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INDUCED STRUCTURES ON SUBMANIFOLDS

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It is well known that hypersurfaces of almost complex manifolds carry almost contact structures [7]. On the other hand submanifolds of codimension 2 of almost complex manifolds and hypersurfaces of almost contact manifolds are not in general almost complex. Previously these submanifolds have been studied from the standpoint of conditions under which they possess an induced almost complex structure or an f-structure [9], [11]. In this paper we obtain a more general structure on these spaces.

In particular, while the odd-dimensional spheres carry a normal contact structure, the even-dimensional spheres are not almost complex except in dimensions 2 and 6; however we show that the even-dimensional spheres do carry the more general structure induced here. Thus even-dimensional spheres can now be studied from a differential geometric point of view.

1. Submanifolds of codimension 2 of almost complex manifolds. Let M^{2n+2} be a $(2n+2)$-dimensional almost Hermitian manifold, that is, M^{2n+2} carries a tensor field J of type $(1,1)$ such that

$$J^2=-I$$

and a metric G satisfying

$$G(JX, JY)=G(X, Y).$$

Suppose that N^{2n} is a C^∞ submanifold with unit normals C and D and induced metric g. Thus, if B denotes the differential of the imbedding and X and Y tangent vector fields on N^{2n}, then

$$G(BX, BY)=g(X, Y),$$
$$G(C, C)=1, \qquad G(D, D)=1, \qquad G(C, D)=0,$$
$$G(BX, C)=0, \qquad G(BX, D)=0.$$

It is easy to see ([1], [6], [8]) that we can define a tensor field f of type $(1,1)$, vector fields E and A, 1-forms η and α, and a function λ on N^{2n} by

$$\left.\begin{aligned} JBX&=BfX+\eta(X)C+\alpha(X)D,\\ JC&=-BE+\lambda D,\\ JD&=-BA-\lambda C.\end{aligned}\right\} \tag{1}$$

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LEMMA 1. 1. *$f, E, A, \eta, \alpha, \lambda$ satisfy*

$$\left.\begin{array}{ll} f^2=-I+\eta\otimes E+\alpha\otimes A, & \\ \eta\circ f=\lambda\alpha, & \alpha\circ f=-\lambda\eta, \\ fE=-\lambda A, & fA=\lambda E, \\ \eta(E)=1-\lambda^2, & \alpha(E)=0, \\ \eta(A)=0, & \alpha(A)=1-\lambda^2. \end{array}\right\} \qquad (2)$$

Proof. Computing J^2BX we have

$$-BX=Bf^2X+\eta(fX)C+\alpha(fX)D-\eta(X)BE+\lambda\eta(X)D-\alpha(X)BA-\lambda\alpha(X)C.$$

Comparing tangential and normal parts we obtain the first three results. Similarly computing J^2C and J^2D, we have

$$-C=-BfE-\eta(E)C-\alpha(E)D-\lambda BA-\lambda^2C,$$
$$-D=-BfA-\eta(A)C-\alpha(A)D+\lambda BE-\lambda^2D,$$

which yield the remaining identities.

Applying Lemma 1. 1 we immediately obtain

LEMMA 1. 2. $\qquad f^3+f=\lambda(\alpha\otimes E-\eta\otimes A).$

In general an *f-structure* of rank $2n$ on a C^∞ manifold M^{2n+s} is a tensor field f of type $(1,1)$ and of constant rank $2n$ such that $f^3+f=0$ [11]. If there exist on M^{2n+s} vector fields $E_1, \cdots, E_s$ such that if $\eta^1, \cdots, \eta^s$ are dual 1-forms, then

$$\eta^x(E_y)=\delta^x_y \qquad (x, y=1, 2, \cdots, s),$$
$$fE_x=0, \qquad \eta^x\circ f=0,$$
$$f^2=-I+\eta^x\otimes E_x,$$

we say that the f-structure has *complemented frames.*

In our case it is clear from the above lemmas that if λ is identically zero, then f is an f-structure with complemented frames.

On the other hand if λ is identically $+1$ or -1 we see from (2) that $E=A=0$, and from (1) $\lambda=G(JC, D)$, $JC=\pm D$ and $JD=\mp C$. Hence by Lemma 1. 1, f is an almost complex structure on N^{2n}.

Conversely if f is an f-structure then by Lemmas 1. 1 and 1. 2 we have $0=(f^3+f)A=\lambda(1-\lambda^2)E$; but from $\eta(E)=1-\lambda^2$ and the above we see that $E=0$ if and only if $\lambda=\pm 1$. Thus we have

THEOREM 1. 3. *The tensor f defines an f-structure if and only if λ is identically* 0, $+1$ *or* -1. *Moreover in the $\lambda=0$ case f is an f-structure of rank $2n-2$ with complemented frames and in the $\lambda=\pm 1$ case f is an almost complex structure.*

In contrast with the above we have the following results.

THEOREM 1.4. *If λ never vanishes, then f is non-singular.*

Proof. Suppose $fX=0$, then $JBX=\eta(X)C+\alpha(X)D$ and hence

$$-BX=J^2BX=\eta(X)(-BE+\lambda D)+\alpha(X)(-BA-\lambda C)$$

$$=-B(\eta(X)E+\alpha(X)A)+\lambda\eta(X)D-\lambda\alpha(X)C,$$

which yields $\eta(X)=0$ and $\alpha(X)=0$. Therefore since X is tangent to N^{2n} and $BX=0$, then $X=0$.

THEOREM 1.5. *If λ is never equal to ± 1, then N^{2n} carries an almost complex structure.*

Proof. Since λ is never equal to ± 1, we see from Lemma 1.1 that E and A are non-zero vector fields on N^{2n}. Now $G(JC, JD)=G(C, D)=0$ so that $0=G(-BE+\lambda D, -BA-\lambda C)=G(BE, BA)=g(E, A)$. Thus BE, BA, C, D span a 4-dimensional invariant subspace of J, and hence its orthogonal complement P is also invariant under J. But P, BE, BA span the tangent spaces of N^{2n}, so for $p\in N^{2n}$ we have

$$N_p^{2n}\cap JN_p^{2n}=P_p.$$

Consequently by a result of one of the authors [11], N^{2n} has an f-structure, say $\bar{f}$, of rank $2n-2$ with complemented frames E and A. Thus if we set $\tilde{f}=\bar{f}+\eta\otimes A-\alpha\otimes E$, then $\tilde{f}^2=-I$.

We now list some properties of the induced metric g.

LEMMA 1.6. *The induced metric g on N^{2n} satisfies*

$$g(X, Y)=g(fX, fY)+\eta(X)\eta(Y)+\alpha(X)\alpha(Y),$$

$$g(X, E)=\eta(X), \qquad g(X, A)=\alpha(X),$$

$$g(E, E)=g(A, A)=1-\lambda^2, \qquad g(E, A)=0,$$

$$g(X, fY)=-g(fX, Y).$$

Proof.

$$g(X, Y)=G(BX, BY)=G(JBX, JBY)$$

$$=G(BfX+\eta(X)C+\alpha(X)D, BfY+\eta(Y)C+\alpha(Y)D)$$

$$=g(fX, fY)+\eta(X)\eta(Y)+\alpha(X)\alpha(Y);$$

$$g(X, fY)=G(BX, BfY)=G(BX, JBY)=-G(JBX, BY)$$

$$=-G(BfX, BY)=-g(fX, Y);$$

similarly

$$\eta(X)=G(JBX, C)=-G(BX, JC)$$
$$=G(BX, BE)=g(X, E)$$

and

$$\alpha(X)=G(JBC, D)=-G(BX, JD)=G(BX, BA)=g(X, A)$$

yield the remaining results.

Now let us apply the Gauss-Weingarten equations

$$(\nabla_{BX}B)Y=h(X, Y)C+k(X, Y)D,$$
$$\nabla_{BX}C=-BHX+l(X)D, \qquad \nabla_{BX}D=-BKX-l(X)C,$$

where h and k are the second fundamental forms, H and K are the corresponding Weingarten maps, l is the third fundamental form, and ∇ denotes covariant differentiation. Moreover, we now assume that the ambient space is Kaehlerian, i.e. $\nabla J=0$. Thus we have

$$\begin{aligned}\nabla_{BX}JBY&=(\nabla_{BX}J)BY+J(\nabla_{BX}B)Y+JB\nabla_X Y\\&=J(h(X, Y)C+k(X, Y)D)+JB\nabla_X Y\\&=-h(X, Y)BE+h(X, Y)\lambda D-k(X, Y)BA-k(X, Y)\lambda C+JB\nabla_X Y.\end{aligned}$$

On the other hand

$$\begin{aligned}\nabla_{BX}JBY&=\nabla_{BX}(BfY+\eta(Y)C+\alpha(Y)D)\\&=h(X,fY)C+k(X,fY)D+B(\nabla_X f)Y+Bf\nabla_X Y\\&\quad+(\nabla_X\eta)(Y)C+\eta(\nabla_X Y)C+\eta(Y)(-BHX+l(X)D)\\&\quad+(\nabla_X\alpha)(Y)D+\alpha(\nabla_X Y)D+\alpha(Y)(-BKX-l(X)C).\end{aligned}$$

Therefore, using (1) and comparing tangential and normal parts we have

$$\left.\begin{aligned}&-h(X, Y)E-k(X, Y)A=(\nabla_X f)Y-\eta(Y)HX-\alpha(Y)KX,\\&\lambda h(X, Y)=k(X,fY)+(\nabla_X\alpha)(Y)+l(X)\eta(Y),\\&-\lambda k(X, Y)=h(X,fY)+(\nabla_X\eta)(Y)-l(X)\alpha(Y).\end{aligned}\right\} \tag{3}$$

The first of these equations gives us an expression for the covariant derivative of f; clearly if N^{2n} is totally geodesic then f is covariant constant. More generally we prove

THEOREM 1.7. *Let N^{2n} be a submanifold of a Kaehler manifold M^{2n+2}. Then if the induced structure $(f, E, A, \eta, \alpha, \lambda)$ has $\lambda\neq\pm1$, f is covariant constant if and only if h and k have the following form*

$$h=\sigma_1\eta\otimes\eta+\sigma_2(\alpha\otimes\eta+\eta\otimes\alpha)+\sigma_3\alpha\otimes\alpha,$$

$$k=\sigma_2\eta\otimes\eta+\sigma_3(\alpha\otimes\eta+\eta\otimes\alpha)+\sigma_4\alpha\otimes\alpha,$$

where

$$(1-\lambda^2)^2\sigma_1=h(E,E),\qquad (1-\lambda^2)^2\sigma_2=h(E,A)=k(E,E),$$

$$(1-\lambda^2)^2\sigma_3=h(A,A)=k(A,E),\qquad (1-\lambda^2)^2\sigma_4=k(A,A).$$

Proof. If f is covariant constant, then the first of equations (3) can be written in the form

$$h(X,Y)\eta(Z)+k(X,Y)\alpha(Z)=h(X,Z)\eta(Y)+k(X,Z)\alpha(Y).$$

Setting $Z=E$ yields

$$(1-\lambda^2)h(X,Y)=h(X,E)\eta(Y)+k(X,E)\alpha(Y);$$

from which, setting $X=E$, $Y=A$ and $X=A$, $Y=A$ we obtain $h(E,A)=k(E,E)$ and $h(A,A)=k(A,E)$ respectively. On the other hand, setting $X=Y=E$ and $X=E$, $Y=A$ respectively gives

$$h(E,E)\eta(Z)+k(E,E)\alpha(Z)=(1-\lambda^2)h(E,Z),$$

$$h(E,A)\eta(Z)+k(E,A)\alpha(Z)=(1-\lambda^2)k(E,Z).$$

Therefore

$$(1-\lambda^2)^2h(X,Y)=h(E,E)\eta(X)\eta(Y)+k(E,E)\alpha(X)\eta(Y)$$
$$+h(E,A)\eta(X)\alpha(Y)+k(E,A)\alpha(X)\alpha(Y)$$

and $k(E,E)=h(E,A)$ giving the first of the desired formulas. Similarly, setting $Z=A$ we have

$$(1-\lambda^2)k(X,Y)=h(X,A)\eta(Y)+k(X,A)\alpha(Y)$$

and setting $X=Y=A$ and $X=A$, $Y=E$ respectively gives

$$h(A,A)\eta(Z)+k(A,A)\alpha(Z)=(1-\lambda^2)k(A,Z),$$

$$h(A,E)\eta(Z)+k(A,E)\alpha(Z)=(1-\lambda^2)h(A,Z).$$

Therefore

$$(1-\lambda^2)^2k(X,Y)=h(A,E)\eta(X)\eta(Y)+k(A,E)\alpha(X)\eta(Y)$$
$$+h(A,A)\eta(X)\alpha(Y)+k(A,A)\alpha(X)\alpha(Y)$$

and $k(A,E)=h(A,A)$ giving the second of the desired formulas.

Conversely if h and k are of the desired form, a direct substitution into the first of equations (3) yields $\nabla f=0$.

In the case λ is identically $+1$ or -1, $E=A=0$ and $f^2=-I$ as we have seen. Hence from Lemma 1.6 $\eta=\alpha=0$ and $g(fX,fY)=g(X,Y)$. Consequently the first of equations (3) yields

THEOREM 1.8. *Let N^{2n} be a submanifold of a Kaehler manifold M^{2n+2}. Then if the induced structure has λ identically $+1$ or -1, N^{2n} is Kaehlerian.*

We next use the integrability condition of the almost complex structure J to define the corresponding notion of normality of the induced structure on N^{2n}. Letting $[J,J]$ and $[f,f]$ denote the Nijenhuis torsion of J and f respectively, a lengthy computation gives

$$\begin{aligned}
[J,J](BX,BY)=B\{&[f,f](X,Y)+d\eta(X,Y)E+d\alpha(X,Y)A\\
&+(\alpha(X)l(Y)-\alpha(Y)l(X))E-(\eta(X)l(Y)-\eta(Y)l(X))A\\
&+\eta(Y)(Hf-fH)X-\eta(X)(Hf-fH)Y+\alpha(Y)(Kf-fK)X\\
&-\alpha(X)(Kf-fK)Y\}+\{(\nabla_{fX}\eta)(Y)-(\nabla_{fY}\eta)(X)-\eta(\nabla_X fY-\nabla_Y fX)\\
&+\eta(HX)\eta(Y)-\eta(HY)\eta(X)+\eta(KX)\alpha(Y)-\eta(KY)\alpha(X)\\
&-l(fX)\alpha(Y)+l(fY)\alpha(X)-\lambda(\eta(X)l(Y)-\eta(Y)l(X))\\
&+\lambda d\alpha(X,Y)\}C+\{(\nabla_{fX}\alpha)(Y)-(\nabla_{fY}\alpha)(X)-\alpha(\nabla_X fY-\nabla_Y fX)\\
&+\alpha(HX)\eta(Y)-\alpha(HY)\eta(X)+\alpha(KX)\alpha(Y)-\alpha(KY)\alpha(X)\\
&+l(fX)\eta(Y)-l(fY)\eta(X)-\lambda(\eta(X)l(Y)-\eta(Y)l(X))\\
&-\lambda d\eta(X,Y)\}D.
\end{aligned}$$

DEFINITION. The structure $(f,E,A,\eta,\alpha,\lambda)$ is said to be *normal* if f commutes with the Weingarten maps H and K and

$$[f,f]+d\eta\otimes E+d\alpha\otimes A+(\alpha\wedge l)\otimes E-(\eta\wedge l)\otimes A=0. \tag{4}$$

REMARK. We could, of course, discuss the geometry of a manifold N^{2n} with an intrinsically defined structure of the type introduced above. We simply say that a C^∞ manifold N^{2n} has an *$(f,E,A,\eta,\alpha,\lambda)$-structure* if there exist on N^{2n} tensors $f,E,A,\eta,\alpha,\lambda$ satisfying the relations of Lemma 1.1. In this case normality is defined by the condition

$$[f,f]+d\eta\otimes E+d\alpha\otimes A=0. \tag{5}$$

We see that in the case λ is identically zero, the f-structure with complemented frames (f,E,A,η,α) gives rise to an almost complex structure J on $N^{2n}\times R^2$ defined by

$$(J)=\begin{pmatrix} f & \eta & \alpha \\ -E & 0 & 0 \\ -A & 0 & 0 \end{pmatrix}.$$

J is integrable if and only if $[f,f]+d\eta\otimes E+d\alpha\otimes A=0$ [5].

In general N^{2n} may be considered as a totally geodesic submanifold of $N^{2n}\times R^2$ such that $l=0$. Thus, in this case, equation (4) reduces to (5).

2. Hypersurfaces of almost contact spaces. In this section we show that a hypersurface N^{2n} of an almost contact manifold M^{2n+1} also has a naturally induced $(f, E, A, \eta, \alpha, \lambda)$-structure. In [3] two of the authors studied conditions under which such a hypersurface carries an almost complex structure or an f-structure of rank $2n-2$.

A C^∞ manifold M^{2n+1} is said to have an *almost contact structure* if there exist on M^{2n+1} a tensor field φ, a vector field ξ, and a 1-form η' such that

$$\eta'(\xi)=1, \qquad \varphi\xi=0,$$

$$\eta'\circ\varphi=0, \qquad \varphi^2=-I+\eta'\otimes\xi;$$

this is equivalent to a reduction of the structural group of the tangent bundle to $U(n)\times 1$. If moreover M^{2n+1} carries a Riemannian metric G satisfying

$$G(\xi, X)=\eta'(X), \qquad G(\varphi X, \varphi Y)=G(X, Y)-\eta'(X)\eta'(Y),$$

we say M^{2n+1} has an *almost contact metric structure.*

Now let N^{2n} be a hypersurface with unit normal C. Let B denote the differential of the imbedding and g the induced metric. Define a tensor field f of type (1, 1), vector fields E, A, 1-forms η, α and a function λ by

$$\varphi BX=BfX+\eta(X)C, \qquad \xi=BA+\lambda C,$$

$$\varphi C=-BE, \qquad \alpha(X)=\eta'(BX).$$

Note that $\lambda=G(\xi, C)=\eta'(C)$. Moreover we have

PROPOSITION 2. 1. *$f, E, A, \eta, \alpha, \lambda$ as defined here, satisfy the relations of Lemma* 1. 1.

Proof. $\varphi^2BX=Bf^2X+\eta(fX)C-\eta(X)BE$, and $\varphi^2BX=-BX+\eta'(BX)\xi=-BX+\alpha(X)BA+\alpha(X)\lambda C$. Comparing we have $f^2=-I+\eta\otimes E+\alpha\otimes A$ and $\eta\circ f=\lambda\alpha$. $\alpha(fX)=\eta'(BfX)=\eta'(\varphi BX)-\eta'(C)\eta(X)=-\lambda\eta(X)$, that is $\alpha\circ f=-\lambda\eta$. $\varphi BA=BfA+\eta(A)C$ and $\varphi BA=\varphi\xi-\lambda\varphi C=\lambda BE$, hence $fA=\lambda E$ and $\eta(A)=0$. $\lambda\alpha(fX)=\eta(f^2X)=\eta(-X+\eta(X)E+\alpha(X)A)=-\eta(X)+\eta(X)\eta(E)$ and $\lambda\alpha(fX)=-\lambda^2\eta(X)$ so that $\eta(E)=1-\lambda^2$. Similarly $\alpha(A)=\eta'(BA)=\eta'(\xi)-\lambda\eta'(C)=1-\lambda^2$ and $\alpha(E)=\eta'(BE)=-\eta'(\varphi C)=0$. Finally $\varphi BE=BfE+(1-\lambda^2)C$ and $\varphi BE=-\varphi^2C=C-\eta'(C)\xi=C-\lambda BA-\lambda^2C$ hence $fE=-\lambda A$.

Calculations similar to those of Lemma 1.6 yield again

$$\eta(X)=g(X, E), \qquad \alpha(X)=g(X, A),$$

$$g(fX, fY)=g(X, Y)-\eta(X)\eta(Y)-\alpha(X)\alpha(Y),$$

$$g(X, fY)=-g(fX, Y).$$

We will now examine some special cases for the hypersurfaces. First of all we note that if A is singular at a point $p\in N^{2n}$ then so is E and conversely. Similarly if A is non-singular at $p\in N^{2n}$ then A and E are linearly independent at p. Thus we have the following result.

LEMMA 2.2. *If $p\in N^{2n}$ and $\lambda(p)\neq 0$, then f is non-singular at p.*

Proof. $f^2E=-\lambda^2E$, $f^2A=-\lambda^2A$, and $f^2X=-X$ if $\eta(X)=\alpha(X)=0$.

We can see from Proposition 2.1 that if E is the zero vector field then λ is identically $+1$ or -1 and f is an almost complex structure on N^{2n} [3]. On the other hand, if λ is identically zero then f defines an f-structure with complemented frames on N^{2n} and hence if we define $\tilde{f}$ by $\tilde{f}=f+\eta\otimes A-\alpha\otimes E$, $\tilde{f}$ is an almost complex structure [3, 4].

PROPOSITION 2.3. a) *If λ is never* 0, *then $\tilde{f}$ defined by $\tilde{f}=f-(1/\lambda)\eta\otimes A$, is an almost complex structure on N^{2n}.* b) *If λ is never* 1, *then $\tilde{f}$ defined by $\tilde{f}=f+(1/(\lambda-1))(\eta\otimes A-\alpha\otimes E)$, is an almost complex structure on N^{2n}.*

Proof. The proof of this proposition is merely computing $\tilde{f}^2$ while making use of the identities of Proposition 2.1.

Proposition 2.3 indicates that the interest in $(f, E, A, \eta, \alpha, \lambda)$-structures lies in the case where λ assumes both of the values 0 and 1 (note that E and A are zero vectors at points where $\lambda=1$) and hence all values between 0 and 1. This must be the case for the even-dimensional sphere which we discuss in the next section.

Applying the Gauss-Weingarten equations

$$(\tilde{\nabla}_{BX}B)Y=h(X, Y)C, \qquad \tilde{\nabla}_{BX}C=-BHX,$$

we have

$$\begin{aligned}\tilde{\nabla}_{BX}(\varphi BY)&=(\tilde{\nabla}_{BX}\varphi)BY+\varphi(\tilde{\nabla}_{BX}B)Y+\varphi B\nabla_X Y\\ &=(\tilde{\nabla}_{BX}\varphi)BY-h(X, Y)BE+\varphi B\nabla_X Y\end{aligned}$$

and

$$\begin{aligned}\tilde{\nabla}_{BX}(\varphi BY)&=\tilde{\nabla}_{BX}(BfY+\eta(Y)C)\\ &=h(X, fY)C+B(\nabla_X f)Y+Bf\nabla_X Y+(\nabla_X\eta)(Y)C+\eta(\nabla_X Y)C-\eta(Y)BHX,\end{aligned}$$

hence

$$(\nabla_{BX}\varphi)BY-h(X, Y)BE=h(X,fY)C+B(\nabla_X f)Y+(\nabla_X\eta)(Y)C-\eta(Y)BHX.$$

In the case that M^{2n+1} is cosymplectic, that is φ and η' are covariant constant with respect to the Riemannian connection of G (cf. [2]), we have

$$\left.\begin{aligned} -h(X, Y)E&=(\nabla_X f)Y-\eta(Y)HX, \\ 0&=h(X,fY)+(\nabla_X\eta)(Y). \end{aligned}\right\} \tag{6}$$

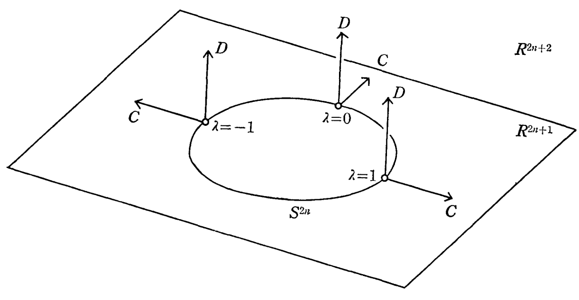
Similar arguments to those giving Theorems 1.7 and 1.8 yield

THEOREM 2.4. *Let N^{2n} be a hypersurface of a cosymplectic manifold M^{2n+1}. Then if the induced structure $(f, E, A, \eta, \alpha, \lambda)$ has $\lambda\neq\pm1$, f is covariant constant if and only if $h=\sigma\eta\otimes\eta$ where $\sigma=h(E,E)/(1-\lambda^2)^2$. On the other hand if λ is identically* $+1$ or -1, *N^{2n} is Kaehlerian.*

3. The even-dimensional sphere. In this section we show that the even-dimensional spheres are non-trivial examples of manifolds with the structure that we have been studying.

Let S^{2n} denote the unit sphere in R^{2n+1} considered as a cosymplectic manifold. Let x be the position vector in R^{2n+1} determining S^{2n}, then $x\cdot x=1$ and $x\cdot x_i=0$, $x_i=\partial_i x$, $i=1,\cdots,2n$. The metric tensor g of S^{2n} being given by $x_i\cdot x_j$. Now the mean curvature vector or outward normal C may be identified with x. Hence we have $0=\nabla_j(x_i\cdot x)=(h_{ji}C)\cdot C+x_i\cdot x_j$ and therefore $h=-g$.

We can also consider S^{2n} in R^{2n+2} regarded as a Kaehler manifold. Again let C be the outer normal to S^{2n} in R^{2n+1} and D the normal to R^{2n+1} in R^{2n+2}. Then $0=\nabla_j(x_i\cdot x)=(h_{ji}C+k_{ji}D)\cdot C+x_i\cdot x_j$ and $0=\nabla_j D=-K_j^i x_i-l_j C$ hence $h=-g$, $k=0$ and $l=0$. As we have seen in section 1, the induced structure on S^{2n} has $\lambda=G(JC,D)$, that is, λ is the cosine of the angle between JC and D. The diagram below gives an interpretation of λ for the sphere example.



In either case we have from equations (3) or (6)

$$(\nabla_X\eta)(Y)=-h(X,fY)=g(X,fY)$$

and from equations (3) or by differentiating $\alpha(X)=\eta'(BX)$ and using $\nabla\eta'=0$ and the Gauss equation we have

$$(\nabla_X\alpha)(Y)=\lambda h(X, Y)=-\lambda g(X, Y).$$

LEMMA 3.1. *The* 1-*form* η *is killing and* $d\eta(X, Y)=2g(X, fY)$; *the* 1-*form* α *is closed.*

Proof. $(\nabla_X\eta)(Y)+(\nabla_Y\eta)(X)=g(X, fY)+g(Y, fX)=0$; further $d\eta(X, Y)=(\nabla_X\eta)(Y)-(\nabla_Y\eta)(X)=2g(X, fY)$. Similarly $d\alpha(X, Y)=(\nabla_X\alpha)(Y)-(\nabla_Y\alpha)(X)=-\lambda g(X, Y)+\lambda g(Y, X)=0$.

While α is not killing we do have the following result for the vector field A which is the contravariant form of α.

PROPOSITION 3.2. *The vector field* A *is a conformal infinitesimal transformation which is not an isometry for* λ *not identically zero.*

Proof. Let $\mathcal{L}$ denote Lie differentiation, then

$$\begin{aligned}(\mathcal{L}_A g)(X, Y)&=g(\nabla_X A, Y)+g(\nabla_Y A, X)\\ &=(\nabla_X\alpha)(Y)+(\nabla_Y\alpha)(X)\\ &=-2\lambda g(X, Y).\end{aligned}$$

REMARK. Proposition 3.2 is both natural and interesting in view of the conjecture of one of the authors [9] that a compact Riemannian manifold of constant scalar curvature admitting a non-killing conformal vector field is isometric to a Euclidean sphere. We note more generally than Proposition 3.2 that if M^{2n+1} has an almost contact metric structure (φ, ξ, η', G) with ξ killing (e.g. quasi-Sasakian [2] and if N^{2n} is a totally umbilical hypersurface, then the conclusion of Proposition 3.2 holds. For

$$\begin{aligned}(\nabla_X\alpha)(Y)&=\nabla_{BX}\eta'(BY)-\eta'(B\nabla_X Y)\\ &=(\nabla_{BX}\eta')(BY)+\eta'(h(X, Y)C)\\ &=(\nabla_{BX}\eta')(BY)+\lambda h(X, Y).\end{aligned}$$

If now $h=\mu g$, $\mu\neq 0$, then

$$(\mathcal{L}_A g)(X, Y)=(\nabla_X\alpha)(Y)+(\nabla_Y\alpha)(X)=2\lambda\mu g(X, Y)$$

using the fact that η' is killing. Note also that if $h=\mu g$, then $\mu F=d\eta$ where F is the covariant form of f.

THEOREM 3.3. *The* $(f, E, A, \eta, \alpha, \lambda)$-*structure on* S^{2n} *is normal.*

Proof. Note that from (6) $(\nabla_X f)Y=-\eta(Y)X+g(X, Y)E$ since $\nabla\varphi=0$ and $H=-I$. Then a direct computation using Lemma 3.1 shows that $[f, f]+d\eta\otimes E+d\alpha\otimes A=0$.

Finally note that $g(X, \nabla_Y A)=(\nabla_Y\alpha)(X)=-\lambda g(X, Y)$ and hence $\nabla_Y A=-\lambda Y$. Letting $R(X, Y)$ denote the curvature transformation we have

$$\begin{aligned} R(A, Y)A &= \nabla_{[A, Y]}A + \nabla_Y \nabla_A A - \nabla_A \nabla_Y A \\ &= -\lambda[A, Y] + \nabla_Y(-\lambda A) - \nabla_A(-\lambda Y) \\ &= -(Y\lambda)A + (A\lambda)Y. \end{aligned}$$

Now letting Y be a unit vector orthogonal to A we have for the sectional curvature $K(Y, A) = A\lambda/(1-\lambda^2)$, but S^{2n} has constant curvature 1, hence we have

PROPOSITION 3. 4. *The function λ on S^{2n} satisfies the differential equation*

$$\frac{A\lambda}{1-\lambda^2} = 1.$$

REMARK. In the case of a totally umbilical hypersurface, $h = \mu g$, of a cosymplectic manifold we have by making similar computations to the above that $K(A, Y) = -A(\mu\lambda)/(1-\lambda^2)$. By computing $R(E, A)E$ using (6) we obtain $R(A, E) = \mu^2 - \lambda(A\mu)/(1-\lambda^2)$ and hence

$$\frac{A\lambda}{1-\lambda^2} = -\mu.$$

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