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GAUSS MAP IN A SPHERE

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0. Introduction.

To a surface M^2 of a Euclidean 3-space $\boldsymbol{E}^3$ there is associated the Gauss map which assigns to a point of M^2 the unit normal vector at the point. This is a mapping of M^2 into the unit sphere S^2 about the origin of $\boldsymbol{E}^3$. Chern and Lashof gave a generalization of this classical Gauss map in [2] as follows. Let M^n be an n-dimensional Riemannian manifold isometrically immersed into a Euclidean $(n+N)$-space E^{n+N} $(N\geqq 1)$ and B be the bundle of unit normal vectors of M^n $(\dim B=n+N-1)$. Then a mapping of B into the unit sphere S^{n+N-1} about the origin of E^{n+N} can be naturally defined.

Furthermore, Willmore and Saleemi [5] and Chen [1] generalized this mapping to the case where M^n is an n-dimensional Riemannian manifold isometrically immersed into an $(n+N)$-dimensional, complete, and simply connected Riemannian manifold M^{n+N} with non-positive sectional curvature. The manner can be stated as follows. Let q be a point of M^n and B be the pseudo-normal bundle of M^n (for the definition, see [1]). The parallel displacement of $\nu\in B$ along the shortest geodesic segment joining the foot point of ν and q gives a mapping of B into the unit sphere in the tangent space of M^{n+N} at q.

With the same ideas as the one of Willmore and Saleemi and Chen we can associate to an n-dimensional Riemannian manifold M^n isometrically immersed into the Euclidean unit $(n+N)$-sphere S^{n+N} the mapping analogous to the above Gauss map in the following way. Let p a point of M^n and B be the bundle of unit normal vectors of $M^n-\{-p\}$ in S^{n+N}. Then the parallel displacement Γ_p of $\nu\in B$ along the shortest geodesic segment joining the foot point of ν and p gives a mapping of B into the unit sphere S_p^{n+N-1} in the tangent space of S^{n+N} at p. We shall call Γ_p the Gauss map associated to M^n immersed into S^{n+N}, and p the base point. The purpose of this note is to relate the Gauss map Γ_p with the geometrical structure of M^n. The main results obtained is the following

THEOREM 1. *Let M^n be an n-dimensional, complete Riemannian manifold isometrically immersed into the Euclidean unit $(n+N)$-sphere S^{n+N}. Let p be a point of S^{n+N} and Γ_p be the Gauss map: $B\to S_p^{n+N-1}$ associated to M^n. Then Γ_p has rank m at $\nu\in B$ if and only if $\langle\nu,p\rangle/(1+\langle x,p\rangle)$ is an eigenvalue of the second fundamental form whose multiplicity is equal to $n+N-1-m$, where $\langle\,,\rangle$ is the canonical inner product of E^{n+N} and x is the foot point of ν.*

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THEOREM 2. *Let* $N=1$ *in addition to the assumption of Theorem* 1. *Suppose that the Jacobian of* Γ_p *has constant rank* $n-m$ *on* B $(0\leqq m\leqq n)$.

(1) *Let* $m=0$. *If* M^n *is compact and* $-p\notin M^n$, *then* M^n *is diffeomorphic to the n-sphere.*

(2) *Let* $1\leqq m\leqq n-1$. *Then* M^n *is a locus of a moving m-sphere.*

(3) *Let* $m=n$. *Then there exist a real number* ξ $(|\xi|<1)$ *and a point* q *of* M^n *such that* $M^n=\{x\in S^{n+1};\ \langle x, q\rangle=\xi\}$ *and* $\langle p, q\rangle=-\xi$. *The converse of this is also true.*

As seen from these results, Γ_p is different from the ordinary Gauss map in that it depends on a choice of the base point p as well as the immersion of M^n and it is not defined for normal vectors at the point $-p$ (if $-p\in M^n$) because p and $-p$ can not be uniquely joined by the shortest geodesic segment on S^{n+1}. A large part of this note is devoted to proofs of Theorem 1 and 2.

1. Moving frames.

Throughout this note, let M^n be an n-dimensional, connected, complete Riemannian manifold isometrically immersed into the unit hypersphere S^{n+N} in a Euclidean $(n+N+1)$-space E^{n+N+1}. We choose a locally defined orthonormal frame field $e_1, \cdots, e_{n+N}$ in S^{n+N} such that, restricted to M^n, $e_1, \cdots, e_n$ are tangent to M^n. We shall agree on the following ranges of indices:

$$1\leqq i, j, k\leqq n, \qquad n+1\leqq \alpha, \beta, \gamma\leqq n+N, \qquad 1\leqq A, B, C\leqq n+N.$$

Let $\omega_1, \cdots, \omega_{n+N}$ be the dual of the frame field chosen above and ω_{AB} be the connection forms for S^{n+N}. Then the structure equations of S^{n+N} are given by

$$(1) \qquad d\omega_A=\sum_B \omega_{AB}\wedge\omega_B, \qquad \omega_{AB}+\omega_{AB}=0,$$

$$(2) \qquad d\omega_{AB}=\sum_C \omega_{AC}\wedge\omega_{CB}-\omega_A\wedge\omega_B.$$

Restricting these forms to M^n, we have

$$(3) \qquad \omega_\alpha=0$$

Hence the equation (1) gives

$$(4) \qquad \sum_i \omega_{\alpha i}\wedge\omega_i=0$$

From this we may write as

$$(5) \qquad \omega_{\alpha i}=\sum_j h_{\alpha ij}\omega_j, \qquad h_{\alpha ij}=h_{\alpha ji}.$$

The quadratic form $\sum_{i,j} h_{\alpha ij}\omega_i\omega_j$ is the second fundamental form of M^n with respect to e_α.

2. Explicit expression of Γ_p and a proof of Theorem 1.

First we shall express explicitly the Gauss map Γ_p: $B \to S_p^{n+N-1}$ associated to M^n. Set $\Gamma = \Gamma_p$ for simplicity and let π: $B \to M^n$ be the projection. We assert that Γ is given by

$$\Gamma(\nu) = \nu - \frac{\langle \nu, p \rangle}{1 + \langle x, p \rangle} p - \frac{\langle \nu, p \rangle}{1 + \langle x, p \rangle} x \tag{6}$$

where $\nu \in B$ and $x = \pi(\nu)$. In fact, if $x = p$, then $\Gamma(\nu) = \nu$. If $x \neq p$, then we decompose into the component $\nu_p p + \nu_x x$ of the 2-plane Π spanned by p and x and the component ν_0 normal to Π: $\nu = \nu_0 + \nu_p P + \nu_x X$. Then it is evident that $\Gamma(\nu) = \nu_0 + Z$, where $Z \in S_p^{n+N-1}$ is the vector parallel to $\nu_p P + \nu_x X$ along the geodesic joining p and x. Since $\nu_p = \langle \nu, p \rangle / (1 - \langle x, p \rangle^2)$,

$$\nu_x = -\langle \nu, p \rangle \langle x, p \rangle / (1 - \langle x, p \rangle^2)$$

and

$$Z = \nu_p (2 \langle x, p \rangle p - x) + \nu_x p,$$

after a simple computation we have (6).

The differential $d(\Gamma(\nu))$ of Γ at ν is given by

$$d(\Gamma(\nu)) = d\nu - \frac{\{(1 + \langle x, p \rangle) \langle d\nu, p \rangle - \langle \nu, p \rangle \langle dx, p \rangle\}(x + p)}{(1 + \langle x, p \rangle)^2} - \frac{\langle \nu, p \rangle}{1 + \langle x, p \rangle} dx. \tag{7}$$

Since (6) is valid for a tangent vector of S^{n+N}, we have

$$\Gamma(e_A) = e_A - \frac{\langle e_A, p \rangle}{1 + \langle x, p \rangle} p - \frac{\langle e_A, p \rangle}{1 + \langle x, p \rangle} x. \tag{8}$$

Set $\nu = e_{n+N}$. Then making use of the fact that

$$dx = \sum_i \omega_i e_i, \qquad x \in M^n \tag{9}$$

$$de_A = \sum_B \omega_{AB} \cdot e_B, \tag{10}$$

$$\langle x, e_A \rangle = 0, \tag{11}$$

we obtain

$$\langle d(\Gamma(\nu)),\ \Gamma(e_i) \rangle = \omega_{n+N,i} - \frac{\langle \nu, p \rangle}{1 + \langle x, p \rangle} \omega_i, \tag{12}$$

$$\langle d(\Gamma(\nu)),\ \Gamma(e_\alpha) \rangle = \omega_{n+N,\alpha}. \tag{13}$$

Since $\Gamma(e_1), \cdots, \Gamma(e_{n+N-1})$ forms a basis for the tangent space of S^{n+N-1} at $\Gamma(\nu)$, (12), (13) and (5) imply that the Jacobian matrix of Γ at ν is of the form

$$
\left(\begin{array}{c|c} H_{n+N}-\dfrac{\langle\nu,p\rangle}{1+\langle x,p\rangle}I_n & 0 \\ \hline 0 & I_{N-1} \end{array}\right) \tag{14}
$$

where $H_{n+N}=(h_{n+N,ij})$ and I_r denotes the identity matrix of degree r. This proves Theorem 1.

3. A proof of Theorem 2.

In this section we assume that the Gauss map Γ associated to M^n has constant rank $n+N-1-m$ $(0\leqq m\leqq n)$, in other words, for every $\nu\in B$, the second fundamental form with respect to ν has the eigenvalue $\lambda=\langle\nu,p\rangle/(1+\langle x,p\rangle)$ of multiplicity m.

Proof of Theorem 2. (1). Since Γ is nonsingular everywhere and $M^n-\{-p\}=M^n$ is compact, the image $\Gamma(B)$ of B under Γ is an open and closed subset of S_p^{n+N-1}, and so $\Gamma(B)=S_p^{n+N-1}$. Hence (Γ, B) is a covering space of $S_p^{n+N=1}$. If $N\geqq 2$, Γ must be one-to-one because B is connected. Hence Γ is a diffeomorphism. If $N=1$, one of two connected components of B is diffeomorphic to S_p^n, and also to M^n. q.e.d.

From now on let $N=1$ and $0<m\leqq n$. In this case there arises an m-dimensional distribution Λ on M^n which assigns to each point x of $M^n-\{-p\}$ the space of principal vectors corresponding to the principal curvature $\lambda=\langle\nu,p\rangle/(1+\langle x,p\rangle)$ at x. To prove (2) and (3) of Theorem 2 we shall establish the following

THEOREM 3. *Let M^n be a hypersurface immersed into the unit $(n+1)$-sphere S^{n+1}. Suppose that the multiplicity m of principal curvature λ is constant. Then the distribution Λ of the space of principal vectors corresponding to λ is completely integrable.*

Proof. We shall agree on the following ranges of indices:

$$1\leqq a, b, c\leqq m, \qquad m+1\leqq r, s, t\leqq n$$

We may choose a frame field $e_1, \cdots, e_{n+1}$ in §1 so that $e_1, \cdots, e_m$ forms a basis for Λ, that is, setting $h_{ab}=h_{n+1,ab}$ and $h_{rs}=h_{n+1,rs}$,

$$h_{ab}=\delta_{ab}\lambda, \qquad h_{ra}=0 \tag{15}$$

or equivalently

$$\omega_{n+1,a}=\lambda\omega_a \tag{16}$$

$$\omega_{n+1,r}=\sum_s h_{rs}\omega_s. \tag{17}$$

Taking exterior differentiation of (16), we have from (1)

$$\begin{aligned} d\omega_{n+1,a} &= d\lambda \wedge \omega_a + \lambda d\omega_a \\ &= \sum_b \lambda_b \omega_b \wedge \omega_a + \sum_r \lambda_r \omega_r \wedge \omega_a + \lambda \sum_b \omega_{ab} \wedge \omega_b + \lambda \sum_r \omega_{ar} \wedge \omega_r \end{aligned} \tag{18}$$

where we set $d\lambda = \sum_b \lambda_b \omega_b + \sum_r \lambda_r \omega_r$. On the other hand, we have from (2), (16) and (17)

$$\begin{aligned} d\omega_{n+1,a} &= \sum_b \omega_{n+1,b} \wedge \omega_{ba} + \sum_r \omega_{n+1,r} \wedge \omega_{ra} - \omega_{n+1} \wedge \omega_a \\ &= \lambda \sum_b \omega_b \wedge \omega_{ba} + \sum_{r,s} h_{rs} \omega_s \wedge \omega_{ra} \end{aligned} \tag{19}$$

since $\omega_{n+1}=0$ on M^n. Comparing (18) and (19) we obtain

$$\sum_b \lambda_b \omega_b \wedge \omega_a = 0, \tag{20}$$

$$\sum_r \Big(\sum_s h_{rs} \omega_{sa} - \lambda \omega_{ra} - \lambda_r \omega_a\Big) \wedge \omega_r = 0 \tag{21}$$

It follows from (21) and Cartan's lemma that we may write as

$$\sum_s h_{rs} \omega_{sa} - \lambda \omega_{ra} - \lambda_r \omega_a = \sum_s \theta_{ars} \omega_s, \qquad \theta_{ars} = \theta_{asr}. \tag{22}$$

Set here $\omega_{ra} = \sum_b \sigma_{rab} \omega_b + \sum_s \sigma_{ras} \omega_s$. Substituting this into (22), we have

$$\sum_s (h_{rs} - \delta_{rs} \lambda) \sigma_{saa} = \lambda_r, \tag{23}$$

$$\sum_s (h_{rs} - \delta_{rs} \lambda) \sigma_{sab} = 0 \qquad (a \neq b). \tag{24}$$

Since $\det(h_{rs} - \lambda I_m) \neq 0$ by the assumption, (23) implies that

$$\sigma_{r11} = \cdots = \sigma_{rmm} \tag{25}$$

and (24) implies that

$$\sigma_{rab} = 0 \qquad (a \neq b). \tag{26}$$

Denoting (25) by σ_r, we found

$$\omega_{ra} = \sigma_r \omega_a + \sum_s \sigma_{ras} \omega_s. \tag{27}$$

Hence

$$\begin{aligned} d\omega_r &= \sum_a \omega_{ra} \wedge \omega_a + \sum_s \omega_{rs} \wedge \omega_s \\ &= \sum_{a,s} \sigma_{ras} \omega_s \wedge \omega_a + \sum_s \omega_{rs} \wedge \omega_s \\ &\equiv 0 \qquad (\text{mod } \omega_t). \end{aligned} \tag{28}$$

This means that Λ is completely integrable. q.e.d.

COROLLARY 4. *Under the assumption of Theorem* 3, *if* m *is greater than* 1, *then* λ *is constant on each integral manifold of* Λ *and each integral manifold of* Λ *is a totally umbilic submanifold of* S^{n+1}.

Proof. The first assersion follows from (20) and the second from (16) and (27). q.e.d.

REMARK 5. In general, Theorem 3 and Corollary 4 hold also in the case where M^n is immersed as a hypersurface into a Riemannian manifold of constant curvature. The proof is entirely analogous. Thus Theorem 3 (resp. Corollary 4) is a slight generalization of a theorem of Ōtsuki ([4] Theorem 2) (resp. his Corollary).

For $x\in M^n-\{-p\}$ we denote by Λ_x the maximal integral manifold of Λ through x. Clearly we may assume that Λ_{-p} is defined if $-p\in M^n$.

LEMMA 6. $\sum_r \sigma_r^2$ *is constant on* Λ_x.

Proof. Restrict the forms under consideration on Λ_x. Then

$$\omega_r=0 \tag{29}$$

and from (27)

$$\omega_{ra}=\sigma_r\omega_a. \tag{30}$$

Taking exterior differentiation of (30) and using (1), (2) and (29) we have

$$(d\sigma_r-\sum_s \sigma_s\omega_{rs})\wedge\omega_a=0 \qquad \text{for all} \quad a. \tag{31}$$

This means that

$$d\sigma_r=\sum_s \sigma_s\omega_{rs}. \tag{32}$$

Hence

$$d\sum_r \sigma_r^2=2\sum_r \sigma_r d\sigma_r=2\sum_{r,s}\sigma_r\sigma_s\omega_{rs}=0. \qquad \text{q.e.d.} \tag{33}$$

Proof of Theorem 2. (2). By Lemma 6 and Corollary 4, Λ_x is either a totally geodesic submanifold of S^{n+1} or else not totally geodesic at every point.

In the former case Λ_x is a unit m-sphere. In the latter case Λ_x is totally umbilic and not totally geodesic at every point also as a submanifold of E^{n+2}. Therefore by menas of a theorem of Ōtsuki ([3], Theorem 1) Λ_x is an m-sphere in a linear subspace E^{m+1}. q.e.d.

There is a following relation between the base point p and an arbitrary Λ_y $(y\in M^n)$

LEMMA 7. *There exist a number* ξ $(|\xi|<1)$ *and a point* $q\in S^{n+1}$ *such that* Λ_y *is contained in a hypersphere* $\{x\in S^{n+1};\ \langle x, q\rangle=\xi\}$ *and* $\langle p, q\rangle=-\xi$.

Proof. Since $\lambda=\langle\nu, p\rangle/(1+\langle x, p\rangle)$ is constant on Λ_y by Corollary 4, $\nu-\lambda x$ is a constant vector on Λ_y by the definition of Λ. Thus we can set on Λ_y

$$\nu-\lambda x=-\sqrt{1+\lambda^2}\, q, \qquad \text{for a} \quad q\in S^{n+1} \tag{34}$$

Taking the inner product of (34) with x and p, we obtain

$$\langle x, q\rangle=\lambda/\sqrt{1+\lambda^2}, \qquad x\in\Lambda_y, \tag{35}$$

$$\langle p, q\rangle=-\lambda/\sqrt{1+\lambda^2}. \quad \text{q.e.d.} \tag{36}$$

Proof of Theorem 2. (3). Let $m=n$ in Lemma 7. Then M^n must be contained in Λ_y $(y\in M^n)$. By completeness, we conclude $M^n=\Lambda_y$. The converse of this is a straightforward computation.

REMARK 8. Consider the special case $m=n-1$ in Theorem 2. In this case the locus of all centers in E^{n+2} of Λ_y $(y\in M^n)$ is a curve C: $E\to E^{n+2}$, where C is parametrized by arc length. λ is a function on C.

We assert that there is no open interval of E on which λ vanishes identically. In fact, assume $\lambda\equiv 0$ on an open interval U.

Then it follows from Lemma 7 that both the base point p and Λ_y whose centers lies in $C(U)$ are contained in a unit hypersphere S^n. Thus M^n must contain an open subset of S^n, which contradicts the assumption $m=n-1$.

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