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ON HYPERSURFACES IN SASAKIAN MANIFOLDS

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§ 1. Introduction.

Recently, Okumura [3] has studied hypersurfaces of an odd dimensional sphere S^{n+1} and obtained a sufficient condition for a hypersurface M in S^{n+1} to be totally umbilical. Also, Watanabe [6] has studied totally umbilical hypersurfaces in a Sasakian manifold and proved

THEOREM (Watanabe). *Let M be a complete orientable connected totally umbilical hypersurface in a Sasakian manifold. If M is of constant mean curvature H, then M is isometric with a sphere of radius $1/\sqrt{1+H^2}$ in the Euclidean space.*

It might be interesting to obtain other sufficient conditions that a hypersurface in a Sasakian manifold is isometric to a sphere. In § 2, we recall first of all the definition of a Sasakian manifold and those parts of the theory of hypersurfaces in a Sasakian manifold which are necessary for what follows. Some general properties of a hypersurface in a Sasakian manifold are derived in § 2. In § 3, taking account of the theorem above, we prove Theorem 3. 3.

This theorem plays an important role in § 5. In § 4 we shall consider a totally umbilical hypersurface in certain Sasakian manifolds. In the last section we prove the main

THEOREM. *Let M $(n>2)$ be a complete orientable connected hypersurface in a Sasakian manifold $\tilde{M}$. If the contact form η over $\tilde{M}$ is not tangent to M almost everywhere and if f commutes with h, then M is isometric with a sphere of radius $1/\sqrt{1+H^2}$ in a Euclidean space.*

§ 2. Preliminaries.

An $(n+1)$-dimensional *contact metric manifold* is by definition a Riemannian manifold admitting a structure $(\varphi, \xi, \eta, \tilde{g})$, $\eta=(\eta_\lambda)$ being a 1-form, $\xi=(\xi^\lambda)$ a contravariant vector field, $\varphi=(\varphi_\lambda{}^\mu)$ a (1, 1)-type tensor field and $\tilde{g}=(\tilde{g}_{\lambda\mu})$ the Riemannian metric tensor, which is positive definite, such that

$$\varphi_\alpha{}^\lambda\xi^\alpha=0, \qquad \varphi_\lambda{}^\alpha\eta_\alpha=0, \qquad \xi^\alpha\eta_\alpha=1, \tag{2. 1}$$

$$\varphi_\mu{}^\alpha\varphi_\alpha{}^\nu=-\delta_\mu^\nu+\eta_\mu\xi^\nu, \tag{2. 2}$$

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(2. 3) $$\eta_\lambda=\tilde{g}_{\lambda\alpha}\xi^\alpha,^{1)}$$

(2. 4) $$\tilde{g}_{\alpha\beta}\varphi_\lambda{}^\alpha\varphi_\mu{}^\beta=\tilde{g}_{\lambda\mu}-\eta_\lambda\eta_\mu,$$

(2. 5) $$\varphi_{\lambda\mu}\equiv\tilde{g}_{\mu\alpha}\varphi_\lambda{}^\alpha=\frac{1}{2}(\partial_\lambda\eta_\mu-\partial_\mu\eta_\lambda),$$

where (η_λ), (ξ^λ), $(\varphi_\lambda{}^\mu)$ and $(\tilde{g}_{\lambda\mu})$ denote respectively the components of η, ξ, φ and $\tilde{g}$ with respect to local coordinates $\{X^\kappa\}$.[2] A contact metric manifold $\tilde{M}$ is said to be Sasakian, if the structure $(\varphi, \xi, \eta, \tilde{g})$ satisfies the conditions

(2. 6) $$\varphi_{\lambda\mu}=\tilde{\nabla}_\lambda\eta_\mu, \qquad \tilde{\nabla}_\mu\varphi_{\lambda\nu}=\eta_\lambda\tilde{g}_{\mu\nu}-\eta_\nu\tilde{g}_{\mu\lambda},$$

where $\tilde{\nabla}$ denotes the operator of the covariant differentiation with respect to $\tilde{g}$.

Let $\tilde{M}$ be a Sasakian manifold and M[3] an orientable hypersurface represented locally by the equations

$$X^\kappa=X^\kappa(x^i),$$

where $\{x^i\}$ are local coordinates of M. If we put

$$B_i{}^\kappa=\frac{\partial X^\kappa}{\partial x^i},$$

$B_i{}^\kappa (i=1, 2, \cdots, n)$ are linearly independent local vector fields tangent to M. The induced Riemannian metric g of the hypersurface M is given by

(2. 7) $$g_{ji}=\tilde{g}_{\beta\alpha}B_j{}^\beta B_i{}^\alpha.$$

Since the Sasakian manifold $\tilde{M}$ and the hypersurface M are both orientable, we can choose a unit normal vector field C^κ along the hypersurface M in such a way that $(C^\kappa, B_i{}^\kappa)$ form a frame having the positive sense of $\tilde{M}$ and $(B_i{}^\kappa)$ form a frame having the positive sense of M. Then we have

(2. 8) $$\tilde{g}_{\beta\alpha}B_i{}^\beta C^\alpha=0, \qquad \tilde{g}_{\beta\alpha}C^\beta C^\alpha=1.$$

The transforms $\varphi_\alpha{}^\kappa B_i{}^\alpha$ of $B_i{}^\alpha$ by $\varphi_\alpha{}^\kappa$ and $\varphi_\alpha{}^\kappa C^\alpha$ of C^α by $\varphi_\alpha{}^\kappa$ are expressed as linear combinations of $B_i{}^\alpha$ and C^α as follows:

$$\varphi_\beta{}^\kappa B_j{}^\beta=f_j{}^r B_r{}^\kappa+f_j C^\kappa,$$

$$\varphi_\alpha{}^\kappa C^\alpha=k^r B_r{}^\kappa+q' C^\kappa,$$

$$\eta^\kappa=p^r B_r{}^\kappa+q C^\kappa,$$

1) In the following we use a notation η^λ in stead of ξ^λ.

2) The indices run over the following ranges respectively:

$$\alpha, \beta, \cdots, \lambda, \mu, \cdots=1, 2, \cdots, n, n+1;$$
$$h, i, \cdots, r, s, \cdots=1, 2, \cdots, n.$$

3) In this paper we assume that M is connected.

from which

$$f_j{}^i = B^i{}_\alpha \varphi_\beta{}^\alpha B_j{}^\beta, \tag{2.9}$$

$$f_j = -k_j = B_j{}^\alpha \varphi_\alpha{}^\beta C_\beta, \tag{2.10}$$

$$p_j = B_j{}^\alpha \eta_\alpha, \tag{2.11}$$

$$q = \eta_\alpha C^\alpha, \qquad q' = 0, \tag{2.12}$$

where we have denoted by $(B^i{}_\alpha, C_\alpha)$ the coframe dual to the frame $(B_i{}^\alpha, C^\alpha)$. By virtue of (2. 1)~(2. 5) and (2. 9)~(2. 10), we have

$$f_{ji} \equiv g_{ir} f_j{}^r = -f_{ij}, \tag{2.13}$$

$$f_j{}^r f_r{}^i = -\delta_j^i + f_j f^i + p_j p^i, \tag{2.14}$$

$$f_j{}^r f_r = q p_j, \qquad f_j{}^r p_r = -q f_j, \tag{2.15}$$

$$f^r f_r = p^r p_r = 1 - q^2, \qquad f^r p_r = 0. \tag{2.16}$$

Now, denoting by ∇ the symbol of the covariant differentiation along the hypersurface M, we have respectively the equations of Gauss and Weingarten

$$\nabla_j B_i{}^\kappa \equiv \partial_j B_i{}^\kappa + B_j{}^\beta B_i{}^\alpha \widetilde{\left\{ {\kappa \atop \beta\ \alpha} \right\}} - B_r{}^\kappa \left\{ {r \atop j\ i} \right\} = h_{ji} C^\kappa,$$

$$\nabla_j C^\kappa \equiv \partial_j C^\kappa + B_j{}^\alpha C^\beta \widetilde{\left\{ {\kappa \atop \alpha\ \beta} \right\}} = -h_j{}^r B_r{}^\kappa,$$

where $\widetilde{\{ {\kappa \atop \beta\ \alpha} \}}$ (resp. $\{ {h \atop j\ i} \}$) are the Christaffels symbols with respect to $\tilde{g}$ (resp. g) and h_{ji} are components of the second fundamental tensor of M. Differentiating covariantly (2. 9)~(2. 12) along the hypersurface M, we obtain

$$\nabla_k f_{ji} = p_j g_{ki} - p_i g_{kj} + f_j h_{ki} - f_i h_{kj}, \tag{2.17}$$

$$\nabla_j f_i = -q g_{ji} - f_i{}^r h_{rj}, \tag{2.18}$$

$$\nabla_j p_i = f_{ji} + q h_{ji}, \tag{2.19}$$

$$\nabla_j q = f_j - p^r h_{rj}. \tag{2.20}$$

We here prove an identity for the later use. Operating ∇_k to (2. 20) and taking account of (2. 18) and (2. 19), we have

$$\nabla_k \nabla_j q = -q g_{kj} - f_j{}^r h_{rk} - (f_k{}^r + q h_k{}^r) h_{rj} - p^r \nabla_k h_{jr}. \tag{2.21}$$

If we subtract (2. 21) from the equation obtained by interchanging the indices k and j in (2. 21), we obtain

$$p^r(\nabla_k h_{jr} - \nabla_j h_{kr}) = 0. \tag{2.22}$$

§ 3. Totally umbilical hypersurfaces.

When, at each point of the hypersurface M, the second fundamental tensor h_{ji} is proportional to the induced Riemannian tensor g_{ji} of M, i.e., when the condition

$$h_{ji}=Hg_{ji} \tag{3. 1}$$

is satisfied, the hypersurface M is called a *totally umbilical hypersurface*. The proportional factor H is the mean curvature of the hypersurface. A totally umbilical hypersurface with vanishing mean curvature is said to be *totally geodesic*. We shall prove now that, for an orientable totally umbilical hypersurface M of a Sasakian manifold $\tilde{M}$ the mean curvature H is constant.

If we put $M_0=\{(x^i)\in M|q^2(x^i)\neq 1\}$, then M_0 is an open set in M. We assume now that M_0 is not empty. Then, substituting (3. 1) into (2. 22), we obtain

$$p_j\nabla_k H-p_k\nabla_j H=0.$$

Contracting this with p^j and making use of (2. 16), we get

$$(1-q^2)\nabla_k H=p_k p^r\nabla_r H,$$

from which

$$\nabla_k H=\alpha p_k \tag{3. 2}$$

in M_0, where α is a certain scalar function defined over M_0. Differentiating this covariantly, we have

$$\nabla_j\nabla_k H=p_k\nabla_j\alpha+\alpha\nabla_j p_k.$$

If we take the skew-symmetric part of this tensor equation and take account of (2. 19), we have

$$p_k\nabla_j\alpha-p_j\nabla_k\alpha+2\alpha f_{jk}=0.$$

Transvecting the last equation with f^{kj} and $f^j p^k$ respectively, we have

$$qf^j\nabla_j\alpha-\alpha(n-2+2q^2)=0,$$

$$(1-q^2)(f^j\nabla_j\alpha-2\alpha q)=0.$$

In M_0, from the two equations above we have $(n-2)\alpha=0$. Thus we get $\alpha=0$ if $n>2$. Therefore from (3. 2) we see that the mean curvature H is locally constant in M_0, that is, it satisfies $\nabla_i H=0$. Consequently we have

LEMMA 3. 1. *If M_0 is not empty, the mean curvature H of a totally umbilical hypersurface M $(n>2)$ is locally constant in M_0.*

Next, if we put $M_1=\{(x^i)\in M|\nabla_k H(x^i)=0\}$, then we see that $M-M_1$ is an open set in M and we have $M-M_1\subset M-M_0$ by virtue of Lemma 3. 1. Hence, by virtue of definition of M_0, we get $q^2=1$ in $M-M_1$. Therefore, we have $f_i=p_i=0$ in $M-M_1$ by virtue of (2. 16). We assume now that $M-M_1$ is not empty. Then $M-M_1$ being an open set in M, we find

$$qg_{ji}+Hf_{ji}=0, \tag{3. 3}$$

if we differentiate $f_i=0$ covariantly and take account of (2. 18) and (3. 1). If we add (3. 3) to the equation obtained by interchanging the indices j and i in (3. 3) and take account of (2. 13), we have $qg_{ji}=0$. Thus we get $q=0$ in $M-M_1$, which contradicts the condition $q^2=1$. Since M is connected, the mean curvature H is constant over M. Consequently we have

THEOREM 3. 2. *Let $M\,(n>2)$ be an orientable connected totally umbilical hypersurface of a Sasakian manifold $\tilde{M}$. Then the mean curvature H is constant over M.*

Combining Theorem (Watanabe) stated in §1 and Theorem 3. 2, we have immediately

THEOREM 3. 3. *Let $M\,(n>2)$ be a complete orientable connected totally umbilical hypersurface in a Sasakian manifold $\tilde{M}$. Then M is isometric with a sphere of radius $1/\sqrt{1+H^2}$ in a Euclidean space.*

§4. Totally umbilical hypersurfaces of a certain Sasakian manifold.

Watanabe [6] has proved

LEMMA 4. 1. *If M is an orientable totally umbilical hypersurface with constant mean curvature in a Sasakian manifold $\tilde{M}$, then the scalar function q is not constant in M.*

This lemma plays an important role in this section.

When the Ricci tensor of a Sasakian manifold $\tilde{M}$ has components of the form

$$\tilde{R}_{\lambda\mu}=a\tilde{g}_{\lambda\mu}+b\eta_\lambda\eta_\mu, \tag{4. 1}$$

then $\tilde{M}$ is called a C-Einstein (η-Einstein) manifold. In such a manifold $\tilde{M}$, a and b are necessarily constants (Cf. Okumura [1]). A C-Einstein manifold is Einstein if $b=0$. When the curvature tensor of a Sasakian manifold $\tilde{M}$ has components of the form

$$\begin{aligned}\tilde{R}_{\lambda\mu\nu\kappa}=&(k+1)(\tilde{g}_{\mu\nu}\tilde{g}_{\lambda\kappa}-\tilde{g}_{\mu\kappa}\tilde{g}_{\lambda\nu})+k(\varphi_{\mu\nu}\varphi_{\lambda\kappa}-\varphi_{\mu\kappa}\varphi_{\lambda\nu}-2\varphi_{\lambda\mu}\varphi_{\nu\kappa})\\&+k(\eta_\mu\eta_\kappa\tilde{g}_{\lambda\nu}-\eta_\mu\eta_\nu\tilde{g}_{\lambda\kappa}+\tilde{g}_{\mu\kappa}\eta_\lambda\eta_\nu-\tilde{g}_{\nu\mu}\eta_\lambda\eta_\kappa),\end{aligned} \tag{4. 2}$$

then $\tilde{M}$ is called a locally C-Fubinian manifold (Tashiro and Tachibana [5]). In such a manifold $\tilde{M}$, k is necessarily constant. A locally C-Fubinian manifold is necessarily C-Einstein, In the first place, we consider a totally umbilical hypersurface M of a C-Einstein manifold $\tilde{M}$. From the Codazzi equation of the hypersurface

$$\nabla_k h_{ji} - \nabla_j h_{ki} = B_k{}^\lambda B_j{}^\mu B_i{}^\nu C^\kappa \tilde{R}_{\lambda\mu\nu\kappa}, \tag{4. 3}$$

we have

$$\nabla_r h_i{}^r - \nabla_i h_r{}^r = -C^\kappa B_i{}^\mu \tilde{R}_{\kappa\mu}.$$

Making use of (4. 1), this reduces to

$$\nabla_r h_i{}^r - \nabla_i h_r{}^r = -bqp_i. \tag{4. 4}$$

As the hypersurface M is totally umbilical, by virtue of Theorem 3. 2, the mean curvature H is constant in M if $n>2$. Thus we obtain $bqp_i=0$. Hence if we assume that a C-Einstein manifold $\tilde{M}$ is not Einstein, i.e., $b\neq 0$, then we have $qp_i=0$. Differentiating this covariantly and making use of (2. 19) and $h_{ji}=Hg_{ji}$, we obtain

$$(f_j - Hp_j)p_i + q(f_{ji} + Hqg_{ji}) = 0.$$

Transvecting this with f^j and making use of $qp_i=0$, (2. 15) and (2. 16), it follows that

$$p_i + Hq^2 f_i = 0.$$

Transvecting this with p^i and taking account of (2. 16), we get $1-q^2=0$. This contradicts Lemma 4. 1. Thus we have $b=0$. Consequently, we have

THEOREM 4. 2. *If an orientable hypersurface M $(n>2)$ in a C-Einstein manifold $\tilde{M}$ is a totally umbilical hypersurface, then a C-Einstein manifold $\tilde{M}$ is necessarily Einstein* (Watanabe [6]).

COROLLARY 4. 3. *Let $\tilde{M}$ be a C-Einstein manifold. If $\tilde{M}$ is not Einstein, then there is no orientable totally umbilical hypersurface M $(n>2)$.*

In the next place, we shall consider a totally umbilical hypersurface in a locally C-Fubinian manifold $\tilde{M}$. If we substitute (4. 2) into (4. 3), it follows that

$$\nabla_k h_{ji} - \nabla_j h_{ki} = k(f_{ji}f_k - f_{ki}f_j - 2f_{kj}f_i) + kq(g_{ki}p_j - g_{ji}p_k).$$

Since, by Theorem 3. 2 the mean curvature H is constant, the equation above reduces to

$$k(f_{ji}f_k - f_{ki}f_j - 2f_{kj}f_i) + kq(g_{ki}p_j - g_{ji}p_k) = 0.$$

Transvecting this with g^{ji} and making use of (2. 15) and of Lemma 4. 1, we get $k=0$. Therefore we have

THEOREM 4. 4. *If an orientable hypersurface M $(n>2)$ in a locally C-Fubinian manifold $\tilde{M}$ is a totally umbilical hypersurface, then a locally C-Fubinian manifold*

$\tilde{M}$ is necessarily of constant curvature.

COROLLARY 4.5. *Let $\tilde{M}$ be a locally C-Fubinian manifold. If $\tilde{M}$ is not of constant curvature, then there is no orientable totally umbilical hypersurface M $(n>2)$.*

§ 5. Determination of the hypersurfaces.

In this section we assume that the 1-form η over a Sasakian manifold $\tilde{M}$ is not tangent to a hypersurface M almost everywhere. Moreover, we assume that f commutes with h, i.e.,

$$f_j{}^r h_r{}^i = h_j{}^r f_r{}^i. \tag{5.1}$$

The following Lemma is known [3].

LEMMA 5.1. *If f commutes with h and if the 1-form η over $\tilde{M}$ is not tangent to M almost everywhere, then we have*

$$h_{ji} f^j p^i = 0, \tag{5.2}$$

$$h_{ji} f^j f^i = h_{ji} p^j p^i. \tag{5.3}$$

Now, transvecting (5.1) with $f_k{}^j$ and making use of (2.14), we get

$$-h_{ki} + p_k p^r h_{ri} + f_k f^r h_{ri} = -h_{rs} f_k{}^r f_i{}^s. \tag{5.4}$$

If we subtract (5.4) from the equation obtained by interchanging the indices k and i in (5.4), it follows that

$$p_k p^r h_{ri} - p_i p^r h_{rk} + f_k f^r h_{ri} - f_i f^r h_{rk} = 0.$$

Transvecting this with p^k and with f^k and taking account of (2.16) and (5.2), we find respectively

$$\begin{aligned} &(1-q^2) p^k h_{kj} - p_j h_{rs} p^r p^s = 0, \\ &(1-q^2) f^k h_{kj} - f_j h_{rs} f^r f^s = 0. \end{aligned} \tag{5.5}$$

Now, if we put $M_0 = \{(x^i) \in M \mid q^2(x^i) \neq 1\}$, then M_0 is an open set in M. We suppose now that M_0 is not empty. Then we have from (5.5)

$$h_{rj} p^r = \alpha p_j, \qquad h_{rj} f^j = \alpha f_j \tag{5.6}$$

in M_0, where α is a differentiable function defined over M_0. Differentiating (5.6) covariantly, we get in M_0

$$(f_k{}^r + q h_k{}^r) h_{rj} + p^r \nabla_k h_{jr} = p_j \nabla_k \alpha + \alpha (f_{kj} + q h_{kj}),$$

because M_0 is open and non-empty. If we take the skew-symmetric part of this tensor equation and take account of (2.22) and (5.1), we obtain

(5. 7) $$2f_k{}^r h_{rj} = p_j \nabla_k \alpha - p_k \nabla_j \alpha + 2\alpha f_{kj}.$$

Transvecting (5. 7) with p^j and making use of (2. 16) and (5. 6), we get in M_0

(5. 8) $$\nabla_k \alpha = \beta p_k,$$

where β is a certain function defined in M_0. Differentiating (5. 8) covariantly, we obtain

$$\nabla_j \nabla_k \alpha = p_k \nabla_j \beta + \beta \nabla_j p_k.$$

If we subtract this from the equation obtained by interchanging the indices j and k in this and take account of (2. 19), we have

$$p_k \nabla_j \beta - p_j \nabla_k \beta + 2\beta f_{jk} = 0.$$

Transvecting this with f^{jk} and with $f^j p^k$, we get respectively

(5. 9) $$q\nabla_j \beta f^j = (n-2+2q^2)\beta, \qquad (1-q^2)(\nabla_j \beta f^j - 2\beta q) = 0,$$

where we have used (2. 13)$\sim$(2. 16). As a consequence of (5. 9), if $n>2$, we have $\beta=0$, which implies together with (5. 8) that α satisfies $\nabla_j \alpha = 0$ in M_0. Thus by virtue of (5. 7), we have

$$f_k{}^r h_{rj} = \alpha f_{kj},$$

from which

(5. 10) $$h_{ji} = \alpha g_{ji}.$$

Therefore we proved

LEMMA 5. 2. *If M_0 is not empty, the hypersurface M $(n>2)$ is umbilical at each point of M_0.*

In the next place, let M_1 be the set of all umbilical point of M. Then, we see that $M-M_1$ is an open set in M and we have from Lemma 5. 2 $M-M_1 \subset M-M_0$. Hence, by definition of M_0, we get $q^2=1$ in $M-M_1$. Thus we get $f_j = p_j = 0$ in $M-M_1$ by virtue of (2. 16). We assume now that $M-M_1$ is not empty. Then $M-M_1$ being open in M, if we differentiate $f_i=0$ covariantly and take account of (2. 18), we obtain

$$q g_{ji} + f_i{}^s h_{sj} = 0.$$

If we take the symmetric part of this tensor equation and take account of (5. 1), we get $q g_{ji} = 0$. Thus we have $q=0$ in $M-M_1$, which contradicts the condition $q^2=1$. Therefore the set $M-M_1$ is necessarily empty.

Summing up the results obtained above, we get

THEOREM 5.3. *Let M $(n>2)$ be an orientable connected hypersurface of a Sasakian manifold $\tilde{M}$. If the contact form η over $\tilde{M}$ is not tangent to M almost everywhere and if f commutes with h, then the hypersurface M is totally umbilical.*

Combining Theorem 3.3 and 5.3, we have immediately the main theorem stated in § 1.

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