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A THEORY OF RIEMANNIAN SUBMANIFOLDS

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As is well known, the most useful method of studying the properties of a curve in a euclidean space or more generally Riemannian space, from the standpoint of differential geometry, is making use of the Frenet formula, in which the first curvature and the second curvature and so forth are the essential quantities for the curve. Regarding a submanifold of dimension $\geqq 2$ in a higher dimensional space, the situation is quite different from the case of curves. We have, at present, only the Gauss and Weingarten formulas and the Gauss and Codazzi equations as the complete integrability conditions for the formers, in which the system of the second fundamental tensors corresponding to the normal unit vectors are performing important rôles. Recently, O'Neill [2～5] obtained many interesting results on submanifolds considering the system of the second fundamental tensors as an operator from the tangent space to the normal space at each point of a submanifold which is called the shape operator. This idea is an interesting method for treating submanifolds but it seems to the author that there remains a direction of exploring analogous formulas and quantities for submanifolds to those of curves.

Let C: $x=x(s)$ be a C^∞ curve in the euclidean n-space E^n parameterized with arclength s and let $(x(s), e_1, \cdots, e_n)$ be the field of its Frenet frames. Then, we have the Frenet formulas:

$$\frac{dx}{ds}=e_1, \qquad \frac{de_1}{ds}=k_1(s)e_2,$$

$$\frac{de_2}{ds}=-k_1(s)e_1+k_2(s)e_3, \quad \cdots,$$

$$\frac{de_i}{ds}=-k_{i-1}(s)e_{i-1}+k_i(s)e_{i+1}, \quad \cdots,$$

$$\frac{de_n}{ds}=-k_{n-1}(s)e_{n-1}.$$

For any normal vector $e=\sum_{\alpha=2}^n \xi_\alpha e_\alpha$ at $x(s)$, we have

$$\frac{d^2x}{ds^2}\cdot e=k_1(s)\xi_1.$$

Hence we can consider the first curvature $k_1(s)$ as

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$$k_1(s)=\max\left\{\frac{d^2x}{ds^2}\cdot e,\ e\in N_{x(s)}=e_1{}^{\perp},\ |e|=1\right\}, \tag{0. 1}$$

where $N_{x(s)}$ denotes the normal space to the tangent line $T_{x(s)}C$ of C at $x(s)$. e_2 is uniquely determined as a normal unit vector by

$$k_1(s)=\frac{d^2x}{ds^2}\cdot e_2 \tag{0. 2}$$

when $k_1(s)\neq 0$. Since $k_1(s)$ is differentiable on a subarc of C in which $k_1(s)\neq 0$, e_2 is also differentiable on it. Then, the field $e_2(s)$ defines a linear transformation

$$\varphi_{\perp}\colon\ T_{x(s)}C\ \to\ N_{x(s)}\cap e_2{}^{\perp} \tag{0. 3}$$

at each point such that $k_1(s)\neq 0$ by

$$\varphi_1(X)=\sum_{\alpha>2}(X\cdot e_1)\left(\frac{de_2}{ds}\cdot e_\alpha\right)e_\alpha=\sum_{\alpha>2}(de_2(X)\cdot e_\alpha)e_\alpha, \tag{0. 4}$$

then the second curvature $k_2(s)$ of C can be considered as

$$k_2(s)=\max\{|\varphi_1(e)|,\ e\in T_{x(s)}C,\ |e|=1\}. \tag{0. 5}$$

Making use of these interpretations of the curvatures of the curve, we may define the curvatures for a submanifold. For simplicity, let M^n be an n-dimensional submanifold in the euclidean space E^{n+N}. Let $(p, e_1, \cdots, e_{n+N})$, $p\in M^n$, be an orthonormal frame of E^{n+N} at p such that $e_1, \cdots, e_n$ are tangent and $e_{n+1}, \cdots, e_{n+N}$ are normal to M^n at p. Then for any normal vector $e=\sum_{\alpha=n+1}^{n+N}\xi_\alpha e_\alpha$ at p, the quantity

$$\bar{m}(e)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{d^2x}{ds^2}\cdot e\right)\bigg|_{\frac{dx}{ds}=e_i}\ {}^{1)}$$

does not depend on the choice of the frame $(p, e_1, \cdots, e_n)$ of M^n and is linear on the normal space N_p to M^n at p. Then, we can define the first curvature of M^n at p by

$$k_1(p)=\max\{\bar{m}(e),\quad e\in N_p,\quad |e|=1\}, \tag{0. 6}$$

which is a continuous scalar of M^n. If $k_1(p)\neq 0$, there exists an uniquely determined normal unit vector $\bar{e}$ such that

$$k_1(p)=\bar{m}(\bar{e}). \tag{0. 7}$$

On the domain of M^n such that $k_1\neq 0$, $\bar{e}$ is differentiable. Then, the field $\bar{e}$ defines a linear transformation

$$\varphi_1\colon\ T_pM^n\ \to\ N_p\cap\bar{e}^{\perp}$$

at each point such that $k_1(p)\neq 0$ by

$$\varphi_1(X)=\sum_{\alpha>n+1}(d\bar{e}(X)\cdot e_\alpha)e_\alpha,\qquad X\in T_pM^n. \tag{0. 8}$$

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1) In the right hand side, $\left(\frac{d^2x}{ds^2}\cdot e\right)\Big|_{\frac{dx}{ds}=e_i}$ denotes $\left(\frac{d^2x(s)}{ds^2}\cdot e\right)_{s=0}$ where $x(s)$ is a smooth curve through p, $x(0)=p$, and $\frac{dx(s)}{ds}\Big|_{s=0}=e_i$.

Then the second curvature $k_2(p)$ of M^n can be defined by

$$k_2(p)=\{\max|\varphi_1(e)|,\quad e\in T_pM^n,\quad |e|=1\}. \tag{0.9}$$

In this paper, the author will show that we can introduce the concepts of the Frenet frame, the Frenet formulas and some kinds of curvatures for submanifolds in the theory of Riemannian submanifolds according to the methods mentioned above and investigate a special immersions of Riemannian manifolds by making use of these concepts.

§1. Preliminaries.

Let $\boldsymbol{S}_n$ be the set of all real symmetric square matrices of order n. We consider it as a vector space over the real field $\boldsymbol{R}$ and the orthogonal group $O(n, \boldsymbol{R})$ of order n operates on it as follows: For any $A\in\boldsymbol{S}_n$, $T\in O(n, \boldsymbol{R})$

$$T(A)=T'AT, \tag{1.1}$$

where T' denotes the transposed matrix of T.

Now we define an inner product of any two elements A, B in $\boldsymbol{S}_n$ by

$$\langle A, B\rangle=\frac{1}{n}\ \mathrm{trace}\,(AB), \tag{1.2}$$

then we have

$$||A||=\sqrt{\langle A, A\rangle}=\sqrt{\frac{\mathrm{trace}\,(A^2)}{n}}\quad \text{and}\quad ||I_n||=1, \tag{1.3}$$

where I_n denotes the unit matrix in $\boldsymbol{S}_n$,

Let m: $\boldsymbol{S}_n\to\boldsymbol{R}$ be a linear transformation defined by

$$m(A)=\frac{1}{n}\mathrm{trace}\,A \tag{1.4}$$

and put $\boldsymbol{M}_n=m^{-1}(0)$, that is

$$\boldsymbol{M}_n=\{A|\mathrm{trace}\,A=0,\quad A\in\boldsymbol{S}_n\}. \tag{1.5}$$

We call an element of $\boldsymbol{M}_n$ *minimal* in the following. Since we have

$$\langle A, I_n\rangle=m(A),\qquad A\in\boldsymbol{S}_n,$$

$\boldsymbol{S}_n$ is decomposed in a direct sum as follows:

$$\boldsymbol{S}_n=\boldsymbol{M}_n\oplus\boldsymbol{R}I_n,\qquad \boldsymbol{M}_n\perp\boldsymbol{R}I_n. \tag{1.6}$$

Let ρ: $\boldsymbol{S}_n\to\boldsymbol{M}_n$ be the projection according to the direct sum (1.6), that is

$$\rho(A)=A-m(A)I_n. \tag{1.7}$$

Now, we define real valued functions P_r: $\boldsymbol{S}_n\to\boldsymbol{R}$ of order r, $r=0, 1, \cdots, n$, by the equality:

$$\det (I_n+tA)=\sum_{r=0}^{n}\binom{n}{r}t^r P_r(A), \qquad A\in \boldsymbol{S}_n,$$

where t is an auxiliary variable. We have especially

$$P_0(A)=1, \quad P_1(A)=m(A), \quad P_n(A)=\det A$$

and

$$P_2(A)=\frac{2}{n(n-1)}\sum_{i<j}(a_{ii}a_{jj}-a_{ij}a_{ji})$$
$$=\frac{1}{n(n-1)}\{(\text{trace } A)^2-\text{trace } A^2\},$$

that is

(1. 8) $$P_2(A)=\frac{n}{n-1}(m(A))^2-\frac{1}{n-1}||A||^2.$$

From the above equation we get easily the following

LEMMA 1. *P_2 is a non-singular quadratic form on $\boldsymbol{S}_n$, negative definite on $\boldsymbol{M}_n$ and of index* $\dim \boldsymbol{M}_n=(n+2)(n-1)/2$ *on $\boldsymbol{S}_n$.*

Substituting $A=\rho(A)+m(A)I_n$ in (1. 8), we get

$$P_2(A)=\frac{n}{n-1}m^2(A)-\frac{1}{n-1}||\rho(A)+m(A)I_n||^2$$
$$=\frac{n}{n-1}m^2(A)-\frac{1}{n-1}\{||\rho(A)||^2+m^2(A)\},$$

that is

(1. 9) $$P_2(A)=m^2(A)-\frac{1}{n-1}||\rho(A)||^2.$$

We get from (1. 9) the classical result.

LEMMA 2. *For any symmetric matrix A of order n, we have*

$$\{P_1(A)\}^2\geqq P_2(A)$$

and the equality holds if and only if $A\in \boldsymbol{R}I_n$.

Lastly, we remark that the mappings, functions, subspaces in this section are all invariant under the group $O(n, \boldsymbol{R})$.

§ 2. The first curvature.

Let $M=M^n$ and $\bar{M}=\bar{M}^{n+N}$ be two differentiable Riemannian manifolds of dimension n and $n+N$ respectively and let ψ: $M\to\bar{M}$ be an isometric immersion. Let $F(M)$ and $F(\bar{M})$ be the bundles of orthonormal frames of M and $\bar{M}$ with the projections π: $F(M)\to M$ and $\bar{\pi}$: $F(\bar{M})\to\bar{M}$. Let B be the set of elements $b=(p, e_1, e_2, \cdots, e_{n+N})$ such that

$$(p, e_1, e_2, \cdots, e_n)\in F(M), \qquad (\psi(p), e_1, e_2, \cdots, e_{n+N})\in F(\bar{M}),$$

identifying e_i with $d\psi(e_i)$, $i=1, 2, \cdots, n$. B is considered as a differentiable manifold in a natural way. Then, let $\bar{\psi}$: $B\to F(\bar{M})$ be the natural immersion covering ψ. We denote the basic forms and connection forms of $\bar{M}$ by

$$\bar{\omega}_A, \quad \bar{\omega}_{AB}=-\bar{\omega}_{BA}, \quad A, B=1, 2, \cdots, n+N$$

and the induced ones on B by $\bar{\psi}$ from $\bar{\omega}_A$, $\bar{\omega}_{AB}$ by

$$\omega_A=\psi^*\omega_A, \qquad \omega_{AB}=\psi^*\bar{\omega}_{AB}.$$

Then we have, as is well known,

$$\omega_\alpha=0, \quad \omega_{i\alpha}=\sum_{j=1}^{n} A_{\alpha ij}\omega_j, \quad \alpha=n+1, \cdots, n+N.\text{[2)]} \tag{2.1}$$

For any normal unit vector $e=\sum_\alpha \xi_\alpha e_\alpha$, $\sum_\alpha \xi_\alpha^2=1$, we have the second fundamental from Φ_e defined by

$$\Phi_e(\omega)=\sum_{\alpha,i,j} \xi_\alpha A_{\alpha ij}\omega_i\omega_j. \tag{2.2}$$

We denote the normal space to $\psi(M)$ at $\psi(p)$ by

$$N_p=\left\{X\colon X=\sum_{\alpha=n+1}^{n+N} \xi_\alpha e_\alpha,\ \xi_\alpha\in R\right\}$$

and define a linear mapping $\bar{m}$: $N_p\to\boldsymbol{R}$ by

$$\bar{m}(X)=\sum_{\alpha=n+1}^{n+N} \xi_\alpha m(A_\alpha), \qquad A_\alpha=(A_{\alpha ij}). \tag{2.3}$$

Since $m(A)$, $A\in\boldsymbol{S}_n$, is invariant under $O(n, \boldsymbol{R})$, the mapping $\bar{m}$ is well defined. We denote the kernel of $\bar{m}$ at p by ${}^{M-}N_p$ and call its element *minimal*. We denote the unit sphere in N_p by ${}^{N-}S_p$ and define the first curvature at p by

$$k_1(p)=\max\{\bar{m}(e), \quad e\in {}^{N-}S_p\}. \tag{2.4}$$

It is clear that the function k_1: $M\to\boldsymbol{R}$ is continuous and differentiable on the domain $U_1=\{p\colon p\in M,\ k_1(p)>0\}$ of M. When M is a curve in E^3, k_1 is clearly the first (principal) curvature of M.

At any point p of U_1, there exists a uniquely determined normal unit vector $\bar{e}_{n+1}\in N_p$ such that

$$k_1(p)=\bar{m}(\bar{e}_{n+1}). \tag{2.5}$$

$\bar{e}_{n+1}$ is a differentiable vector field on U_1. On U_1, we take only the frame $(p, e_1, \cdots, e_{n+N})$ such that $e_{n+1}=\bar{e}_{n+1}$. Then we have

2) In this note, Latin indices $i, j, \cdots$ run from 1 to n and Greek indices $\alpha, \beta, \cdots$ take values in $\{n+1, n+2, \cdots, n+N\}$.

(2. 6) $$\bar{m}(e_\alpha)=0, \qquad n+1<\alpha.$$

Accordingly for any normal unit vector $e=\sum_{\alpha=n+1}^{n+N}\xi_\alpha e_\alpha$ at $p\in U_1$, we have

(2. 7) $$\bar{m}(e)=\xi_{n+1}k_1(p),$$

hence

$$-k_1(p)\leqq\bar{m}(e)\leqq k_1(p), \qquad p\in M.$$

The condition that M is a minimal submanifold in $\bar{M}$ is its first curvature $k_1\equiv 0$ on M.

§ 3. The M-index of a submanifold.

At any point $p\in M$, we take a frame $b=(p, e_1, \cdots, e_{n+N})\in B$. Let ψ_b: $N_p\to \boldsymbol{S}_n$ be the linear mapping defined by

(3. 1) $$\psi_b\left(\sum_{\alpha=n+1}^{n+N}\xi_\alpha e_\alpha\right)=\sum_{\alpha=n+1}^{n+N}\xi_\alpha A_\alpha, \qquad A_\alpha=(A_{\alpha ij}).$$

Making use of the functions P_r on $\boldsymbol{S}_n$, $r=0, 1, \cdots, n$, defined in § 1, we define functions $\bar{P}_r$: $N_p\to\boldsymbol{R}$ by

(3. 2) $$\bar{P}_r\left(\sum_{\alpha=n+1}^{n+N}\xi_\alpha e_\alpha\right)=P_r\left(\sum_{\alpha=n+1}^{n+N}\xi_\alpha A_\alpha\right).$$

Since P_r on $\boldsymbol{S}_n$ is invariant under $O(n, \boldsymbol{R})$, the above defined $\bar{P}_r$ on N_p is well defined, that is, independent of the choice of the frame b at p. $\bar{P}_1$ is identical with $\bar{m}$ in § 2.

By means of Lemma 1, $\bar{P}_2$ is a quadratic form on the vector space N_p and negative semi-definite on ${}^{M-}N_p$. We call the dimension of $\psi_b({}^{M-}N_p)$ the M-*index* of M at p and denote it by M-index ${}_pM$.

LEMMA 3. M-index ${}_pM\leqq\min\left\{\dim {}^{M-}N_p, \dfrac{(n-1)(n+2)}{2}\right\}$.

Proof. For a frame $b=(p, e_1, \cdots, e_{n+N})\in B$, $\psi_b({}^{M-}N_p)\subset\boldsymbol{M}_n$ and

$$\dim \boldsymbol{M}_n=\dim \boldsymbol{S}_n-1=\frac{n(n+1)}{2}-1=\frac{(n-1)(n+2)}{2}.$$

Since P_2 is negative semi-definite on $\boldsymbol{M}_n$, we get easily the above inequality.

Now, we put M-index ${}_pM=\iota$. If p is not a minimal point, using only such frame $b=(p, e_1, \cdots, e_{n+N})$ that $e_{n+1}=\bar{e}_{n+1}$ in § 2, then $\{e_{n+2}, \cdots, e_{n+N}\}$ is an orthonormal base of ${}^{M-}N_p$. For any $X=\sum_{\alpha=n+2}^{n+N}\xi_\alpha e_\alpha\in{}^{M-}N_p$, we get by (1. 8)

(3. 3) $$\bar{P}_2(X)=-\frac{1}{n-1}\langle A, A\rangle=-\frac{1}{n-1}\sum_{\alpha,\beta>n+1}\langle A_\alpha, A_\beta\rangle\xi_\alpha\xi_\beta,$$

where $A=\sum\xi_\alpha A_\alpha$. Accordingly we can choose a frame such that

$$(3.4)\qquad \begin{cases} A_{n+2}\neq 0, \quad A_{n+3}\neq 0, \cdots, A_{n+\iota+1}\neq 0, \\ A_{n+\iota+2}=A_{n+\iota+3}=\cdots=A_{n+N}=0, \\ \langle A_\alpha, A_\beta\rangle=0, \quad \alpha\neq\beta, \quad \alpha,\beta=n+2,\cdots,n+\iota+1. \end{cases}$$

If p is a minimal point, we can choose a frame analogously such that

$$(3.5)\qquad \begin{cases} A_{n+1}\neq 0, \quad A_{n+2}\neq 0, \cdots, A_{n+\iota}\neq 0, \\ A_{n+\iota+1}=A_{n+\iota+2}=\cdots=A_{n+N}=0, \\ \langle A_\alpha, A_\beta\rangle=0, \quad \alpha\neq\beta, \quad \alpha,\beta=n+1,\cdots,n+\iota. \end{cases}$$

§ 4. The second curvature and the Frenet decomposition.

On U_1, we have the mean curvature normal unit vector field $\bar{e}_{n+1}$. Let B_1 be the subset of all $b\in B$ such that $b=(p, e_1, e_2, \cdots, e_{n+N})$, $p\in U_1$, $e_{n+1}=\bar{e}_{n+1}$. B_1 can be considered as a submanifold of B. Making use of B_1, we have

$$(4.1)\qquad \bar{D}\bar{e}_{n+1}=\sum_i \omega_{\underline{n+1}\,i}e_i+\sum_{n+1<\beta}\omega_{\underline{n+1}\,\beta}e_\beta,$$

where $\bar{D}$ denotes the covariant derivative of $\bar{M}$ along M. Clearly

$$\omega_{\underline{n+1}\,\beta}=0 \quad (\mathrm{mod}\ \omega_1, \cdots, \omega_n).$$

Making use of these relations, we define a linear mapping

$$\varphi_1\colon\ M_p=T_p(M)\to {}^{M-}N_p$$

by

$$(4.2)\qquad \varphi_1(X)=\sum_{\beta=n+2}^{n+N}\omega_{\underline{n+1}\,\beta}(X)e_\beta, \quad X\in M_p.$$

We can easily see that φ_1 is well defined. We denote the tangent unit sphere of M_p by $S_p=\{X\colon X\in M_p, \|X\|=1\}$. φ_1 is linear, hence $\varphi_1(S_p)$ is an elliptic surface with some dimension ($\leqq n-1$) in ${}^{M-}N_p$. We define the second curvature of M at $p\in U_1$ by

$$(4.3)\qquad k_2(p)=\max\{\|\varphi_1(e)\|, \quad e\in S_p\}.$$

Clearly k_2 is continuous on U_1 and differentiable on

$$U_2=\{p\colon p\in U_1, \quad k_2(p)\neq 0\}.$$

If M is a curve in E^3, then k_2 is its torsion (or the second curvature). Since we have

$$\|\varphi_1(e)\|^2=\sum_{i,j,\beta}\omega_{\underline{n+1}\,\beta}(e_i)\omega_{\underline{n+1}\,\beta}(e_j)\xi_i\xi_j, \qquad e=\sum_i \xi_i e_i,$$

we can choose a frame $(p, e_i, \cdots, e_n)\in F(M)$ such that

$$\|\varphi_1(e)\|^2 = k_{21}^2(\xi_{n+2}^2 + \cdots + \xi_{n+\beta_1+1}^2) + \cdots$$
$$+ k_{22}^2(\xi_{n+\beta_1+2}^2 + \cdots + \xi_{n+\beta_1+\beta_2+1}^2) + \cdots.$$

Accordingly, we get a decomposition of M_p and a decomposition ${}^{M-}N_p$ as follows:

$$(4.4) \qquad \begin{cases} M_p = E_{p,1}^{\beta_1} \oplus E_{p,2}^{\beta_2} \oplus \cdots \oplus E_{p,\sigma}^{\beta_\sigma} \oplus E_{p,0}^{n-N_2}, \\ {}^{M-}N_p = \bar{E}_{p,1}^{\beta_1} \oplus \bar{E}_{p,2}^{\beta_2} \oplus \cdots \oplus \bar{E}_{p,\sigma}^{\beta_\sigma} \oplus \bar{E}_{p,\infty}^{N-N_2-1} \end{cases}$$

where $\oplus$ denote the orthogonal direct sum and $\beta_1 + \cdots + \beta_\sigma = N_2 \leqq n$, such that

$$\varphi_1 | E_{p,\tau}^{\beta_\tau}\colon E_{p,\tau}^{\beta_\tau} \to \bar{E}_{p,\tau}^{\beta_\tau}$$

is a homothety with magnification $k_{2\tau}$, $\tau = 1, 2, \cdots, \sigma$ and

$$(4.5) \qquad k_2 = k_{21} > k_{22} > \cdots > k_{2\sigma} > 0$$

and

$$\varphi_1(E_{p,0}^{n-N_2}) = 0.$$

If $\beta_1, \beta_2, \cdots, \beta_\sigma$ are constants, then $k_{21}, k_{22}, \cdots, k_{2\sigma}$ are scalars on U_1. In such case, we take a frame $b = (p, e_1, \cdots, e_{n+N}) \in B_1$ such that

$$\{e_1, \cdots, e_{\beta_1}\},\ \{e_{\beta_1+1}, \cdots, e_{\beta_1+\beta_2}, \cdots\},$$
$$\{e_{\beta_1+\cdots+\beta_{\sigma-1}+1}, \cdots, e_{\beta_1+\cdots+\beta_\sigma}\}$$

are the orthonormal bases of $E_{p,1}^{\beta_1}, \cdots, E_{p,\sigma}^{\beta_\sigma}$, respectively and if $e_i \in E_{p,\tau}^{\beta_\tau}$, $\varphi_1(e_i) = k_{2\tau} e_{n+1+i}$, $\tau = 1, 2, \cdots, \sigma$, then we get

$$(4.6) \qquad \begin{aligned} D\bar{e}_{n+1} = \sum_i \omega_{\underline{n+1}i} e_i &+ k_{21}(e_{n+2}\omega_1 + \cdots + e_{n+\beta_1+1}\omega_{\beta_1}) \\ &+ k_{22}(e_{n+\beta_1+2}\omega_{\beta_1+1} + \cdots + e_{n+\beta_1+\beta_2+1}\omega_{\beta_1+\beta_2}) + \cdots \\ &+ k_{2\sigma}(e_{n+N_2-\beta_\sigma+2}\omega_{N_2-\beta_\sigma+1} + \cdots + e_{n+N_2+1}\omega_{N_2}). \end{aligned}$$

Furthermore, making use of LEMMA 1, we can take frames b such that

$$(4.7) \qquad \begin{gathered} \langle \psi_b(e_\alpha),\ \psi_b(e_\beta) \rangle = 0, \qquad \|\psi_b(e_\alpha)\| \geqq \|\psi_b(e_\beta)\|, \\ e_\alpha,\ e_\beta \in \bar{E}_{p,\tau}^{\beta_\tau}, \quad \tau = 1, 2, \cdots, \sigma, \quad \text{or} \quad \bar{E}_{p,\infty}^{N-N_2-1}, \quad \alpha < \beta. \end{gathered}$$

We call a frame b satisfying the condition (4.6) and (4.7) a *Frenet frame* at p and the decomposition of M_p in (4.4) the *Frenet decomposition* of the tangent space at p.

§ 5. Relations between the Riemannian curvature and the scalars of a submanifold $M \Subset \overline{M}$.

On B, we denote the curvature forms of M by Ω_{ij} and the induced forms from the curvature forms of $\bar{M}$ on $F(\bar{M})$ through $\bar{\phi}\colon B \to F(\bar{M})$ by $\bar{\Omega}_{AB}$. Then we have

$$\begin{aligned}\Omega_{ij}&=d\omega_{ij}-\sum_{k=1}^{n}\omega_{ik}\wedge\omega_{kj}\\&=d\omega_{ij}-\sum_{B=1}^{n+N}\omega_{iB}\wedge\omega_{Bj}+\sum_{\alpha>n}\omega_{i\alpha}\wedge\omega_{\alpha j}\\&=\bar{\Omega}_{ij}+\sum_{\alpha>n}\omega_{i\alpha}\wedge\omega_{\alpha j}=\bar{\Omega}_{ij}-\sum_{\alpha,h,k}A_{\alpha ih}A_{\alpha jk}\omega_h\wedge\omega_k\end{aligned}$$

which are written in components as

$$R_{ijhk}=\bar{R}_{ijhk}+\sum_{\alpha=n+1}^{n+N}(A_{\alpha ik}A_{\alpha jh}-A_{\alpha ih}A_{\alpha jk}), \tag{5.1}$$

where R_{ijhk} are defined by $\Omega_{ij}=(1/2)\sum_{h,k}R_{ijhk}\omega_h\wedge\omega_k$ and $\bar{R}_{ijhk}$ are the functions on B induced by $\bar{\phi}$ from the components of the curvature forms of $\bar{M}$ on $F(\bar{M})$. Contracting with respect to j and h, we get

$$R_{ik}=\bar{R}_{ik}-\sum_{\alpha}\bar{R}_{i\alpha\alpha k}+\sum_{\alpha}\{nm(A_\alpha)A_{\alpha ik}-(A_\alpha{}^2)_{ik}\}, \tag{5.2}$$

where R_{ik} and $\bar{R}_{AB}$ are the components of Ricci tensors of M and $\bar{M}$. Furthermore, contracting (5.2), we get

$$R=\bar{R}-\sum_{\alpha}\bar{R}_{\alpha\alpha}+\sum_{i,\alpha}\bar{R}_{i\alpha i\alpha}+\sum_{\alpha}\{n^2m^2(A_\alpha)-n\|A_\alpha\|^2\}.$$

By means of (1.8), we have

$$R=\bar{R}-\sum_{\alpha}\bar{R}_{\alpha\alpha}+\sum_{i,\alpha}\bar{R}_{i\alpha i\alpha}+n(n-1)\sum P_2(A_\alpha),$$

that is

$$\operatorname{trace}_{N_p}P_2=\frac{1}{n(n-1)}\left\{R-\bar{R}+\sum_{\alpha}\bar{R}_{\alpha\alpha}-\sum_{i,\alpha}\bar{R}_{i\alpha i\alpha}\right\}. \tag{5.3}$$

From this formula and LEMMA 1, we get easily

THEOREM 1. *A Riemannian manifold with positive scalar curvature can not be isometrically imbedded (immersed) in a euclidean space as a minimal submanifold.*

Proof. If M with positive scalar curvature can be isometrically immersed in a Euclidean space E^{n+N} as a minimal submanifold, then we have $N_p={}^{M-}N_p$ at any point $p\in M$, hence $\operatorname{trace}_{N_p}P_2\leqq 0$. The right hand side of (5.3) is positive in the case. This is a contradiction.

Furthermore, we can generalize THEOREM 1 as follows.

THEOREM 2. *An n-dimensional Riemannian manifold M whose scalar curvature is everywhere greater than a constant c can not be isometrically imbedded (immersed) in an $(n+N)$-dimensional Riemannian manifold $\bar{M}$ of constant curvature $(n+N)(n+N-1)c/n(n-1)$ as a minimal submanifold.*

Proof. Let us suppose M is isometrically immersed in $\bar{M}$ as a minimal submanifold. Since $\bar{M}$ is a constant curvature, on B we have

$$\bar{R}_{ABCD}=\frac{\bar{R}}{(n+N)(n+N-1)}(\delta_{AD}\delta_{BC}-\delta_{AC}\delta_{BD}),$$

hence

$$\bar{R}_{i\alpha j\beta}=\frac{-\bar{R}}{(n+N)(n+N-1)}\delta_{ij}\delta_{\alpha\beta}$$

and

$$\bar{R}_{\alpha\beta}=\frac{\bar{R}}{n+N}\delta_{\alpha\beta}.$$

Accordingly, we get

$$\begin{aligned}
&R-\bar{R}+\sum_{\alpha}\bar{R}_{\alpha\alpha}-\sum_{i,\alpha}\bar{R}_{i\alpha i\alpha}\\
&=R-\bar{R}+\frac{N}{n+N}\bar{R}+\frac{nN}{(n+N)(n+N-1)}\bar{R}\\
&=R-\frac{n(n-1)}{(n+N)(n+N-1)}\bar{R}\\
&>c-\frac{n(n-1)}{(n+N)(n+N-1)}\bar{R}=0.
\end{aligned}$$

On the other hand, at any point p of M we have trace$_{N_p}P_2\leqq 0$, since $N_p={}^{M\text{-}}N_p$. This contradicts (5.3).

REMARK. If $R\geqq c$, then M can not be also isometrically immersed into an $(n+N)$-dimensional Riemannian manifold $\bar{M}$ of constant curvature $(n+N)(n+N-1)c/n(n-1)$ as a minimal submanifold with a positive M-index at some point of M.

Now, we consider the case that M is not minimal in $\bar{M}$ at each point. Using the notation in §4, assume that the M-index $i(p)$ of the immersion $M\Subset\bar{M}$ and the dimensions of the components of the decompositions of M_p and ${}^{M\text{-}}N_p$ in (4.4) are all constants.

Then, making use of Frenet frames, we may put

$$\text{(5.4)}\qquad \begin{cases}\omega_{\underline{n+1}\alpha}=\tilde{k}_\alpha\omega_{\alpha-n-1} & (\alpha=n+2,\cdots,n+1+N_2),\\ \omega_{\underline{n+1}\beta}=0 & (n+1+N_2<\beta).\end{cases}$$

Differentiating $\omega_{i\underline{n+1}}=\sum A_{\underline{n+1}ij}\omega_j$ and using the structure equations, we get

$$\begin{aligned}
d\omega_{i\underline{n+1}}&=\sum_j\omega_{ij}\wedge\omega_{j\underline{n+1}}+\sum_{n+1<\alpha}\omega_{i\alpha}\wedge\omega_{\alpha\underline{n+1}}+\bar{\Omega}_{i\underline{n+1}}\\
&=\sum_j A_{\underline{n+1}jh}\omega_{ij}\wedge\omega_h-\sum_{j,n+1<\alpha}\tilde{k}_\alpha A_{\alpha ij}\omega_j\wedge\omega_{\alpha-n-1}+\bar{\Omega}_{i\underline{n+1}},\\
d\omega_{i\underline{n+1}}&=\sum_j dA_{\underline{n+1}ij}\wedge\omega_j+\sum_{j,h}A_{n+1ij}\omega_h\wedge\omega_{hj}
\end{aligned}$$

hence

$$DA_{\underline{n+1}ij}\wedge\omega_j=-\sum_{j,\,n+1<\alpha\leqq n+1+N_2}\tilde{k}_\alpha A_{\alpha ij}\omega_j\wedge\omega_{\alpha-n-1}+\bar{\Omega}_{i\underline{n+1}}, \tag{5.5}$$

where

$$DA_{\underline{n+1}ij}=dA_{\underline{n+1}ij}+\sum_h A_{\underline{n+1}hj}\omega_{hi}+\sum_h A_{\underline{n+1}ih}\omega_{hj}$$

is the covariant differential of the tensor field $A_{\underline{n+1}ij}\,e_i\otimes e_j$ of M. Putting

$$DA_{\underline{n+1}ij}=\sum_h A_{\underline{n+1}ij;\,h}\omega_h,$$

we get from (5.5)

$$\begin{aligned}\sum_{j,h} A_{n+1ij;\,h}\omega_j\wedge\omega_h-\sum_{j,\,n+1<\alpha\leqq n+1+N_2}\tilde{k}_\alpha A_{\alpha ij}\omega_j\wedge\omega_{\alpha-n-1}&\\ +\frac{1}{2}\sum \bar{R}_{i\underline{n+1}jh}\omega_j\wedge\omega_h=0,&\end{aligned} \tag{5.5'}$$

that is

$$\begin{cases}A_{\underline{n+1}ia;\,b}-A_{\underline{n+1}ib;\,a}-\tilde{k}_{n+b+1}A_{\underline{n+b+1}ia}\\ \qquad +\tilde{k}_{n+a+1}A_{\underline{n+a+1}ib}+\bar{R}_{i\underline{n+1}ab}=0,\quad 1\leqq a<b\leqq N_2;\\ A_{\underline{n+1}ia;\,r}-A_{\underline{n+1}r;\,a}+\tilde{k}_{n+a+1}A_{\underline{n+a+1}ir}+\bar{R}_{i\underline{n+1}ar}=0,\quad 1\leqq a\leqq N_2<r\leqq n;\\ A_{\underline{n+1}ir;\,t}-A_{\underline{n+1}it;\,r}+\bar{R}_{i\underline{n+1}rt}=0,\quad N_2<r<t\leqq n.\end{cases} \tag{5.6}$$

This formula is a generalization of the Mainardi-Codazzi's equation in the case M is a hypersurface.

§ 6. Submanifolds of M-index 0 in euclidean spaces.

Let M be isometrically immersed in the euclidean space E^{n+N} and assume that M is everywhere of M-index 0 and not minimal. Let $\bar{e}_{n+1}$ be the mean curvature normal unit vector field on M. Then, on B_1 (in § 4), we have

$$\omega_{i\alpha}=0,\qquad i=1,2,\cdots,n,\qquad \alpha=n+2,\cdots,n+N. \tag{6.1}$$

By means of the structure equations, we get

$$0=d\omega_{i\alpha}=\sum_j\omega_{ij}\wedge\omega_{j\alpha}+\omega_{i\underline{n+1}}\wedge\omega_{\underline{n+1}\alpha}+\sum_{n+1<\beta}\omega_{i\beta}\wedge\omega_{\beta\alpha},$$

that is

$$\omega_{i\underline{n+1}}\wedge\omega_{\underline{n+1}\alpha}=0. \tag{6.2}$$

Since $k_1(p)=m(A_{n+1})=(1/n)\operatorname{trace}A_{n+1}\neq 0$, we have

$$\operatorname{rank}A_{n+1}\geqq 1.$$

Let ν be the index of relative nullity of $M\Subset E^{n+N}$ in the sense of Chern and Kuiper [1], then by virtue of (6.1) we have

$$\operatorname{rank}A_{n+1}=n-\nu. \tag{6.3}$$

Case I: $\nu \leqq n-2$.

From (6.2), we have

$$\omega_{\underline{n+1}\alpha}=0, \quad \alpha=n+2, \cdots, n+N.$$

$$dp=\sum_i \omega_i e_i, \quad de_i=\sum_j \omega_{ij} e_j+\omega_{in+1} e_{n+1}, \quad de_{n+1}=-\sum_i \omega_{i\underline{n+1}} e_i.$$

This follows that there exists an $(n+1)$-dimensional linear subspace E^{n+1} in E^{n+N} such that $M^n \Subset E^{n+1}$.

Now, we suppose that ν is constant. We use only such frames $b=(p, e_1, \cdots, e_{n+N})$ that

(6.4) $$A_{\underline{n+1}ai}=0, \quad a=1, 2, \cdots, \nu, \quad i=1, \cdots, n.$$

Then we have

$$\omega_{a\underline{n+1}}=0,$$

from which

$$0=d\omega_{a\underline{n+1}}=\sum_{b\leqq\nu} \omega_{ab} \wedge \omega_{b\underline{n+1}}+\sum_{\nu<r\leqq n} \omega_{ar} \wedge \omega_{r\underline{n+1}}+\sum_{\beta>n+1} \omega_{a\beta} \wedge \omega_{\beta\underline{n+1}}$$

that is

$$\sum_{\nu<r\leqq n} \omega_{ar} \wedge \omega_{r\underline{n+1}}=0.$$

Hence, by means of (6.3) and (6.4), we have

$$\omega_{ar} \equiv 0 \quad (\mathrm{mod}\ \omega_{\nu+1}, \omega_{\nu+2}, \cdots, \omega_n).$$

Accordingly we get

$$d\omega_r=\sum \omega_i \wedge \omega_{ir} \equiv 0 \quad (\mathrm{mod}\ \omega_{\nu+1}, \omega_{\nu+2}, \cdots, \omega_n).$$

Hence, the system of Pfaff equations:

(6.5) $$\omega_{\nu+1}=\omega_{\nu+2}=\cdots=\omega_n=0$$

is completely integrable. Let Q be an integral submanifold of (6.5), then we have along Q the equations:

$$dp=\sum_{a\leqq\nu} \omega_a e_a, \quad de_a=\sum_{b\leqq\nu} \omega_{ab} e_b \quad de_r=\sum_{t>\nu} \omega_{rt} e_t, \quad de_{n+1}=0.$$

These follow that Q is a ν-dimensional linear subspace and $\bar{e}_{n+1}$ is parallel along Q in E^{n+N}. We denote the integral submanifold through $p \in M$ by $E^{\nu}(p)$.

Case II: $\nu=n-1$.

We use only such frames $b=(p, e_1, \cdots, e_{n+N})$ that

(6.6) $$\omega_{a\underline{n+1}}=0 \ (a=1, \cdots, n-1), \ \omega_{n\underline{n+1}}=\lambda\omega_n \ (\lambda \neq 0).$$

Then, from (6.2), $\omega_{\underline{n+1}\alpha}$ can be written as

(6.7) $$\omega_{\underline{n+1}\alpha}=\rho_\alpha \omega_n (n+1<\alpha).$$

From the first part of (6.6), we get

$$0=d\omega_{\underline{an+1}}=\sum_j \omega_{aj}\wedge\omega_{j\underline{n+1}}+\sum_{\beta>n+1}\omega_{a\beta}\wedge\omega_{\beta n+1}=\lambda\omega_{an}\wedge\omega_n.$$

Hence ω_{an} can be written as

$$\omega_{an}=\mu_a\omega_n \qquad (a=1, 2, \cdots, n-1).$$

Analogously as in Case I, the Pfaff equation

$$\omega_n=0 \tag{6.8}$$

is completly integrable. In this case, we have the equations

$$dp=\sum_i \omega_i e_i, \qquad de_a=\sum_{b\leqq n-1}\omega_{ab}e_b+\mu_a\omega_n e_n,$$

$$de_n=\Big(-\sum_{a\leqq n-1}\mu_a e_a+\lambda e_{n+1}\Big)\omega_n,$$

$$de_{n+1}=\Big(-\lambda e_n+\sum_{\alpha>n+1}\rho_\alpha e_\alpha\Big)\omega_n.$$

These show that an integral submanifold of (6.8) is an $(n-1)$-dimensional linear subspace in E^{n+N}. We denote the integral submanifold through p by $E^{n-1}(p)$. Along $E^{n-1}(p)$, e_n and $\bar{e}_{n+1}$ are parallelly displaced in E^{n+N}.

In general, for any submanifold $M^n \Subset E^{n+N}$ which is not minimal at every point, we define a mapping Φ: $M^n \to S_0^{n+N-1}$ (the unit hypersphere in E^{n+N} with center at the origin) by $\Phi(p)=\bar{e}_{n+1}(p)$, $p\in M$. We call *Φ the spherical mean curvature mapping* of M^n.

Now, returning to Case II, the mapping Φ is constant on each integral submanifold. Therefore the image M under Φ is a curve on S_0^{n+N-1} and its tangent vector is $-\lambda e_n+\sum_{\alpha>n+1}\rho_\alpha e_\alpha$. In order that there exists an $(n+1)$-dimensional linear subspace E^{n+1} such that $M\Subset E^{n+1}$, it is necessary and sufficient that

$$\rho_{n+2}=\rho_{n+3}=\cdots=\rho_{n+N}=0,$$

that is

$$k_2(p)=\Big(\sum_{\alpha>n+1}\rho_\alpha^2\Big)^{1/2}=0,$$

where k_2 is the second curvature vector of M. In other words, we can say that any orthogonal trajectory of the family of $E^{n-1}(p)$ and its image under Φ have the parallel tangents at the corresponding points.

THEOREM 3. *Let M^n be an n-dimensional isometrically immersed submanifold in E^{n+N} which is everywhere not minimal and of M-index* 0. *Let assume that the index of relative nullity of M^n in E^{n+N} ν is constant.*

Then, $\nu\leqq n-1$ and there exists a ν-dimensional assymptotic linear submanifold $E^\nu(p)\subset M^n$ through any point $p\in M^n$. Along $E^\nu(p)$, the mean curvature normal unit vector field $\bar{e}_{n+1}$ is parallel in E^{n+N}. Furthermore, the following holds:

I) *If $\nu\leqq n-2$, then there exists an $(n+1)$-dimensional linear subspace E^{n+1} such that $E^{n+1}\Supset M^n$.*

II) *If* $\nu=n-1$, *then the normal unit vector* e_n *along* $E^{n-1}(p)$ *in* M^n *is also parallel in* E^{n+N} *and in order that there exists an* $E^{n+1}\supseteq M^n$, *it is necessary and sufficient that* $k_2\equiv 0$ *or the image of an orthogonal trajectory of the family of* $E^{n-1}(p)$ *under the spherical mean curvature mapping has the same tangent direction with the one of the trajectory at the corresponding points.*

COROLLARY. *For any immersed submanifold* M^n *in* E^{n+N} *which is everywhere not minimal, the necessary and sufficient conditions in order that there exists an* $(n+1)$*-dimensional linear subspace* $E^{n+1}\supseteq M^n$ *are*

i) *the M-index of* M^n *is every where zero, and*

ii) *the second curvature* $k_2\equiv 0$.

REMARK. If $M^n\subseteq E^{n+N}$ is everywhere of M-index 0 and minimal, then M^n is totally geodesic and so M^n is an n-dimensional Euclidian space E^n or its subdomain.

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