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COMPLETE SURFACES IN E^4 WITH CONSTANT MEAN CURVATURE

By Takehiro Itoh

On surfaces in Euclidean 3-space with constant mean curvature, Klotz and Osserman [4] proved an interesting result as follows:

Any complete immersed surface with constant mean curvature on which the Gaussian curvature does not change its sign is either a sphere, a minimal surface or a right circular cylinder.

Ōtsuki [7] introduced a notion that submanifolds in higher dimensional Euclidean spaces are pseudo-umbilical or not, and stated some suggestive results with respect to the above one.

In the present paper, the author will study surfaces in Euclidean 4-space E^4 which are pseudo-umbilical at each point and have non-vanishing constant mean curvature. Our main result is

Theorem. *A complete, connected, oriented and pseudo-umbilical surface immersed in E^4 with non-vanishing constant mean curvature H and the Gaussian curvature K which does not change its sign is necessarily either a Clifford flat torus in E^4 or a sphere with radius $1/H$ in a hyperplane E^3.*

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§ 1. Preliminaries. In this section we define basic concepts, in the sense of [6] and [7], for surfaces in E^4. Let M^2 be a 2-dimensional complete oriented Riemannian manifold immersed in Euclidean 4-space E^4 with the induced Riemannian structure through the immersion $x: M^2 \to E^4$. Let $F(M^2)$ and $F(E^4)$ be the bundles of all orthonormal frames over M^2 and E^4 respectively. Let B be the set of element $b=(p, e_1, e_2, e_3, e_4)$ such that $(p, e_1, e_2) \in F(M^2)$ and $(p, e_1, e_2, e_3, e_4) \in F(E^4)$ whose orientation is coherent with the one of E^4, identifying $p \in M^2$ with $x(p)$ and e_i with $dx(e_i)$, $(i=1, 2)$. Then B is naturally considered as a differentiable submanifold in $F(E^4)$. We have, as is well known, a system of differential 1-forms $\omega_i, \omega_{ij}, \omega_{i\alpha}, \omega_{34}$ on B associated with the immersion $x: M^2 \to E^4$ such that

$$(1.1)\quad \begin{cases} dp = \sum_i \omega_i e_i, \qquad de_i = \omega_{ij} e_j + \sum_\alpha \omega_{i\alpha} e_\alpha, \\ de_\alpha = \sum_i \omega_{\alpha i} e_i + \omega_{\alpha\beta} e_\beta, \\ d\omega_i = \omega_{ij} \wedge \omega_j, \end{cases}$$

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$$\begin{cases} d\omega_{ij}=\sum_{\alpha}\omega_{i\alpha}\wedge\omega_{\alpha j}, & d\omega_{i\alpha}=\omega_{ij}\wedge\omega_{j\alpha}+\omega_{i\beta}\wedge\omega_{\beta\alpha}, \\ d\omega_{34}=\sum_{i}\omega_{3i}\wedge\omega_{i4}, & (i\neq j \text{ and } \alpha\neq\beta), \end{cases}$$

and

$$\omega_{i\alpha}=-\omega_{\alpha i}=\sum_{j}A_{\alpha ij}\omega_j, \qquad A_{\alpha ij}=A_{\alpha ji}, \tag{1. 2}$$

where we use the following ranges of indices: $1\leqq i,\ j\leqq 2,\ 3\leqq\alpha,\ \beta\leqq 4$, throughout this paper. Let N_p be the normal space of M^2 at $p\in M^2$, then N_p is clearly spanned by e_3 and e_4. For any normal unit vector $e=\sum_\alpha\xi_\alpha e_\alpha$, $\sum_\alpha\xi_\alpha^2=1$, the second fundamental form corresponding to e is given by $\Phi_e=\sum_{\alpha,i,j}\xi_\alpha A_{\alpha ij}\omega_i\omega_j$. We define a linear map $\bar{m}$ of N_p into the set R of all real numbers by $\bar{m}(e)=(1/2)\sum_{\alpha,i}\xi_\alpha A_{\alpha ii}$ for any normal vector $e=\sum_\alpha\xi_\alpha e_\alpha$. Let ${}^{m-}N_p$ be the kernel of $\bar{m}$ which is called *the minimal normal space* at p.

For any unit normal vector e at p, the maximum of $\bar{m}(e)$ is called *the first curvature* at p and denoted by $k_1(p)$, i.e., $k_1(p)=\max\{\bar{m}(e)|e\in N_p,\ \|e\|=1\}$. We say that M^2 is *minimal* at p if $\bar{m}(e)=0$ for any $e\in N_p$. Then M^2 is minimal at p if and only if $k_1(p)=0$. It is clear that the function k_1: $M^2\to R$ is continuous on M^2 and differentiable on the domain $D=\{p\in M^2|k_1(p)>0\}$. At a point p of D, we denote by $\bar{e}(p)$ the mean curvature normal unit vector uniquely defined by the equation $\sum_{\alpha,i}A_{\alpha ii}e_\alpha=k_1(p)\bar{e}(p)$. Then $\bar{e}$ determines a differentiable normal unit vector field on D. The first curvature $k_1(p)$ is called the *mean curvature* and denoted by H, i.e., $H=k_1(p)$.

Now, for any frame $b\in B$, consider a linear mapping ψ_b of N_p into the set S_2 of all real symmetric matrices defined by $\psi_b(\sum_\alpha\xi_\alpha e_\alpha)=\sum_\alpha\xi_\alpha A_\alpha$, A_α being the matrix $(A_{\alpha ij})$. The dimension of the image of ${}^{m-}N_p$ by ψ_b is called the *m-index* of M^2 at p and denoted by m-index$_pM$. By the definition of m-index, we have

$$m\text{-index}_pM\leqq\dim{}^{m-}N_p. \tag{1. 3}$$

When a surface M^2 is not minimal at a point p of M^2, M^2 is called *pseudo-umbilical* at p if the second fundamental form corresponding to the mean curvature normal unit vector $\bar{e}(p)$ satisfies the equation $\Phi_{\bar{e}(p)}=k_1(p)\sum_i\omega_i\omega_i$. When a surface M^2 is not minimal and pseudo-umbilical at each point, the immersion x: $M^2\to E^4$, or rather, the surface M^2, is said to be *pseudo-umbilical.*

Supposing that a surface M^2 is not minimal at each point, we can consider frames $b\in B$ such that $e_3=\bar{e}(p)$. If we denote the set of all such frames by B_1, then B_1 can be considered as a differentiable submanifold of B. Then we have $\bar{m}(e_3)=k_1(p)=H$ and $\bar{m}(e_4)=0$. Therefore, in this case, we have by (1. 3)

$$m\text{-index}_pM\leqq 1 \tag{1. 4}$$

at any point $p\in M^2$, because $\dim{}^{m-}N_p=1$. Let us suppose that M^2 is pseudo-umbilical. Then the unit normal vector fields e_3 and e_4 are differentiable and defined globally on M^2, since both of M^2 and E^4 are orientable. In this case, we have

(1. 5) $$\begin{cases} A_4=0 & \text{when} \quad m\text{-index}_p M=0, \\ A_4\neq 0 & \text{when} \quad m\text{-index}_p M=1. \end{cases}$$

We may consider M^2 as a Riemann surface by Chern [2], because M^2 is a 2-dimensional oriented Riemannian manifold. According to Ahlfors and Sario [1], we call M^2 to be *parabolic* if there are no non-constant negative subharmonic functions on M^2. Then, when M^2 is parabolic, if a subharmonic function f on M^2 is bounded from above on M^2, f must be constant on M^2. Furthermore, we can easily show that the entire plane E^2 is parabolic.

In order to prove main theorem, we shall provide some basic lemmas in § 2 and prove it for the case (i) $K\leqq 0$ in § 3 and the case (ii) $K\geqq 0$ in § 4.

In the following, we may restrict ourselves to the set B_1 of frames, because we consider only surfaces which are not minimal at each point.

§ 2. **Basic lemmas.** Let M_0 be the set of all points whose m-index are zero According to (1. 5), we see that $m\text{-index}_p M=0$ if and only if a global differentiable function $\det A_4$ vanishes at p. Hence M_0 is closed in M^2. By virtue of (1. 4), for any point $p\in M^2-M_0$, we have

$$m\text{-index}_p M=1.$$

Hence, using this fact, we have the following

LEMMA 2. 1. *For a pseudo-umbilical surface M^2 in E^4 the mean curvature H is constant if and only if the form ω_{34} vanishes identically.*

Proof. First we assume that the ω_{34} vanishes identically. Then, using the structure equations (1. 1) for $\omega_{i3}=k_1\omega_i$, $(i=1, 2)$, we have

(2. 1) $$\begin{cases} dk_1\wedge\omega_1=\omega_{14}\wedge\omega_{43}=0, \\ dk_1\wedge\omega_2=\omega_{24}\wedge\omega_{43}=0, \end{cases}$$

which imply that the mean curvature $H=k_1$ should be constant. Conversely, we assume that the mean curvature $H=k_1$ is constant. Then we have easily from (2.1) the equation $\omega_{34}=0$ in the open subset $M_1=M^2-M_0$ of M^2. When the set M_1 is dense in M^2, we have $\omega_{34}=0$ everywhere on M^2 because of the continuity of the form ω_{34} itself. In the next step, we consider the case where M_0 has non-empty open set Ω as its open kernel. Then we have $\omega_{i4}=0$ in Ω by (1. 5). Hence, by means of the structure equations (1.1) for ω_{i4}, we have $k_1\omega_1\wedge\omega_{34}=0$ and $k_1\omega_2\wedge\omega_{34}=0$, which imply immediately $\omega_{34}=0$ in Ω. Summing up, we can conclude by virtue of the continuity of ω_{34} that the differentiable form ω_{34} vanishes identically on M^2 when the mean curvature $H=k_1$ is constant. Thus we have proved Lemma 2. 1.

LEMMA 2. 2. *If a surface M^2 in E^4 is pseudo-umbilical and the mean curvature $H=k_1$ is constant, then M^2 is contained in a hypersphere S^3 in E^4 with radius $1/H$.*

Proof. Consider the mapping ψ: $M^2 \to E^4$ defined by $\psi(p)=p+e_3/k_1$ for any point p of M^2, where p in the right hand side denotes the position vector of the image M^2 in E^4. Then, taking account of $\omega_{34}=0$ which is a direct consequence of Lemma 2. 1, we have $d(\psi(p))=0$. This means that $\psi(p)$ denotes a fixed point in E^4. Thus M^2 lies on a hypersphere S^3 with radius $1/H$.

§ 3. The proof of the main theorem in the case (i) $K \leqq 0$.

PROPOSITION 1. *A complete, connected, oriented and pseudo-umbilical surface M^2 immersed in E^4 with constant mean curvature $H \neq 0$ and the Gaussian curvature K which is nowhere positive is a Clifford flat torus $S^1(1/\sqrt{2}H)\times S^1(1/\sqrt{2}H)$ in E^4.*

Proof. The Gaussian curvature K is given by the equation $d\omega_{12}=-K\omega_1\wedge\omega_2$. On the other hand, the structure equations (1. 1) imply $d\omega_{12}=-(k_1^2+\det A_4)\omega_1\wedge\omega_2$, because of $\omega_{i3}=k_1\omega_i$, $(i=1, 2)$. Hence we get

$$K=k_1^2+\det A_4=k_1^2-(A_{411}^2+A_{412}^2), \tag{3. 1}$$

by means of trace $A_4=0$. Since $K\leqq 0$ and $k_1\neq 0$, we have

$$0<k_1^2\leqq A_{411}^2+A_{412}^2. \tag{3. 2}$$

Thus, we find

$$m\text{-index}_p M=1 \tag{3. 3}$$

at each point p of M^2. Accordingly, we can choose locally such frames $b\in B_1$ that the matrix A_4 is given by

$$A_4=\begin{pmatrix} h & 0 \\ 0 & -h \end{pmatrix}$$

the function h being differentiable and defined globally on M^2, because $\det A_4=-h^2$ is a global differentiable function on M^2, furthermore we may suppose $h>0$ on M^2. Then, using the structure equations (1. 1) for ω_{i4} and Lemma 2. 1, we get

$$\begin{cases} 2hd\omega_1+dh\wedge\omega_1=0, \\ 2hd\omega_2+dh\wedge\omega_2=0. \end{cases} \tag{3. 4}$$

Hence we can consider local coordinates (u, v) in an open neighbourhood U of a point $p\in M$ such that

$$I=E\,du^2+G\,dv^2, \qquad \omega_1=\sqrt{E}\,du, \qquad \omega_2=\sqrt{G}\,dv,$$

where I is the first fundamental form and E and G are local positive functions on U. Then the equations (3. 4) are reduced to

$$(3.5)\qquad \begin{cases} d(hE)\wedge du=0, \\ d(hG)\wedge dv=0, \end{cases}$$

which show that there exists a neighbourhood V of each point $p\in M^2$ such that there exist isothermal coordinates (u, v) in V such that

$$(3.6)\qquad \begin{cases} I=\lambda\{du^2+dv^2\}, \qquad \omega_1=\sqrt{\lambda}\,du, \qquad \omega_2=\sqrt{\lambda}\,dv, \\ h\lambda=1, \end{cases}$$

where $\lambda=\lambda(u, v)$ is a positive function defined on V.

Now, we get the following

LEMMA. *The universal covering surface $\tilde{M}$ of M^2 is conformally equivalent to the entire plane, and hence M^2 is parabolic.*

Proof of Lemma. Since k_1 is positive constant, the conformal metric $k_1 I$ is complete on M^2. However, since $k_1 \leqq h$, the conformal metric hI is also complete on M^2. Furthermore, the metric hI is flat from (3.6). Hence, the covering surface $\tilde{M}$ with the lifted metric from hI on M^2 is isometric to the entire plane. Thus, $\tilde{M}$ is conformally equivalent the entire plane with respect to I.

We shall prove that M^2 is parabolic. Suppose that M^2 is not parabolic. Then, by the definition of parabolicity stated in §1, there is a non-constant negative subharmonic function on M^2. Since the existence of a negative non-constant subharmonic function on M^2 implies the one on $\tilde{M}$, $\tilde{M}$ must be also non parabolic. On the other hand, $\tilde{M}$ is conformally equivalent to the entire plane and hence $\tilde{M}$ is parabolic, contradicting the consequence induced from the assumption that M^2 is not parabolic. Thus, M^2 is parabolic.

Going back to the proof of Proposition 1, as is well known, the Gaussian curvature K is given by

$$K=-\frac{1}{2\lambda}\Delta \log \lambda$$

with respect to the isothermal coordinates (u, v). Hence the condition $K\leqq 0$ with $h\lambda=1$ implies

$$(3.7)\qquad \Delta \log h=-\Delta \log \lambda \leqq 0.$$

This inequality (3.7) implies that the function $\log(1/h)$ is a subharmonic function on M^2. On the other hand, since $K=k_1^2-h^2$ from (3.1), we have $0<k_1\leqq h$. Hence the subharmonic function $\log(1/h)$ on M^2 is bounded from above by $\log(1/k_1)$, i.e.,

$$\log\frac{1}{h}\leqq \log\frac{1}{k_1}.$$

However, since M^2 is parabolic by Lemma, a subharmonic function bounded from above on M^2 must be constant. Hence the subharmonic function $\log(1/h)$ on M^2 is identically constant on M^2. Therefore

$$K=\frac{-1}{2\lambda}\Delta\log\lambda=\frac{h}{2}\Delta\log h$$

is identically zero on M^2. Since $\log\lambda=\log(1/h)$ is constant, λ is also constant on V. Hence we can choose the isothermal coordinates (u, v) such that

$$I=du^2+dv^2, \qquad \omega_1=du, \qquad \omega_2=dv.$$

Then, taking account of $\omega_{12}=0$ and $h=k_1$, we get the following Frenet formulas:

$$\text{(3. 8)} \qquad \begin{cases} dp=e_1du+e_2dv, \\ de_1=k_1(e_3+e_4)du, \\ de_2=k_1(e_3-e_4)dv, \\ de_3=-k_1(e_1du+e_2dv), \\ de_4=-k_1(e_1du-e_2dv). \end{cases}$$

Now, we introduce new frames $b^*=(p, e_1^*, e_2^*, e_3^*, e_4^*)$ such that $e_1^*=e_1$, $e_2^*=e_2$, $e_3^*=(1/\sqrt{2})(e_3+e_4)$ and $e_4^*=(1/\sqrt{2})(-e_3+e_4)$. Then the equations (3. 8) reduce to

$$\text{(3. 9)} \qquad \begin{cases} dp=e_1^*du+e_2^*dv, \\ de_1^*=\sqrt{2}\,k_1e_3^*du, \\ de_2^*=-\sqrt{2}\,k_1e_4^*dv, \\ de_3^*=-\sqrt{2}\,k_1e_1^*du, \\ de_4^*=\sqrt{2}\,k_1e_2^*dv. \end{cases}$$

These equations (3. 9) clearly show that $e_1^*\wedge e_3^*$ and $e_2^*\wedge e_4^*$ determine constant bivectors respectively. Hence the plane $E_1^2(p)$ spanned by e_1^* and e_3^* is parallel to a fixed plane E_1^2 in E^4. The plane $E_2^2(p)$ spanned by e_2^* and e_4^* is also parallel to a fixed plane E_2^2 in E^4. E_1^2 and E_2^2 are clearly perpendicular to each other. Along the u-curve defined by $v=$const., we have from (3. 9)

$$dp=e_1^*du, \qquad de_1^*=\sqrt{2}\,k_1e_3^*du, \qquad de_3^*=-\sqrt{2}\,k_1e_1^*du.$$

Thus p describes a circle in $E_1^2(p)$ with radius $1/\sqrt{2}\,k_1$, because the u-curve is geodesic through p and M^2 is complete. Analogously, along the v-curve defined by $u=$const., p describes a circle in $E_2^2(p)$ with radius $1/\sqrt{2}\,k_1$.

Now, to find the loci of the centers of these two circles, we consider two mapping ψ_1 and ψ_2: $M^2\to E^4$ defined respectively by $\psi_1(p)=p+e_3^*/\sqrt{2}\,k_1$ and by $\psi_2(p)=p-e_4^*/\sqrt{2}\,k_1$ for any point p of M^2, where p denotes the position vector indicating a point p of M^2 in E^4. Using (3. 9), we have

$$(3.10)\qquad \begin{cases} d(\psi_1(p))=e_2^*dv, & de_2^*=-\sqrt{2}\,k_1e_4^*dv, & de_4^*=\sqrt{2}\,k_1e_2^*dv, \\ d(\psi_2(p))=e_1^*du, & de_1^*=\sqrt{2}\,k_1e_3^*du, & de_3^*=-\sqrt{2}\,k_1e_1^*du. \end{cases}$$

We can easily verify that $\psi_1(p)$ (resp $\psi_2(p)$) describes a circles with radius $1/\sqrt{2}\,k_1$ in a plane parallel to $E_2^2(p)$ (resp $E_1^2(p)$). Furthermore, we can find that the center of the circle described by $\psi_1(p)$ coincides with the one of the circle described by $\psi_2(p)$. Thus M^2 in E^4 may be considered as a Riemannian product of two circles lying on planes parallel to E_1^2 and E_2^2 respectively.

§ 4. The proof of the main theorem in the case (ii) $K\geqq 0$.

PROPOSITION 2. *A complete, connected, oriented and pseudo-umbilical surface M^2 immersed in E^4 with the constant mean curvature $H\neq 0$ and the Gaussian curvature K which is nowhere negative must be either a Clifford flat torus in E^4 or a sphere with radius $1/H$ in a hyperplane E^3.*

Proof. We first prove

LEMMA. *$-K$ is a subharmonic function on M^2.*

Proof of Lemma. $M_0=\{p\in M^2|m\text{-index}_pM=0\}$ is closed in M^2. Hence $M_1=M^2-M_0$ is open in M^2. Then, analogously in § 3, we can choose a neighbourhood U of a point $p\in M_1$ in which there exist isothermal coordinates (u, v) such that

$$(4.1)\qquad \begin{cases} I=\lambda\{du^2+dv^2\}, & \omega_1=\sqrt{\lambda}\,du, \qquad \omega_2=\sqrt{\lambda}\,dv, \\ A_4=\begin{pmatrix} h & 0 \\ 0 & -h \end{pmatrix}, & h>0, \qquad h\lambda=1, \end{cases}$$

where h is a differentiable function in U. Hence, as is well known,

$$K=-\frac{1}{2\lambda}\Delta\log\lambda=\frac{h}{2}\Delta\log h\geqq 0$$

from (4. 1), which implies $\Delta h\geqq 0$ because $h>0$. Then we have in U

$$(4.2)\qquad \Delta h^2=2\left\{\left(\frac{\partial h}{\partial u}\right)^2+\left(\frac{\partial h}{\partial v}\right)^2\right\}+2h\Delta h\geqq 0.$$

On the other hand, since $K=k_1^2-h^2$ in U, we have

$$(4.3)\qquad \Delta K=-\Delta h^2\leqq 0 \quad \text{in} \quad U.$$

Thus we have $\Delta K\leqq 0$ in M_1. Finally, we shall prove that $\Delta K\leqq 0$ at any point of M_0. Take a point p_1 of M_0 and consider the isothermal coordinates (u, v) in a neighbourhood U_1 of p_1 such that

(4. 4) $$I=\lambda\{du^2+dv^2\}, \qquad \omega_1=\sqrt{\lambda}\,du, \qquad \omega_2=\sqrt{\lambda}\,dv.$$

In this case, A_4 may be represented by

$$A_4=\begin{pmatrix} h_1 & h_2 \\ h_2 & -h_1 \end{pmatrix},$$

where h_1 and h_2 are functions in U_1. Then we have

(4. 5) $$K=k_1^2-(h_1^2+h_2^2) \qquad \text{in} \quad U.$$

Hence we have in U_1

(4. 6) $$\Delta K=-2\left\{\left(\frac{\partial h_1}{\partial u}\right)^2+\left(\frac{\partial h_1}{\partial v}\right)^2+\left(\frac{\partial h_2}{\partial u}\right)^2+\left(\frac{\partial h_2}{\partial v}\right)^2\right\}-2h_1\Delta h_1-2h_2\Delta h_2$$

with respect to the isothermal coordinates (u, v). Since h_1 and h_2 attain zero at p_1, we have

(4. 7) $$\Delta K\leqq 0.$$

Thus we have $\Delta K\leqq 0$ at a point of M_0. Since the sign of the Laplacian ΔK is invariant with respect to the isothermal coordinates, we have

(4. 8) $$\Delta K\leqq 0 \qquad \text{on} \quad M^2$$

with respect to the isothermal coordinates. Thus $-K$ is a subharmonic function on M^2. We have proved Lemma.

Going back to the proof of Proposition 2, if M^2 is compact, the subharmonic function $-K$ on M^2 attains its maximum at some point on M^2. Hence, $-K$ must be constant on M^2, and hence K is constant on M^2.

On the other hand, if M^2 is not compact, M^2 is parabolic by Theorem 15 in Huber [3], since $K\geqq 0$. Since $-K$ is a negative subharmonic function on M^2, $-K$ must be constant on M^2 by the definition of parabolicity stated in § 1. Thus, whether M^2 is compact or not, K is constant on M^2. Since K is given by $K=k_1^2+\det A_4=k_1^2-||A_4||^2$, $\det A_4=-||A_4||^2$ must be constant on M^2. Hence we can cosider the following two cases:

Case (a): M_0 is not empty

and

Case (b): M_0 is empty.

Case (a). If M_0 is not empty, $\det A_4=-||A_4||^2$ attains zero at points of M_0. Hence $\det A_4=-||A_4||^2$ is identically zero on M^2, because $\det A_4=-||A_4||^2$ is constant on M^2. This implies $M_0=M^2$. Hence $de_4=0$ on M^2 because of $\omega_{43}=0$. Therefore, there exists a hyperplane E^3 in E^4 such that M^2 is immersed in E^3 with the immersion x: $M^2\rightarrow E^4$. Since $\omega_{i3}=k_1\omega_i$, $(i=1, 2)$, M^2 is umbilical in E^3.

Furthermore, M^2 is complete by our assumption. Hence, as is well known, M^2 in E^4 is immersed in E^3 as a sphere with radius $1/H$.

Case (b). If M_0 is empty in M^2, m-index$_pM=1$ at any point $p\in M^2$. Analogously in § 3, we can choose a neighbourhood U of a point p of M^2 in which there exist isothermal coordinates (u, v) such that

$$\text{(4. 9)} \qquad \begin{cases} I=\lambda\{du^2+dv^2\}, & \omega_1=\sqrt{\lambda}\,du, \qquad \omega_2=\sqrt{\lambda}\,dv, \\ A_4=\begin{pmatrix} h & 0 \\ 0 & -h \end{pmatrix}, & h>0, \qquad h\lambda=1, \end{cases}$$

where h is a positive and differentiable function on M^2 Then, since $\det A_4=-h^2$ is constant on M^2, h is also constant on M^2. On the other hand, as stated in § 3, K is given by $K=(h/2)\Delta \log h$. Hence K is identically zero on M^2. Therefore M^2 is immersed in E^4 as a Clifford flat torus in E^4 by means of Proposition 1. Thus we have proved Proposition 2.

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Department of Mathematics,
Tokyo Institute of Technology.