=-I$, and J is integrable if M is normal.

More recently [3], we considered the case where E is always tangent to $i(P)$, so that it can no longer play the role of "affine normal". However, we showed that a vector field N exists along the hypersurface such that

$$\phi i_* X=i_* fX+\alpha(X)N \tag{1.2}$$

and

$$\phi N=-i_* A, \qquad \eta(N)=0, \tag{1.3}$$

for some vector field A on P and (1, 1) tensor field f, so, in this case, N plays the role of "affine normal". The structure induced on P is an f-structure [8], that is $f^3+f=0$ and f has the same rank at each point of P, but it is not almost complex. (Observe that $\phi E=0$ and $\eta(E)=1$, whereas ϕN has a nonzero tangential component

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and $\eta(N)=0$.)

In this paper, we show the existence of a *quartic structure* f, that is a (1, 1) tensor field f of constant rank satisfying the algebraic condition

$$f^4+af^3+bf^2+cf+dI=0,$$

where I is the identity transformation field, which is not an f-structure. As in [3] and [4] we study its properties which turn out to be strikingly similar to those of a globally framed f-manifold.

2. Quartic structures. Let $i(P)$ be a noninvariant hypersurface of the almost contact manifold $M(\phi, E, \eta)$. We wish to choose an affine normal N on $i(P)$ in such a way that the vector field ϕN is always tangent to the hypersurface, that is

$$\phi N=-i_* U, \tag{2. 1}$$

for some vector field U on P.

(The vector field N will not be the metric normal with respect to the Riemannian metric G of the almost contact metric structure (ϕ, E, η, G). For, $\eta=G(E, \cdot)$ and the condition $\eta(N)=0$ imposed below is not possible unless E is tangent to the hypersurface.)

Since a vector field N not tangent to the hypersurface can be represented as

$$N=\frac{1}{\lambda}(-i_*X+E),$$

for a certain vector field X and scalar field $\lambda\neq 0$, we have

$$\phi N=-\frac{1}{\lambda}(i_*JX+\alpha(X)E)$$

by virtue of (1. 1). Thus, for (2. 1) to hold, we must have $\alpha(X)=0$. We therefore assume that a global vector field V exists which satisfies this equation, that is, $\alpha(V)=0$. Putting $X=V$ in the above equation, (2. 1) holds with $U=(1/\lambda)JV$, and

$$E=i_* V+\lambda N, \qquad \lambda\neq 0. \tag{2. 2}$$

Hence, by setting $\beta(X)=\alpha(JX)$, we have, since $J^2=-I$,

$$\alpha(V)=0, \qquad \beta(U)=0, \tag{2. 3}$$

$$JV=\lambda U, \qquad JU=-\frac{1}{\lambda}V. \tag{2. 4}$$

From (1. 1), it is easily seen that $\beta=i^*\eta$.

From (1. 1), (2. 1), (2. 2) and (2. 4),

$$-N+\eta(N)(i^* V+\lambda N)=\frac{1}{\lambda}i_* V-\alpha(U)(i_* V+\lambda N),$$

so that

$$\eta(N)=\frac{1}{\lambda}-\alpha(U),$$

or equivalently,

$$\lambda\eta(N)+\beta(V)=1.$$

From (1. 1) and (2. 2),

$$\phi i_{*}X=i_{*}(J+\alpha\otimes V)X+\lambda\alpha(X)N,$$

that is

$$\phi i_{*}X=i_{*}J'X+\alpha'(X)N, \tag{2. 5}$$

where

$$J'=J+\alpha\otimes V, \qquad \alpha'=\lambda\alpha. \tag{2. 6}$$

Thus,

$$J'^{2}=-I+\alpha'\otimes U+\beta\otimes V.$$

(Observe that J' is an almost complex structure on P, if and only if, $U=0$, $V=0$, that is, if and only if $E=\lambda N$.)

THEOREM 1. *Let $P(J, \alpha)$ be a noninvariant hypersurface of the almost contact manifold $M(\phi, E, \eta)$. If there is a global vector field V on P such that $\alpha(V)=0$ then, the tensor fields J', U, α', V and β on P satisfy the relations*

$$J'^{2}=-I+\alpha'\otimes U+\beta\otimes V,$$

$$J'U=-\eta(N)V, \qquad J'V=\lambda U,$$

$$\alpha'\circ J'=\lambda\beta, \qquad \beta\circ J'=-\eta(N)\alpha',$$

$$\alpha'(U)=1-\lambda\eta(N), \qquad \alpha'(V)=0,$$

$$\beta(U)=0, \qquad \beta(V)=1-\lambda\eta(N).$$

COROLLARY 1. *If the vector fields E and N are distinct affine normals, then the structure on P is a quartic structure.*

For, J' has constant rank and

$$f^{4}+(1+\lambda\eta(N))f^{2}+\lambda\eta(N)I=0,$$

where $f=J'$.

The left side of this equation may be factored, that is

$$(f^{2}+\lambda\eta(N)I)(f^{2}+I)=0.$$

We treat two cases, namely, $\lambda=1$, $\eta(N)=0$ and $\lambda\eta(N)=1$, the former giving rise to the quartic structure $f^4+f^2=0$ and the latter to $(f^2+I)^2=0$.

Case I. $\lambda=1$, $\eta(N)\neq 1$. Then

$$(f^2+\eta(N)I)(f^2+I)=0.$$

By putting

$$\tilde{U}=\frac{1}{1-\eta(N)}U, \qquad \tilde{V}=\frac{1}{1-\eta(N)}V, \qquad \tilde{\alpha}=\alpha \quad \text{and} \quad \tilde{\beta}=\beta,$$

we obtain

COROLLARY 2. *Let P be a noninvariant hypersurface of an almost contact manifold. Then if $\lambda=1$ and $\eta(N)\neq 1$, P is not globally framed, that is*

$$f^2=-I+(1-\eta(N))[\tilde{\alpha}\otimes\tilde{U}+\tilde{\beta}\otimes\tilde{V}],$$

$$\tilde{\alpha}(\tilde{U})=1, \qquad \tilde{\alpha}(\tilde{V})=0,$$

$$\tilde{\beta}(\tilde{U})=0, \qquad \tilde{\beta}(\tilde{V})=1.$$

Moreover,

$$f\tilde{U}=-\eta(N)\tilde{V}, \qquad f\tilde{V}=\tilde{U},$$

$$\tilde{\alpha}\circ f=\tilde{\beta}, \qquad \tilde{\beta}\circ f=-\eta(N)\tilde{\alpha}.$$

However, by choosing $\eta(N)=0$, we obtain

COROLLARY 3. *Let P be a noninvariant hypersurface of an almost contact manifold. Then, if $\lambda=1$ and $\eta(N)=0$, P is globally framed, that is*

$$\text{(2.7)} \qquad \begin{cases} f^2=-I+\alpha\otimes U+\beta\otimes V, & \\ fU=0, & fV=U, \\ \alpha\circ f=\beta, & \beta\circ f=0, \\ \alpha(U)=1, & \alpha(V)=0, \\ \beta(U)=0, & \beta(V)=1. \end{cases}$$

As an example of a noninvariant hypersurface with $\lambda=1$ and $\eta(N)=0$ consider the plane $z=y$ in R^3 with

$$\phi X=a\frac{\partial}{\partial y}-b\frac{\partial}{\partial x},$$

where

$$X=a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}+c\frac{\partial}{\partial z}, \qquad E=\frac{\partial}{\partial z} \quad \text{and} \quad \eta=dz.$$

For N choose the vector field $\partial/\partial y$ and take $U=\partial/\partial x$. Hence, $\phi N=-i_*U$. Set

$$V=\frac{\partial}{\partial z}-\frac{\partial}{\partial y}.$$

Thus, $JU=-V$ and $JV=U$. (Observe that the almost contact structure so defined on R^3 is cosymplectic.)

The structure given on P by Corollary 3 is not an f-structure since $f^3+f=\beta\otimes U$. However, it is globally framed and, since $fU=0$,

$$f^4+f^2=0. \tag{2.8}$$

Clearly, an f-structure satisfies (2.8). Observe that by putting $s=-f^2$, $s^2=s$. In the general case,

$$s^2-(1+\lambda\eta(N))s+\lambda\eta(N)I=0, \tag{2.9}$$

the roots of which are 1 and $\lambda\eta(N)$.

Case II. $\lambda=1/\eta(N)$. Then, the roots of (2.9) are equal, so that $(f^2+I)^2=0$, $\alpha(U)=0$ and $\beta(V)=0$. Moreover, by (2.2), $\eta(N)$ is nowhere zero.

COROLLARY 4. *Let P be a noninvariant hypersurface of an almost contact manifold. Then, if $\lambda=1/\eta(N)$, α and β vanish on the distribution determined by U and V, and*

$$(f^2+I)^2=0,$$

$$f^2=-I+\alpha\otimes U'+\beta\otimes V,$$

$$fU'=-V, \qquad fV=U',$$

$$\alpha\circ f=\beta, \qquad \beta\circ f=-\alpha,$$

where $U'=\lambda U$.

This structure on P is clearly not globally framed.

Let X be a vector field on $P(f, U, \alpha, V, \beta)$ which is annihilated by the (1, 1) tensor field f. Then $f^2X=-X+\alpha(X)U+\beta(X)V$, so X is a linear combination of U and V. Further applications of f then yield

THEOREM 2. *The structure $P(f, U, \alpha, V, \beta)$ of Corollary 3 is a quartic structure of rank $2n-1$ whereas the structure $P(f, U', \alpha, V, \beta)$ of Corollary 4 is also of degree 4 but it has maximal rank.*

We call the former the *restricted quartic structure.*

Although equations (1.2) and (1.3) are formally the same as (2.5) and (2.1), respectively, the polynomial structures they give rise to, due to the different embeddings, are not the same, the former being cubic and the latter quartic.

We denote as usual by L_X the operator of Lie derivation with respect to the vector field X and by $[X, Y]$ the Lie bracket of the vector fields X and Y. The following result is valid for any globally framed manifold (see §6).

LEMMA 1. *On the hypersurface with the restricted quartic structure* (f, U, α, V, β)

(a) $$X(\eta^a(Y))=(L_X\eta^a)(Y)+\eta^a([X, Y]),$$

(b) $$d\eta^a(E_b, X)=(L_{E_b}\eta^a)(X),$$

(c) $$d\eta^a(fX, Y)=(L_{fX}\eta^a)(Y)-Y(\eta^a(fX)),$$

$a=1, 2$, *where* $\eta^1=\alpha$, $\eta^2=\beta$, $E_1=U$, $E_2=V$.

In the sequel, only those quartic structures which are globally framed are studied.

3. Hypersurfaces of affinely cosymplectic spaces. If $M(\phi, E, \eta)$ is an affinely cosymplectic manifold, then $\nabla\phi=0$ and $\nabla\eta=0$ where ∇ denotes covariant differentiation with respect to a symmetric affine connection on M (see [1]). Since $\phi^2=-I+\eta\otimes E$, the vector field E is also parallel with respect to ∇. Denoting by D the induced connection on the hypersurface P with respect to the affine normal N, the equations of Gauss and Weingarten are

$$(D_X i_*)Y=h(X, Y)N$$

and

$$D_X N\equiv\nabla_{i_*X}N=-i_*HX+\omega(X)N,$$

respectively, where h and H are the second fundamental tensors (of types (0, 2) and (1, 1), respectively) of P with respect to the affine normal N, the tensor h being symmetric, and ω is a 1-form on P defining the connection in the affine normal bundle.

Covariant differentiation of both sides of (2. 5) along P gives

$$-h(X, Y)i_*U+\phi i_*D_YX$$

$$=h(Y, J'X)N+i_*(D_YJ')X+i_*J'(D_YX)+[(D_Y\alpha')X+\alpha'(D_YX)]N-\alpha'(X)(i_*HY-\omega(Y)N),$$

so that

$$(D_YJ')X=\alpha'(X)HY-h(X, Y)U$$

and

$$(D_Y\alpha')(X)=-h(Y, J'X)-\alpha'(X)\omega(Y).$$

Differentiating (2. 1) along P yields

$$i_*J'HY+\alpha'(HY)N+\omega(Y)i_*U=h(Y, U)N+i_*D_YU,$$

from which

$$D_YU=J'HY+\omega(Y)U$$

and

$$h(Y, U)=\alpha'(HY).$$

From (2. 2), we obtain

$$\nabla_{i_*X}E=h(X,\ V)N+i_*D_X V-\lambda i_*HX+\lambda\omega(X)N+(X\lambda)N,$$

so that, $\nabla_{i_*X}E$ being zero,

$$D_X V=\lambda HX$$

and

$$h(X,\ V)=-X\lambda-\lambda\omega(X).$$

Differentiating both sides of the relation $\beta=i^*\eta$ gives

$$(D_X\beta)(Y)+\beta(D_X Y)=(\nabla_{i_*X}\eta)(i_* Y)+h(X,\ Y)\eta(N)+\beta(D_X Y),$$

that is

$$(D_X\beta)(Y)=h(X,\ Y)\eta(N).$$

Summarizing, we have

$$(3.1)\qquad \begin{cases} (D_X J')Y=\alpha'(Y)HX-h(X,\ Y)U, \\ D_X V=\lambda HX,\ D_X U=J'HX+\omega(X)U, \\ (D_X\beta)(Y)=h(X,\ Y)\eta(N), \\ (D_X\alpha')(Y)=-h(X,\ J'Y)-\omega(X)\alpha'(Y), \\ h(X,\ V)=-X\lambda-\lambda\omega(X),\ h(X,\ U)=\alpha'(HX). \end{cases}$$

THEOREM 3. *Let P be a noninvariant hypersurface of an affinely cosymplectic manifold with the restricted quartic structure $(f,\ U,\ \alpha,\ V,\ \beta)$. Then, with respect to the induced connection D on P, the quartic structure on P satisfies the relations*

$$(3.2)\qquad \begin{cases} (D_X f)Y=\alpha(Y)HX-h(X,\ Y)U, \\ D_X V=HX, \qquad D_X U=fHX+\omega(X)U, \\ D_X\beta=0, \qquad (D_X\alpha)(Y)=-h(X, fY)-\omega(X)\alpha(Y), \\ h(X,\ V)=-\omega(X), \qquad h(X,\ U)=\alpha(HX), \\ \beta(HX)=0. \end{cases}$$

Proof. Put $f=J'$, $\lambda=1$ and $\eta(N)=0$ in (3. 1). The last formula follows by differentiating $\eta(N)=0$.

If for every vector field X on P, $HX=0$, then, by Weingarten's equation, $D_X N$ and N are proportional. Hence, the affine normals are parallel along the hypersurface. In this case, P is said to be *totally flat*.

We are now able to deduce the following facts.

THEOREM 4. *If the hypersurface P is endowed with the restricted quartic structure and if it is an affinely umbilical hypersurface of an affinely cosymplectic manifold, then it is totally flat.*

Proof. Since P is affinely umbilical, $H=\mu I$. Hence, $0=\beta(HX)=\mu\beta(X)$. But $\beta(V)=1$, so μ must vanish.

The following lemma will be required in the proof of the corollary to Theorem 8.

LEMMA 2. *Let P be a noninvariant hypersurface of an affinely cosymplectic manifold with the restricted quartic structure (f, U, α, V, β). Then, if the linear transformation field f is a parallel field,*

$$h=\mu\alpha\otimes\alpha,$$

$$H=\mu\alpha\otimes U$$

and

$$\omega=0$$

for some function μ depending on U and h.

To see this, we first observe that from (3. 2), $\alpha(Y)\alpha(HX)=h(X, Y)$, from which, since h is symmetric $\alpha(Y)\alpha(HX)=\alpha(X)\alpha(HY)$, so $\alpha(X)h(U, U)=\alpha(HX)$. Setting $\mu=h(U, U)$, we get $h(X, Y)=\mu\alpha(X)\alpha(Y)$, from which $h(X, U)=\mu\alpha(X)$. Thus, putting $Y=U$ in $\alpha(Y)HX-h(X, Y)U=0$, we find that $HX=\mu\alpha(X)U$. On the other hand, $\omega(X)=-h(X, V)=-\mu\alpha(X)\alpha(V)=0$.

Let D' be the induced connection on $i(P)$ with respect to the fundamental vector field E of the almost contact ambient space $M(\phi, E, \eta)$. Then the equations of Gauss and Weingarten are

$$(D'_X i_*)Y=h'(X, Y)E \tag{3. 3}$$

and

$$\nabla_{i_*X}E=-i_*H'X+\omega'(X)E. \tag{3. 4}$$

From

$$(D_X i_*)Y=\nabla_{i_*X}i_*Y-i_*D_X Y,$$

$$(D'_X i_*)Y=\nabla_{i_*X}i_*Y-i_*D'_X Y,$$

and

$$(D_X i_*)Y=h(X, Y)N,$$

$$(D'_X i_*)Y=h'(X, Y)(i_*V+\lambda N),$$

we have

$$i_*(D'-D)(X, Y)=-h'(X, Y)i_*V-[\lambda h'(X, Y)-h(X, Y)]N.$$

Therefore,

$$D'=D-h'\otimes V, \qquad h=\lambda h'. \tag{3.5}$$

Thus, $D'=D$, if and only if $h=0$ since $V\neq 0$. Moreover, the condition that the hypersurface be *totally geodesic* ($h=0$) is independent of the choice of affine normal.

On the other hand, from (3.4),

$$\nabla_{i_*X}(i_*V+\lambda N)=-i_*H'X+\omega'(X)(i_*V+\lambda N).$$

But, $\nabla_{i_*X}(i_*V+\lambda N)=i_*(D_XV-\lambda HX)+[\lambda h'(X, V)+X\lambda+\lambda\omega(X)]N$, so

$$H'=\lambda H-DV+\omega'\otimes V$$

and

$$\lambda\omega'=\lambda\omega+\lambda h'(\cdot, V)+d\lambda.$$

Thus, the hypersurface is totally flat with respect to the affine normals E and N, if and only if

$$DV=\omega'\otimes V.$$

For almost complex manifolds, in general, it is known that if J is an integrable almost complex structure, then there exists a symmetric affine connection with respect to which it is parallel [6]. For noninvariant hypersurfaces we have the following explicit result.

THEOREM 5. *Let $P(J, \alpha)$ be a noninvariant hypersurface of an affinely cosymplectic manifold with the restricted quartic structure. Then, if P is totally geodesic, J is parallel with respect to the induced connection.*

Proof. By (3.5), $h=h'=0$ and $D'=D$. In a recent paper [1], the authors showed that $D'J=0$ in a noninvariant hypersurface of an affinely cosymplectic space, so J is parallel with respect to D.

4. Normal quartic structures. Although the globally framed quartic manifold $P(f, U, \alpha, V, \beta)$ is not an f-structure, it does have an underlying f-structure given by the (1, 1) tensor field

$$f_1=f-\beta\otimes U. \tag{4.1}$$

For, $f_1^2X=-X+\alpha(X)U+\beta(X)V$, and hence $f_1^3X=-f_1X+\alpha(X)f_1U+\beta(X)f_1V=-f_1X$. Moreover, $f_1X=0$ implies $X=\alpha(X)U+\beta(X)V$, so rank $f_1=2n-2$.

The globally framed quartic structure (f, U, α, V, β) on P will be called *normal* if the underlying globally framed f-structure $(f_1, U, \alpha, V, \beta)$ on P is normal. The condition for this is that the tensor field S_{f_1} of type (1, 2) given by

$$S_{f_1}=[f_1, f_1]+d\alpha\otimes U+d\beta\otimes V$$

vanish [4]. In this case, U and V are infinitesimal automorphisms of the structure

$(f_1, U, \alpha, V, \beta)$, and $d\alpha$ and $d\beta$ are of bidegree (1, 1) with respect to f_1 (see [4], Lemma 2).

We express S_{f_1} in terms of f. To this end, we need only evaluate the Nijenhuis torsion $[f_1, f_1]$ in terms of f, U and β. For any vector fields X and Y,

$$\begin{aligned}
[f_1, f_1](X, Y) &= [f-\beta\otimes U, f-\beta\otimes U](X, Y) \\
&= [fX-\beta(X)U, fY-\beta(Y)U]-(f-\beta\otimes U)[fX-\beta(X)U, Y] \\
&\quad +(f-\beta\otimes U)[fY-\beta(Y)U, X]+f_1^2[X, Y] \\
&= [fX, fY]+\beta(Y)\{(L_U f)X-f[X, U]\}-(fX)(\beta(Y))U \\
&\quad +\beta(X)[fY, U]+(fY)(\beta(X))U \\
&\quad +\beta(X)\{(L_U\beta)(Y)+\beta([U, Y])\}U-\beta(Y)\{(L_U\beta)(X) \\
&\quad +\beta([U, X])\}U-f[fX, Y]+\beta([fX, Y])U+f[\beta(X)U, Y] \\
&\quad -\beta([\beta(X)U, Y])U+f[fY, X]-\beta([fY, X])U-f[\beta(Y)U, X] \\
&\quad -\beta([\beta(Y)U, X])U+f^2[X, Y] \\
&= [f, f](X, Y)+\beta(Y)(L_U f)X-\beta(X)(L_U f)Y+\{(fY)(\beta(X)) \\
&\quad -(fX)(\beta(Y))-\beta(Y)(L_U\beta)(X)+\beta(X)(L_U\beta)(Y) \\
&\quad +\beta([fX, Y])-\beta([fY, X])\}U \\
&= [f, f](X, Y)+\beta(Y)\{(L_U f)X-(L_U\beta)(X)U\} \\
&\quad -\beta(X)\{(L_U f)Y-(L_U\beta)(Y)U\} \\
&\quad +\{d\beta(fY, X)-d\beta(fX, Y)\}U.
\end{aligned}$$

If $P(f_1, U, \alpha, V, \beta)$ is normal, then, since $(L_U f_1)X=(L_U f)X-(L_U\beta)(X)U$ and $d\beta(fX, Y)=d\beta(f_1X, Y)+\beta(X)(L_U\beta)(Y)=d\beta(f_1X, Y)$,

$$\begin{aligned}
S_{f_1} &= [f_1, f_1]+d\alpha\otimes U+d\beta\otimes V \\
&= [f, f]+d\alpha\otimes U+d\beta\otimes V \\
&= S_f.
\end{aligned}$$

Thus, if $P(f, U, \alpha, V, \beta)$ is normal, the tensor field S_f given by

$$S_f=[f, f]+d\alpha\otimes U+d\beta\otimes V \tag{4. 2}$$

vanishes.

LEMMA 3. *If the hypersurface $P(f, U, \alpha, V, \beta)$ is normal, then*

(i) $L_{E_a}\eta^b=0$,

(ii) $[E_a, E_b]=0$,

(iii) $L_{E_a}f=0$,

(iv) $d\eta^a(fX, Y)+d\eta^a(X, fY)=0$,

for any vector fields X and Y on P.

Proof. The proof is contained in the above discussions. For the sake of completeness, however, we proceed as follows. Since the structure on P is normal

$$[f, f](X, Y)+d\eta^a(X, Y)E_a=0, \tag{4.3}$$

the summation convention being employed here and in the sequel. Putting $Y=E_b$ in (4.3), we find

$$-f[fX, E_b]+f^2[X, E_b]+d\eta^a(X, E_b)E_a=0,$$

that is,

$$f(L_{E_b}f)X+d\eta^a(X, E_b)E_a=0. \tag{4.4}$$

Taking the interior product of both sides of (4.4) with α and then β, we obtain

$$\beta((L_Uf)X)+d\alpha(X, U)=0,$$

$$\beta((L_Vf)X)+d\alpha(X, V)=0,$$

and

$$d\beta(X, U)=0, \qquad d\beta(X, V)=0.$$

Formula (4.4) reads

$$f(L_Uf)X+d\alpha(X, U)U=0$$

by means of the last relation. Hence, $d\alpha(X, U)=-\beta((L_Uf)X)=(L_U\beta)(fX)=d\beta(U, fX)=0$. Similarly, $d\alpha(X, V)=0$. Applying (b) of Lemma 1, this proves (i).

Substituting $X=U$, $Y=V$ in (4.3), we have $f^2[U, V]=0$, so by (2.7) and (i), $[U, V]=\alpha([U, V])U+\beta([U, V])V=0$.

From (4.4), $fL_{E_a}f=0$, so $L_Uf=\mu\otimes U$ for some 1-form μ. Consequently, $\alpha((L_Uf)X=\mu(X)$. Thus, since $0=(L_U(\alpha\circ f))X=((L_U\alpha)f)X+\alpha((L_Uf)X)=\mu(X)$, L_Uf vanishes. Similarly, $L_Vf=0$.

From (4.3), we find

$$\beta([fX, fY])+d\beta(X, Y)=0, \tag{4.5}$$

so that

$$0=\beta([fX, f^2Y])+d\beta(X, fY)$$

$$=-\beta([fX, Y])+\beta([fX, \alpha(Y)U])+\beta([fX, \beta(Y)V])+d\beta(X, fY)$$
$$=-\beta([fX, Y])+fX(\beta(Y))+d\beta(X, fY).$$

On the other hand,

$$fX(\beta(Y))-\beta([fX, Y])=d\beta(fX, Y),$$

so $d\beta(fX, Y)+d\beta(X, fY)=0$. Again, from (4. 3), we find

$$\alpha([fX, fY])-\beta([fX, Y])-\beta([X, fY])+d\alpha(X, Y)=0,$$

so that

$$\begin{aligned}0&=\alpha([fX, f^2Y])-\beta([fX, fY])-\beta([X, f^2Y])+d\alpha(X, fY)\\&=-\alpha([fX, Y])+\alpha([fX, \alpha(Y)U])+\alpha([fX, \beta(Y)V])-\beta([fX, fY])\\&\quad+\beta([X, Y])-\beta([X, \alpha(Y)U])-\beta([X, \beta(Y)V])+d\alpha(X, fY)\\&=-\alpha([fX, Y])+fX(\alpha(Y))-\alpha(Y)\beta([U, X])+\beta(Y)\beta([X, V])\\&\quad-\beta([fX, fY])+\beta([X, Y])-\alpha(Y)\beta([X, U])-\beta(Y)\beta([X, V])\\&\quad-X(\beta(Y))+d\alpha(X, fY)\\&=-\alpha([fX, Y])+fX(\alpha(Y))-\beta([fX, fY])+\beta([X, Y])\\&\quad-X(\beta(Y))+d\alpha(X, fY).\end{aligned}$$

On the other hand, $d\alpha(fX, Y)=fX(\alpha(Y))-Y(\beta(X))-\alpha([fX, Y])$. Hence,

$$d\alpha(fX, Y)+Y(\beta(X))-\beta([fX, fY])+\beta([X, Y])-X(\beta(Y))+d\alpha(X, fY)=0,$$

that is,

$$d\alpha(fX, Y)-d\beta(X, Y)-\beta([fX, fY])+d\alpha(X, fY)=0,$$

so by (4. 5), $d\alpha(fX, Y)+d\alpha(X, fY)=0$.

Formulae (i)—(iii) say that U and V are infinitesimal automorphisms of the structure (f, U, α, V, β) while (iv) says that $d\alpha$ and $d\beta$ are of bidegree (1, 1) with respect to f.

THEOREM 6. *Let $P(J, \alpha)$ be a noninvariant hypersurface of an almost contact manifold with the restricted quartic structure. If this structure is normal, then the almost complex structure J is integrable.*

Proof. We relate the torsion $[J, J]$ of J to $[f, f]$. From (2. 6)

$$\begin{aligned}[J, J](X, Y)&=[JX, JY]-J[JX, Y]-J[X, JY]-[X, Y]\\&=[fX, fY]-[fX, \alpha(Y)V]+[fY, \alpha(X)V]+[\alpha(X)V, \alpha(Y)V]\end{aligned}$$

$$
\begin{aligned}
&-f[fX, Y]+f[\alpha(X)V, Y]+\alpha([fX, Y])V-\alpha([\alpha(X)V, Y)])V\\
&+f[fY, X]-f[\alpha(Y)V, X]-\alpha([fY, X])V+\alpha([\alpha(Y)V, X])V-[X, Y]\\
=&[fX, fY]-fX(\alpha(Y))V-\alpha(Y)[fX, V]\\
&+fY(\alpha(X))V+\alpha(X)[fY, V]\\
&+\alpha(X)V(\alpha(Y))V-\alpha(Y)V(\alpha(X))V\\
&-f[fX, Y]+\alpha(X)f[V, Y]-Y(\alpha(X))U\\
&\qquad+\alpha([fX, Y])V-\alpha(X)\alpha([V, Y])V\\
&+f[fY, X]-\alpha(Y)f[V, X]+X(\alpha(Y))U\\
&\qquad-\alpha([fY, X])V+\alpha(Y)\alpha([V, X])V-[X, Y]\\
=&[f, f](X, Y)-\alpha([X, Y])U-\alpha(f[X, Y])V\\
&+\{fY(\alpha(X))-fX(\alpha(Y))+\alpha(X)d\alpha(V, Y)-\alpha(Y)d\alpha(V, X)\\
&\qquad+\alpha([fX, Y])-\alpha([fY, X]\}V\\
&+\alpha(X)([fY, V]+f[V, Y])-\alpha(Y)([fX, V]+f[V, X])\\
&+d\alpha(X, Y)U+\alpha([X, Y])U\\
=&[f, f](X, Y)+d\alpha(X, Y)U\\
&+\{\alpha(X)(L_V\alpha)(Y)-\alpha(Y)(L_V\alpha)(X)\}V\\
&-\alpha(X)(L_Vf)Y+\alpha(Y)(L_Vf)X\\
&+\alpha([fX, Y]-[fY, X]-f[X, Y])V\\
&+\{fY(\alpha(X))-fX(\alpha(Y))\}V\\
=&[f, f](X, Y)+d\alpha(X, Y)U+d\beta(X, Y)V\\
&-\{d\alpha(X, fY)+d\alpha(fX, Y)\}V\\
&+\alpha(Y)(L_VJ)X-\alpha(X)(L_VJ)Y.
\end{aligned}
$$

Theorem 6 is now a consequence of Lemma 3.

We state the following converse.

THEOREM 7. *Let $P(J, \alpha)$ be a noninvariant hypersurface of an almost contact manifold with the restricted quartic structure. Then, if J is integrable, the structure (f, U, α, V, β) on P is normal provided $d\alpha$ is of bidegree* (1, 1) (*with respect to f*) *and the vector field V is holomorphic.*

Since J is a complex structure if the ambient space is normal ([1], Theorem 1), we have

COROLLARY 1. *Let $P(J, \alpha)$ be a noninvariant hypersurface of a normal almost contact manifold with the restricted quartic structure. Then, if $d\alpha$ is of bidegree* (1, 1) (*with respect to* f), *and* V *is a holomorphic vector field* (*with respect to* J), *the restricted quartic structure on* P *is normal.*

REMARK. The direct product of the quartic structures $P_i(f_i, U_i, \alpha_i, V_i, \beta_i)$, $i=1, 2$, has a naturally induced almost complex structure $\tilde{J}$ on $P_1 \times P_2$ given by

$$\tilde{J}_{(p_1, p_2)}(X_1, X_2)=(f_1X_1+\beta_2(X_2)V_1-\alpha_2(X_2)U_1, f_2X_2-\beta_1(X_1)V_2+\alpha_1(X_1)U_2).$$

The tensor field $\bar{J}$ defined by

$$\begin{aligned}\bar{J}_{(p_1, p_2)}(X_1, X_2)=(&f_1X_1-[\beta_1(X_1)+\alpha_2(X_2)]U_1-\beta_2(X_2)V_1,\\ &f_2X_2-[\beta_2(X_2)-\alpha_1(X_1)]U_2+\beta_1(X_1)V_2)\end{aligned}$$

is also an almost complex structure on $P_1 \times P_2$. Since the $(f'_i, U_i, \alpha_i, V_i, \beta_i)$, where $f'_i=f_i-\beta_i \otimes U_i$, $i=1, 2$, are framed f-structures on P_i and

$$\bar{J}_{(p_1, p_2)}(X_1, X_2)=(f'_1X_1-\alpha_2(X_2)U_1-\beta_2(X_2)V_1, f'_2X_2+\alpha_1(X_1)U_2+\beta_1(X_1)V_2),$$

$\bar{J}$ is integrable if the quartic structures are normal, and conversely.

5. Normal quartic hypersurfaces. In this section, we seek necessary and sufficient conditions for the normality of the restricted structure on P when the ambient space M is cosymplectic. We compute $\tilde{S}(i_*X, i_*Y)$ for any vector fields X and Y on P, where $\tilde{S}$ is the torsion tensor of the almost contact structure $M(\phi, E, \eta)$, that is

$$\begin{aligned}\tilde{S}(x, y)&=[\phi, \phi](x, y)+d\eta(x, y)E\\ &=(\nabla_{\phi x}\phi)y-(\nabla_{\phi y}\phi)x-\phi\{(\nabla_x\phi)y-(\nabla_y\phi)x\}+\{(\nabla_x\eta)(y)-(\nabla_y\eta)(x)\}E,\end{aligned}$$

where x and y are vector fields on M. Thus, after a lengthy conputation not unlike that in [3] in which the equations (3. 2) are vital

$$\begin{aligned}\tilde{S}(i_*X, i_*Y)=i_*\{&[f, f](X, Y)+d\alpha(X, Y)U+d\beta(X, Y)V\\ &+\alpha(X)(Hf-fH)Y-\alpha(Y)(Hf-fH)X\\ &+[\omega(X)\alpha(Y)-\omega(Y)\alpha(X)]U\}\\ &+\{(D_{fX}\alpha)(Y)-(D_{fY}\alpha)(X)-\alpha(D_X(fY))+\alpha(D_Y(fX))\\ &+\alpha(HX)\alpha(Y)-\alpha(HY)\alpha(X)+\beta([X, Y])\\ &+[\omega(fX)\alpha(Y)-\omega(fY)\alpha(X)]\}N\\ &+(\nabla_N\phi)(\alpha(X)i_*Y-\alpha(Y)i_*X),\end{aligned}$$

so since ϕ is a parallel field and $\tilde{S}$ vanishes when M is cosymplectic,

$$([f,f]+d\alpha\otimes U+d\beta\otimes V)(X,Y)+\alpha(X)(Hf-fH-\omega\otimes U)Y-\alpha(Y)(Hf-fH-\omega\otimes U)X=0.$$

THEOREM 8. *Let P be a noninvariant hypersurface of an affinely cosymplectic manifold with the restricted quartic structure. Then, a necessary and sufficient condition that the structure on P be normal is that*

$$Hf-fH=\omega\otimes U+\alpha\otimes Z, \tag{5.1}$$

where Z is the vector field $-D_U U$.

Theorem 8 may also be obtained by computing S directly from the relations (3.2).

COROLLARY. *Let P be a noninvariant hypersurface of an affinely cosymplectic manifold with the restricted quartic structure. Then, if the structure on P is normal and f is parallel, P is totally flat and the structure is covariant constant.*

For, from (5.1) and Lemma 2, $\mu\alpha(fX)U-\mu\alpha(X)fU=-\alpha(X)D_U U$. Hence, from the equations (2.7), $\mu\beta(X)U=-\alpha(X)D_U U$. Putting $X=V$, we get $\mu=0$, so by Lemma 2, $H=0$.

6. Globally framed quartic manifolds. A C^∞ manifold is said to be *globally framed* if there exists a (1, 1) tensor field f, global vector fields E_a and linear differential forms η^a, $a=1,\cdots,\nu$ satisfying the relations

$$\eta^a(E_b)=\delta^a_b \tag{6.1}$$

and

$$f^2=-I+\eta^a\otimes E_a. \tag{6.2}$$

Clearly, the only globally framed polynomial structures defined by f are those given by $f^3+f=0$ and $f^4+f^2=0$, the former arising by assuming that $fE_a=0$, $a=1,\cdots,\nu$.

In the sequel, a manifold P with a quartic structure f of rank r such that $f^4+f^2=0$ will simply be called a *quartic manifold*. Put

$$s=-f^2,\qquad t=f^2+I,$$

where I is the identity field. Then,

$$s+t=I,$$
$$s^2=s,\qquad t^2=t,$$
$$st=0,\qquad ts=0.$$

Thus, s and t are complementary projection operators defining distributions S and T in P corresponding to s and t, respectively. (f acts as an almost complex structure on S; however, since it is not an f-structure, it is not a null operator on T.)

The distribution S is r-dimensional and $\dim T=m-r$, $m=\dim P$. If there are $m-r$ vector fields E_a spanning the distribution T at each point of P, and $m-r$ linear differential forms η^a satisfying the relations (6. 1) and (6. 2) with $\nu=m-r$, then P is a globally framed manifold. From (6. 1) and (6. 2), one easily obtains

$$f^2E_a=0, \qquad \eta^a\circ f^2=0. \tag{6. 3}$$

Since $\operatorname{rank} f=r$, it is not difficult to show that a basis $\{E'_a\}$, $a=1, \cdots, m-r$, of T_p can be found, with dual basis $\{\eta'^a\}$, which satisfies (6. 1) and (6. 2) such that

$$fE_{2i-1}=0, \qquad fE_{2i}=E_{2i-1} \tag{6. 4}$$

$$\eta^{2i-1}\circ f=\eta^{2i}, \qquad \eta^{2i}\circ f=0, \qquad i=1, \cdots, \left[\frac{m-r}{2}\right], \tag{6. 5}$$

where we have dropped the primes. If we put

$$J=f-\eta^{2i-1}\otimes E_{2i} \tag{6. 6}$$

at p, we see that $\operatorname{rank} J=2n$ and $J^2=-I$ if $\dim P=2n$, and $\operatorname{rank} J=2n-2$ and $J^2=-I+\eta^{2n-r-1}\otimes E_{2n-r-1}$ if $\dim P=2n-1$. In fact, by (6. 4) and (6. 5), in the even dimensional case,

$$\begin{aligned} J^2X&=(f-\eta^{2i-1}\otimes E_{2i})(fX-\eta^{2j-1}(X)E_{2j})\\ &=f^2X-\eta^{2j-1}(X)E_{2j-1}-\eta^{2j}(X)E_{2j}\\ &=-X+\eta^a(X)E_a-\eta^{2j-1}(X)E_{2j-1}-\eta^{2j}(X)E_{2j}\\ &=-X, \end{aligned}$$

and in the odd dimensional case,

$$\begin{aligned} J^2X&=(f-\eta^{2i-1}\otimes E_{2i})(fX-\eta^{2j-1}(X)E_{2j})\\ &=f^2X-\eta^{2j-1}(X)E_{2j-1}-\eta^{2j}(X)E_{2j}\\ &=-X+\eta^{2n-r-1}(X)E_{2n-r-1}. \end{aligned}$$

THEOREM 9. *An even dimensional* (*respectively, odd dimensional*) *globally framed quartic manifold carries an almost complex* (*respectively, almost contact*) *structure.*

Although the globally framed quartic manifold $P(f, E_a, \eta^a)$, $a=1, \cdots, m-r$, $m-r\geqq 2$, is not a cubic structure, it does possess an underlying cubic structure (f_1, E_a, η^a) defined by

$$f_1=f-\eta^{2i}\otimes E_{2i-1}.$$

For, at p, by (6. 4) and (6. 5), $f_1^2X=-X+\eta^a(X)E_a$, and so $f_1^3X=-f_1X$. Moreover,

$f_1X=0$ implies $X=\eta^a(X)E_a$, so rank $f_1=r=\text{const.}\leqq 2n-2$. Conversely, a globally framed f-manifold $P(f, E_a, \eta^a)$ possesses an underlying globally framed quartic structure defined by $f_1=f+\eta^{2i}\otimes E_{2i-1}$. In fact, $f_1^2X=(f+\eta^{2i}\otimes E_{2i-1})(fX+\eta^{2j}(X)E_{2j-1})=-X+\eta^a(X)E_a$, $f_1^3X=-f_1X+\eta^{2i}(X)E_{2i-1}$ and $f_1^4X=-f_1^2X$. (Moreover, $f_1E_{2i-1}=0$, $f_1E_{2i}=E_{2i-1}$ and $\eta^{2i-1}\circ f_1=\eta^{2i}$, $\eta^{2i}\circ f_1=0$.)

THEOREM 10. *A globally framed quartic manifold of dimension m and rank r with $m-r\geqq 2$ possesses an underlying globally framed f-structure, and conversely.*

COROLLARY. *A noninvariant hypersurface of an almost contact manifold with the restricted quartic structure possesses a globally framed f-structure.*

Proof. Immediate from Theorem 1.

THEOREM 11. *Let $P(f, E_a, \eta^a)$ be a globally framed quartic manifold of dimension m and rank r with $m-r\geqq 2$. Then, the* (1, 1) *tensor field $f_1=f+\eta^{2i-1}\otimes E_{2i}$ gives rise to a quartic structure (f_1, E_a, η^a) of maximal rank which is not globally framed.*

Indeed, $f_1^4=I$. Moreover, rank $f_1=m$. For $f_1X=0$ implies $f_1^2X=-X+2\eta^a(X)E_a=0$, $a=1, \cdots, m-r$. Applying (6. 4), we get $\eta^a\otimes E_a=0$, so $X=0$. That (f_1, E_a, η^a) is not globally framed is a consequence of the relation $f_1^2=-I+2\eta^a\otimes E_a$.

The globally framed quartic structure (f, E_a, η^a) is said to be *normal* if the underlying globally framed f-structure (f_1, E_a, η^a), $f_1=f-\eta^{2i}\otimes E_{2i-1}$, is normal (cf. § 4). The condition for this is given by the vanishing of the tensor field S_{f_1} of type (1, 2) given by

$$S_{f_1}=[f_1, f_1]+d\eta^a\otimes E_a$$

(see [4]). In this case, the E_a are infinitesimal automorphisms of the structure (f_1, E_a, η^a) and the differentials $d\eta^a$ are of bidegree (1, 1) with respect to f_1 (see [4], Lemma 2). A calculation identical to that in § 4 shows that $P(f, E_a, \eta^a)$ is normal, if the tensor field S_f given by

$$S_f=[f, f]+d\eta^a\otimes E_a$$

is zero.

THEOREM 12. *The almost complex* (*respectively, almost contact*) *structure in Theorem* 9 *is integrable* (*respectively, normal*) *if the quartic structure is normal.*

Proof. Similar to that of Theorem 6 if dim P is even. The computation in the odd dimensional case is analogous to the former case, that is, the torsion is evaluated in terms of the structure tensors of the almost contact manifold.

Theorem 7 lends itself to the following generalization.

THEOREM 13. *Let $P(f, E_a, \eta^a)$ be an even dimensional globally framed quartic manifold of rank r, $a=1, \cdots, m-r$, whose induced almost complex structure $J=f-\eta^{2i-1}\otimes E_{2i}$, $i=1, \cdots, [(m-r)/2]$, is integrable. Then, if the $d\eta^{2i-1}$ are of bidegree* (1, 1) *with respect to f and the vector fields E_{2i} are holomorphic, the quartic structure is normal.*

We also have the following odd dimensional analogue.

THEOREM 14. *Let $P(f, E_a, \eta^a)$ be an odd dimensional globally framed quartic manifold whose induced almost contact structure $J=f-\eta^{2i-1}\otimes E_{2i}$ is normal. Then, if the $d\eta^{2i-1}$ are of bidegree* (1, 1) *with respect to f and the $L_{E_{2i}}J$ vanish, $i=1, \cdots, [(m-r)/2]$, the quartic structure is normal.*

7. **Quartic metric manifolds.** The manifold $P(f, E_a, \eta^a)$, $a=1, \cdots, m-r$, is called a *globally framed quartic metric manifold* if P carries a Riemannian metric g such that (i) $\eta^a=g(E_a, \cdot)$, $a=1, \cdots, m-r$, and (ii) f is skew symmetric with respect to g. In this case, we denote the structure by $P(f, \eta^a, g)$. Unlike f-manifolds, a globally framed manifold does not, in general, carry a metric with these properties. To see this, consider the hypersurface $P(f, U, \alpha, V, \beta)$ of the almost contact manifold $M(\phi, \eta, G)$ with the restricted quartic structure. Since $G(\phi x, \phi y) =G(x, y)-\eta(x)\eta(y)$ and $\eta(N)$ is zero, $G(N, i_*Y)=G(\phi N, \phi i_*Y)=-G(i_*U, i_*fY+\alpha(Y)N)=-g(U, fY)+\alpha(Y)G(\phi N, N)=-g(U, fY)$ where $g=i^*G$. If f is skew symmetric with respect to g, then $g(U, fY)=-g(fU, Y)=0$ by (2. 7). But this is impossible since N is not the metric normal with respect to G.

We put $F(X, Y)=g(fX, Y)$ and call it the *fundamental* 2-*form* of $P(f, \eta^a, g)$.

Let P be a globally framed metric manifold of dimension $m=2n$ with quartic structure tensor f. Then, by Theorem 9, an almost complex structure $J=f-\eta^{2i-1}\otimes E_{2i}$ is defined on P in terms of which the metric $\tilde{g}$ is hermitian where $\tilde{g}(X, Y) =(1/2)[g(X, Y)+g(JX, JY)]$. Setting $\Omega(X, Y)=\tilde{g}(JX, Y)$, we obtain

$$\Omega=F+\sum_i \eta^{2i}\wedge\eta^{2i-1}.$$

If the fundamental form F and the η^a, $a=1, \cdots, 2n-r$, are closed forms, the almost hermitian structure on P is almost Kaehlerian. It is Kaehlerian, if and only if J has vanishing covariant derivative with respect to $\tilde{g}$. Thus, $P(f, \eta^a, g)$ has an underlying Kaehlerian structure if its structure tensors are covariant constant (with respect to g).

THEOREM 15. *An even dimensional globally framed metric manifold $P(f, \eta^a, g)$ with a quartic structure f carries a Kaehler structure $(J, \tilde{g})$, where $J=f-\eta^{2i-1}\otimes E_{2i}$ and $\tilde{g}(X, Y)=(1/2)[g(X, Y)+g(JX, JY)]$, if f and the η^a, $a=1, \cdots, 2n-r$, are parallel fields with respect to g.*

In the odd dimensional case the globally framed metric manifold $P(f, \eta^a, g)$ gives rise to the almost contact metric manifold $P(J, \eta^{2n-r-1}, \tilde{g})$. For, $\tilde{g}(JX, JY) =g(X, Y)-\eta^{2n-r-1}(X)\eta^{2n-r-1}(Y)$. If $\Phi(X, Y)=g(JX, Y)$, then

$$\Phi = F + \sum_i \eta^{2i} \wedge \eta^{2i-1}.$$

If the fundamental 2-form Φ and the 1-form η^{2n-r-1} are closed, the almost contact structure on P is almost cosymplectic [2]. It is cosymplectic, if and only if, the almost contact structure is normal.

THEOREM 16. *An odd dimensional globally framed metric manifold* $P(f, \eta^a, g)$ *with a quartic structure* f *carries a cosymplectic structure* $(J, \eta^{2n-r-1}, \tilde{g})$, *where* $J = f - \eta^{2i-1} \otimes E_{2i}$ *and* $\tilde{g}(X, Y) = (1/2)[g(X, Y) + g(JX, JY)]$, *if* f *and the* η^a, $a = 1, \cdots, 2n-r-1$, *are parallel fields.*

Proof. Since f and the η^a have vanishing covariant derivatives with respect to the Riemannian connection of g, so does J. Hence, the torsion $(\tilde{D}_{JX}J)Y - (\tilde{D}_{JY}J)X + J(\tilde{D}_Y J)X - J(\tilde{D}_X J)Y + \{(\tilde{D}_X \eta^{2n-r-1})(Y) - (\tilde{D}_Y \eta^{2n-r-1})(X)\}E_{2n-r-1} = 0$, where $\tilde{D}$ denotes covariant differentiation with respect to the Riemannian connection of $\tilde{g}$. Thus, $P(J, \eta^{2n-r-1}, \tilde{g})$ is normal.

THEOREM 17. *Let* P *be a complete simply connected globally framed quartic metric manifold. Then, if its structure tensors are parallel fields, it is a product manifold with one of its factors Kaehlerian.*

Proof. If $\dim P$ is even this is immediate from Theorem 15. If P is odd dimensional, this is a consequence of Theorem 16. For, since $D\Phi = 0$, $P'_p = \{X \in P_p | \Phi(X, P_p) = 0\}$ defines a parallel distribution. Thus, the orthogonal complement P''_p (with respect to g) also gives a parallel distribution. Note that the $E_a(p)$, $a = 1, \cdots, m-r$, do not belong to P''_p. By the de Rham decomposition theorem $P = P' \times P''$, where $\Phi = 0$ on P' and Φ has maximal rank on P''. Since f and the η^a, $a = 1, \cdots, m-r$, are parallel fields, $[J, J]$ must vanish. Hence, the almost complex structure on P'', obtained by restricting J to P'', is integrable. Since Φ is closed, P'' is symplectic; in fact, since $D\Phi = 0$, P'' is a Kaehler manifold.

8. **Automorphisms.** Let $M(f, E_a, \eta^a)$ and $M'(f', E'_a, \eta'^a)$ be globally framed quartic manifolds whose structures have the same rank. A diffeomorphism μ of M onto M' is called an *isomorphism* of M onto M' if

$$\mu_* \circ f = f' \circ \mu_*$$

and

$$\mu_* E_a = E'_a,$$

where μ_* denotes the induced map on tangent spaces. If $M' = M$ and $f' = f$, $E'_a = E_a$, $\eta'^a = \eta^a$, $a = 1, \cdots, m-r$, then μ is said to be an *automorphism* of M. The set of all automorphisms of M clearly forms a group which we denote by $A(f, E_a, \eta^a)$.

LEMMA 4. *Let* $\mu \in A(f, E_a, \eta^a)$. *Then,*

$$\mu^* \eta^a = \eta^a,$$

where μ^* is the induced map on forms.

LEMMA 5. *Let $\mu \in A(f, E_a, \eta^a)$. Then, $\mu \in A(f_1, E_a, \eta^a)$, that is*

$$\mu_* \circ f_1 = f_1 \circ \mu_*,$$

where $f_1 = f - \eta^{2i} \otimes E_{2i-1}$. Conversely, if $\mu \in A(f_1, E_a, \eta^a)$, then $\mu \in A(f, E_a, \eta^a)$.

Thus, an automorphism of the globally framed quartic structure (f, E_a, η^a) on M is also an automorphism of the induced globally framed f-structure (f_1, E_a, η^a) on M, and conversely. Since $A(f_1, E_a, \eta^a)$ is a Lie group (see [4], [5]) and $M(f, E_a, \eta^a)$ is normal if and only if $M(f_1, E_a, \eta^a)$ is normal, we have

THEOREM 18. *The group of automorphisms of a compact normal globally framed quartic structure is a Lie group.*

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