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ON $|C, \alpha|_k$ SUMMABILITY FACTORS OF FOURIER SERIES

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1. Let $\sum a_n$ be a given infinite series with its n-th partial sum S_n, and let $t_n=t_n^0=na_n$. By $\{\sigma_n^\alpha\}$ and $\{t_n^\alpha\}$ we denote the n-th Cesàro means of order α $(\alpha>-1)$ of the sequences $\{S_n\}$ and $\{t_n\}$ respectively. The series $\sum a_n$ is said to be absolutely summable (C, α) with index k, or simply summable $|C, \alpha|_k$ $(k\geqq 1)$, if

$$\sum n^{k-1}|\sigma_n^\alpha-\sigma_{n-1}^\alpha|^k<\infty \qquad ([5]). \tag{1.1}$$

Summability $|C, \alpha|_1$ is the same as summability $|C, \alpha|$. Since

$$t_n^\alpha=n(\sigma_n^\alpha-\sigma_{n-1}^\alpha),$$

condition (1.1) can also be written as

$$\sum \frac{|t_n^\alpha|^k}{n}<\infty. \tag{1.2}$$

A sequence $\{\lambda_n\}$ is said to be convex [7], if $\Delta^2\lambda_n\geqq 0$, $n=1, 2, \cdots$, where $\Delta\lambda_n=\lambda_n-\lambda_{n+1}$ and $\Delta^2\lambda_n=\Delta(\Delta\lambda_n)$.

2. Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. We assume without loss of generality that the constant term in Fourier series is zero such that

$$f(t)\sim\sum_{n=1}^{\infty}(a_n \cos nt+b_n \sin nt)=\sum_{n=1}^{\infty}A_n(t),$$

and

$$\int_{-\pi}^{\pi}f(t)dt=0.$$

We write

$$\phi(t)=\frac{1}{2}\{f(x+t)+f(x-t)-2f(x)\}.$$

3. Cheng [2] established:

THEOREM A. *If*

$$\int_0^t|\phi(u)|du=O\left\{t\left(\log\frac{1}{t}\right)^\beta\right\}, \qquad \beta\geqq 0 \tag{3.1}$$

as $t\to 0$, *then the series*

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$$\sum \frac{A_n(t)}{n^{1-\alpha}(\log n)^{1+\beta+\varepsilon}}, \qquad \varepsilon>0,$$

at the point $t=x$ *is summable* $|C, \alpha|$, $0\leqq\alpha<1$.

Extending the above theorem, Dikshit [4] proved:

THEOREM B. *If* $\{\lambda_n\}$ *is a convex sequence such that the series* $\sum \lambda_n/n$ *is convergent, then the series*

$$\sum \frac{\lambda_n A_n(t)}{n^{1-\alpha}(\log n)^{\beta}}$$

at $t=x$ *is summable* $|C, \alpha|$, $0\leqq\alpha<1$, *whenever condition* (3. 1) *is satisfied.*

The object of this paper is to generalise theorem B for summability $|C, \alpha|_k$. We prove:

THEOREM. *If* $\{\lambda_n\}$ *is a convex sequence such that* $\sum \lambda_n/n$ *is convergent, then the series*

$$\sum \frac{\lambda_n A_n(t)}{n^{1-\alpha}(\log n)^{\beta/k}}$$

at $t=x$ *is summable* $|C, \alpha|_k$ *where* $0\leqq\alpha<1$ *and* $k\geqq1$, *provided that*

$$\int_0^t |\phi(u)|^k\,du = O\left\{t\left(\log\frac{1}{t}\right)^{\beta}\right\}, \qquad \beta\geqq0. \tag{3. 2}$$

4. We require the following lemmas for the proof of our theorem.

LEMMA 1. [3]. *If* $0<\alpha<1$, $0<t<2\pi$ *and*

$$T_n^{\alpha}(t)=\sum_{\mu=1}^{n} A_{n-\mu}^{\alpha-1}\mu\cos\mu t,$$

then

$$T_n^{\alpha}(t)=\begin{cases} O(n^2) & \text{for all } t>0. \\ O(nt^{-\alpha}) & \text{for } t>\dfrac{1}{n}. \end{cases}$$

LEMMA 2. [1]. *If* $0\leqq\alpha\leqq1$ *and* $0\leqq m\leqq n$, *then*

$$\left|\sum_{\nu=0}^{m} A_{n-\nu}^{\alpha-1}a_\nu\right| \leqq \max_{0\leqq\mu\leqq m}\left|\sum_{\nu=0}^{\mu} A_{\mu-\nu}^{\alpha-1}a_\nu\right|.$$

LEMMA 3. *Let* $0<\alpha<1$ *and* $0<t\leqq2\pi$. *We write*

$$M_n^{\alpha}(t)=\frac{1}{A_n^{\alpha}}\sum_{\nu=2}^{n} A_{n-\nu}^{\alpha-1}\frac{\lambda_\nu}{(\log\nu)^{\beta/k}}\nu^{\alpha}\cos\nu t, \qquad \beta\geqq0,\ k\geqq1.$$

Then

$$M_n^\alpha(t)=\begin{cases} O\Big\{n^{-\alpha}\sum_{\nu=2}^{n}\nu^{1+\alpha}(\log\nu)^{-\beta/k}\Delta\lambda_\nu\Big\}+O\{n\lambda_n(\log n)^{-\beta/k}\}+O(n^{-1}) & \textit{for } 0<t\leqq\dfrac{1}{n}. \quad (4.1)\\ O\Big\{(nt)^{-\alpha}\sum_{\nu=2}^{n}\nu^{\alpha}(\log\nu)^{-\beta/k}\Delta\lambda_\nu\Big\}+O\{\lambda_n t^{-\alpha}(\log n)^{-\beta/k}\}+O(n^{-1}) & \textit{for } t>\dfrac{1}{n}. \quad (4.2)\end{cases}$$

Proof. By Abel's transformation, we have

$$M_n^\alpha(t)=\frac{1}{A_n^\alpha}\Big\{\sum_{\nu=2}^{n-1}\Delta\Big(\frac{\lambda_\nu}{\nu^{1-\alpha}(\log\nu)^{\beta/k}}\Big)\sum_{\mu=1}^{\nu}A_{n-\mu}^{\alpha-1}\mu\cos\mu t\Big\}$$
$$+\frac{T_n^\alpha(t)\lambda_n}{A_n^\alpha n^{1-\alpha}(\log n)^{\beta/k}}-\frac{1}{A_n^\alpha}\,\frac{\lambda_2}{2^{1-\alpha}(\log 2)^{\beta/k}}\,A_{n-1}^{\alpha-1}\cos t,$$

where $T_n^\alpha(t)$ is the sequence defined in Lemma 1. So by Lemmas 1 and 2, for $0<t\leqq 1/n$,

$$M_n^\alpha(t)=O\Big\{\frac{1}{A_n^\alpha}\sum_{\nu=2}^{n}\Delta\Big(\frac{\lambda_\nu}{\nu^{1-\alpha}(\log\nu)^{\beta/k}}\Big)\cdot\nu^2\Big\}+O\Big\{\frac{n^2\lambda_n}{A_n^\alpha n^{1-\alpha}(\log n)^{\beta/k}}\Big\}+O(n^{-1})$$
$$=O\Big\{n^{-\alpha}\Big[\sum_{\nu=2}^{n}\frac{\nu^{1+\alpha}\Delta\lambda_\nu}{(\log\nu)^{\beta/k}}+\sum_{\nu=2}^{n}\frac{\nu^{\alpha}\lambda_\nu}{(\log\nu)^{\beta/k}}+\sum_{\nu=2}^{n}\frac{\nu^{\alpha}\lambda_\nu}{(\log\nu)^{\beta/k+1}}\Big]\Big\}+O\Big\{\frac{n\lambda_n}{(\log n)^{\beta/k}}\Big\}+O(n^{-1}).$$

But

$$\sum_{\nu=2}^{n}\frac{\nu^{\alpha}\lambda_\nu}{(\log\nu)^{\beta/k}}=O\Big\{\sum_{\nu=2}^{n}\Delta\lambda_\nu\sum_{m=2}^{\nu}\frac{m^\alpha}{(\log m)^{\beta/k}}\Big\}+O\Big\{\lambda_n\sum_{m=2}^{n}\frac{m^\alpha}{(\log m)^{\beta/k}}\Big\}.$$

Hence, for $0<t\leqq 1/n$, this gives (4.1).

Also, for $t>1/n$,

$$M_n^\alpha(t)=O\Big\{\frac{1}{A_n^\alpha}\sum_{\nu=2}^{n}\nu t^{-\alpha}\Delta\Big(\frac{\lambda_\nu}{\nu^{1-\alpha}(\log\nu)^{\beta/k}}\Big)\Big\}+O\Big\{\frac{n\lambda_n t^{-\alpha}}{A_n^\alpha n^{1-\alpha}(\log n)^{\beta/k}}\Big\}+O(n^{-1})$$
$$=O\Big\{(nt)^{-\alpha}\Big[\sum_{\nu=2}^{n}\frac{\nu\Delta\lambda_\nu}{\nu^{1-\alpha}(\log\nu)^{\beta/k}}+\sum_{\nu=2}^{n}\frac{\nu\lambda_\nu}{\nu^{2-\alpha}(\log\nu)^{\beta/k}}+\sum_{\nu=2}^{n}\frac{\nu\lambda_\nu}{\nu^{2-\alpha}(\log\nu)^{\beta/k+1}}\Big]\Big\}$$
$$+O\Big\{\frac{\lambda_n}{t^\alpha(\log n)^{\beta/k}}\Big\}+O(n^{-1}).$$

But

$$\sum_{\nu=2}^{n}\frac{\lambda_\nu}{\nu^{1-\alpha}(\log\nu)^{\beta/k}}=O\Big\{\sum_{\nu=2}^{n}\Delta\lambda_\nu\sum_{\mu=2}^{\nu}\frac{1}{\mu^{1-\alpha}(\log\mu)^{\beta/k}}\Big\}+O\Big\{\lambda_n\sum_{\nu=2}^{n}\frac{1}{\nu^{1-\alpha}(\log\nu)^{\beta/k}}\Big\}$$
$$=O\Big\{\sum_{\nu=2}^{n}\frac{\nu^\alpha\Delta\lambda_\nu}{(\log\nu)^{\beta/k}}\Big\}+O\Big\{\frac{\lambda_n n^\alpha}{(\log n)^{\beta/k}}\Big\}.$$

This establishes (4.2).

LEMMA 4. *If* (3.2) *holds, then, for* $k\geqq 1$,

(i) $$\Big\{\int_0^{1/n}|\phi(t)|\,dt\Big\}^k=O\{n^{-k}(\log n)^\beta\}$$

and, for $k \geqq 1$ *and* $0<\alpha<1$,

(ii) $$\left\{\int_{1/n}^{\pi} \frac{|\phi(t)|}{t^{\alpha}} dt\right\}^{k} = O\{(\log n)^{\beta}\}.$$

Proof of (i). By Hölder's inequality, we have

$$\left\{\int_0^{1/n} |\phi(t)|\, dt\right\}^k \leqq \left\{\int_0^{1/n} |\phi(t)|^k dt\right\}\left\{\int_0^{1/n} dt\right\}^{k-1}$$
$$= O\left\{\frac{(\log n)^{\beta}}{n} \cdot \frac{1}{n^{k-1}}\right\} = O\left\{\frac{(\log n)^{\beta}}{n^k}\right\}.$$

Proof of (ii). Again, Hölder's inequality gives that

$$\left\{\int_{1/n}^{\pi} \frac{|\phi(t)|}{t^{\alpha}} dt\right\}^k = \left\{\int_{1/n}^{\pi} \frac{|\phi(t)|}{t^{\alpha/k}} \cdot \frac{1}{t^{\alpha(1-1/k)}} dt\right\}^k$$
$$\leqq \left\{\int_{1/n}^{\pi} \frac{|\phi(t)|^k}{t^{\alpha}} dt\right\}\left\{\int_{1/n}^{\pi} \frac{dt}{t^{\alpha}}\right\}^{k-1}$$

But

$$\left\{\int_{1/n}^{\pi} \frac{|\phi(t)|^k}{t^{\alpha}} dt\right\} = O\left\{\left[t^{1-\alpha}\left(\log\frac{1}{t}\right)^{\beta}\right]_{1/n}^{\pi}\right\} + O\left\{\int_{1/n}^{\pi} \frac{1}{t^{\alpha}}\left(\log\frac{1}{t}\right)^{\beta} dt\right\}$$
$$= O(1) + O\{n^{\alpha-1}(\log n)^{\beta}\} + O\left\{(\log n)^{\beta}\int_{1/n}^{\pi} \frac{dt}{t^{\alpha}}\right\}$$
$$= O\{(\log n)^{\beta}\}, \qquad \text{since } 0<\alpha<1.$$

Thus (ii) is evident.

LEMMA 5 [6]. *If* $\{\lambda_n\}$ *is a convex sequence such that* $\sum \lambda_n/n<\infty$, *then*

$$\sum_{n=1}^{m} \log(n+1)\Delta\lambda_n = O(1), \qquad m\to\infty.$$

5. *Proof of the theorem.* Since the case $k=1$ of the theorem is due to Dikshit [4], we prove the theorem for $k>1$ only.

The case $\alpha=0$ being trivial, we take $0<\alpha<1$. We denote the n-th Cesàro mean of order α of the sequence $\{n^{\alpha}\lambda_n A_n(t)(\log n)^{-\beta/k}\}$ by $C_n^{\alpha}(t)$. Then we have to show that

(5.1) $$\sum \frac{|C_n^{\alpha}(t)|^k}{n} < \infty.$$

Now,

$$C_n^{\alpha}(t) = \frac{2}{\pi}\int_0^{\pi} \phi(t) \cdot \frac{1}{A_n^{\alpha}} \sum_{\nu=2}^{n} \frac{A_{n-\nu}^{\alpha-1}\nu^{\alpha}\lambda_{\nu}\cos\nu t}{(\log\nu)^{\beta/k}} dt$$
$$= \frac{2}{\pi}\left\{\int_0^{1/n} + \int_{1/n}^{\pi}\right\}\phi(t) M_n^{\alpha}(t) dt$$
$$= L_n^1 + L_n^2, \qquad \text{say}.$$

By Minkowski's inequality, it is therefore sufficient to prove that

$$\sum \frac{|L_n^1|^k}{n} < \infty, \tag{5. 2}$$

and

$$\sum \frac{|L_n^2|^k}{n} < \infty. \tag{5. 3}$$

Proof of (5. 2). Using (4. 1), we have

$$\sum_{n=2}^{m} \frac{|L_n^1|^k}{n} \leqq \sum_{n=2}^{m} \frac{1}{n}\left[\frac{2}{\pi}\int_0^{1/n} |\phi(t)|\,dt\left\{O\left(n^{-\alpha}\sum_{\nu=2}^{n}\frac{\nu^{1+\alpha}\Delta\lambda_\nu}{(\log\nu)^{\beta/k}}\right)+O\left(\frac{n\lambda_n}{(\log n)^{\beta/k}}\right)+O\left(\frac{1}{n}\right)\right\}\right]^k$$

$$=\left\{O\left[\sum_{n=2}^{m}\frac{1}{n}\left(\int_0^{1/n}\frac{|\phi(t)|}{n^\alpha}\sum_{\nu=2}^{n}\frac{\nu^{1+\alpha}}{(\log\nu)^{\beta/k}}\Delta\lambda_\nu dt\right)^k\right]^{1/k}\right.$$

$$\left.+O\left[\sum_{n=2}^{m}\frac{1}{n}\left(\int_0^{1/n}\frac{|\phi(t)|n\lambda_n}{(\log n)^{\beta/k}}dt\right)^k\right]^{1/k}+O\left[\sum_{n=2}^{m}\frac{1}{n^{1+k}}\left(\int_0^{1/n}|\phi(t)|\,dt\right)^k\right]^{1/k}\right\}^k$$

$$=\{M_1^{1/k}+M_2^{1/k}+M_3^{1/k}\}^k, \qquad \text{say.}$$

Now, applying lemmas 4 (i) and 5, we get

$$M_1=O\left[\sum_{n=2}^{m}\frac{1}{n^{1+k\alpha}}\left(\int_0^{1/n}|\phi(t)|\,dt\right)^k\left(\sum_{\nu=2}^{n}\frac{\nu^{1+\alpha}\Delta\lambda_\nu}{(\log\nu)^{\beta/k}}\right)^k\right]$$

$$=O\left[\sum_{n=2}^{m}\frac{1}{n^{1+k\alpha}}\cdot\frac{(\log n)^\beta}{n^k}\left(\sum_{\nu=2}^{n}\frac{\nu^{1+\alpha}\Delta\lambda_\nu}{(\log\nu)^{\beta/k}}\right)^k\right]$$

$$=O\left[\sum_{n=2}^{m}\frac{(\log n)^\beta}{n^{1+k(1+\alpha)}}\left\{\sum_{\nu=2}^{n}\left(\frac{\nu^{1+\alpha}(\Delta\lambda_\nu)^{1/k}}{(\log\nu)^{\beta/k}}\right)(\Delta\lambda_\nu)^{1-1/k}\right\}^k\right]$$

$$=O\left[\sum_{n=2}^{m}\frac{(\log n)^\beta}{n^{1+k(1+\alpha)}}\left\{\sum_{\nu=2}^{n}\frac{\nu^{k(1+\alpha)}\Delta\lambda_\nu}{(\log\nu)^\beta}\right\}\left\{\sum_{\nu=2}^{n}\Delta\lambda_\nu\right\}^{k-1}\right]$$

$$=O\left[\sum_{n=2}^{m}\frac{(\log n)^\beta}{n^{1+k(1+\alpha)}}\left\{\sum_{\nu=2}^{n}\frac{\nu^{k(1+\alpha)}\Delta\lambda_\nu}{(\log\nu)^\beta}\right\}\right]$$

$$=O\left[\sum_{\nu=2}^{m}\frac{\nu^{k(1+\alpha)}\Delta\lambda_\nu}{(\log\nu)^\beta}\sum_{n=\nu}^{m}\frac{(\log n)^\beta}{n^{1+k(1+\alpha)}}\right]$$

$$=O\left[\sum_{\nu=2}^{m}\Delta\lambda_\nu\right]$$

$$=O(1), \qquad \text{as } m\to\infty.$$

Next, using lemma 4 (i) again, we write

$$M_2=O\left[\sum_{n=2}^{m}\frac{n^{k-1}\lambda_n^k}{(\log n)^\beta}\cdot\left(\int_0^{1/n}|\phi(t)|\,dt\right)^k\right]$$

$$=O\left[\sum_{n=2}^{m}\frac{n^{k-1}\lambda_n^k}{(\log n)^\beta}\cdot\frac{(\log n)^\beta}{n^k}\right]$$

$$=O\left[\sum_{n=2}^{m}\frac{\lambda_n^k}{n}\right]$$

$$=O(1),\qquad \text{as } m\to\infty,$$

and

$$M_3=O\left(\sum_{n=2}^{m}\frac{(\log n)^\beta}{n^{1+2k}}\right)=O(1),\qquad \text{as } m\to\infty.$$

This proves (5. 2).

Proof of (5. 3). Applying (4. 2), we have

$$\sum_{n=2}^{m}\frac{|L_n^2|^k}{n}\leqq\sum_{n=2}^{m}\frac{1}{n}\left[\frac{2}{\pi}\int_{1/n}^{\pi}|\phi(t)|\,dt\left\{O\left((nt)^{-\alpha}\sum_{\nu=2}^{n}\frac{\nu^\alpha\Delta\lambda_\nu}{(\log\nu)^{\beta/k}}\right)+O\left(\frac{\lambda_n}{t^\alpha(\log n)^{\beta/k}}\right)+O\left(\frac{1}{n}\right)\right\}\right]^k$$

$$=\left\{O\left[\sum_{n=2}^{m}\frac{1}{n}\left(\frac{1}{n^\alpha}\int_{1/n}^{\pi}\frac{|\phi(t)|}{t^\alpha}\sum_{\nu=2}^{n}\frac{\nu^\alpha\Delta\lambda_\nu}{(\log\nu)^{\beta/k}}\,dt\right)^k\right]^{1/k}+O\left[\sum_{n=2}^{m}\frac{1}{n}\left(\int_{1/n}^{\pi}\frac{|\phi(t)|}{t^\alpha}\cdot\frac{\lambda_n}{(\log n)^{\beta/k}}\,dt\right)^k\right]^{1/k}+O\left[\sum_{n=2}^{m}\frac{1}{n^{1+k}}\left(\int_{1/n}^{\pi}|\phi(t)|\,dt\right)^k\right]^{1/k}\right\}^k$$

$$=\{N_1^{1/k}+N_2^{1/k}+N_3^{1/k}\}^k,\qquad \text{say.}$$

Using lemmas 4 (ii) and 5, we get

$$N_1=O\left[\sum_{n=2}^{m}\frac{1}{n^{1+k\alpha}}\left(\int_{1/n}^{\pi}\frac{|\phi(t)|}{t^\alpha}\,dt\right)^k\left(\sum_{\nu=2}^{n}\frac{\nu^\alpha\Delta\lambda_\nu}{(\log\nu)^{\beta/k}}\right)^k\right]$$

$$=O\left[\sum_{n=2}^{m}\frac{(\log n)^\beta}{n^{1+k\alpha}}\left(\sum_{\nu=2}^{n}\frac{\nu^\alpha\Delta\lambda_\nu}{(\log\nu)^{\beta/k}}\right)^k\right]$$

$$=O\left[\sum_{n=2}^{m}\frac{(\log n)^\beta}{n^{1+k\alpha}}\left\{\sum_{\nu=2}^{n}\left(\frac{\nu^\alpha(\Delta\lambda_\nu)^{1/k}}{(\log\nu)^{\beta/k}}\right)(\Delta\lambda_\nu)^{1-1/k}\right\}^k\right]$$

$$=O\left[\sum_{n=2}^{m}\frac{(\log n)^\beta}{n^{1+k\alpha}}\left\{\sum_{\nu=2}^{n}\frac{\nu^{k\alpha}\Delta\lambda_\nu}{(\log\nu)^\beta}\right\}\left\{\sum_{\nu=2}^{n}\Delta\lambda_\nu\right\}^{k-1}\right]$$

$$=O\left[\sum_{n=2}^{m}\frac{(\log n)^\beta}{n^{1+k\alpha}}\sum_{\nu=2}^{n}\frac{\nu^{k\alpha}\Delta\lambda_\nu}{(\log\nu)^\beta}\right]$$

$$=O\left[\sum_{\nu=2}^{m}\frac{\nu^{k\alpha}\Delta\lambda_\nu}{(\log\nu)^\beta}\sum_{n=\nu}^{m}\frac{(\log n)^\beta}{n^{1+k\alpha}}\right]$$

$$=O\left[\sum_{\nu=2}^{m}\Delta\lambda_\nu\right]$$

$$=O(1),\qquad \text{as } m\to\infty.$$

Lastly, applying lemma 4 (ii) again, we have

$$N_2=O\left[\sum_{n=2}^{m}\frac{\lambda_n^k}{n}\cdot\frac{1}{(\log n)^\beta}\left(\int_{1/n}^{\pi}\frac{|\phi(t)|}{t^\alpha}dt\right)^k\right]$$

$$=O\left[\sum_{n=2}^{m}\frac{\lambda_n^k}{n}\cdot\frac{1}{(\log n)^\beta}(\log n)^\beta\right]$$

$$=O\left[\sum_{n=2}^{m}\frac{\lambda_n^k}{n}\right]$$

$$=O(1),\qquad \text{as } m\to\infty.$$

And obviously

$$N_3=O\left(\sum_{n=2}^{m}\frac{1}{n^{1+k}}\right)=O(1),\qquad \text{as } m\to\infty.$$

This proves (5. 3).

Thus the proof of the theorem is complete.

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