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SUBMANIFOLDS OF A KÄHLERIAN MANIFOLD

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Dedicated to Professor Hitoshi Hombu on his sixtieth birthday

1. Introduction. The following theorem is well known:

THEOREM A (Cf. Yano [3]). *A holomorphic submanifold of a Kählerian manifold is minimal.*

Thus it would be natural to ask whether a minimal submanifold of a Kählerian manifold is holomorphic. As to a very special case of a totally geodesic submanifold we have shown in [1] the following

THEOREM B. *A totally geodesic submanifold S in a $2n$-dimensional Fubinian manifold M is holomorphic if we assume that $n \neq 2$ and the codimension of S is 2. In the exceptional case $n=2$, S is a holomorphic or an anti-holomorphic submanifold of M.*

A submanifold S is said to be *anti-holomorphic at a point $p \in S$*, if $T_p(S)$ and $N_p(S)$ are transformed under F into each other, where F is an almost complex structure of M, $T_p(S)$ and $N_p(S)$ denoting respectively the tangent space to S at p and the normal space to S at p. S is called an *anti-holomorphic submanifold* if it is anti-holomorphic at each point of S.

Theorem B shows that the converse of Theorem A is not true in general. Now we shall study, in this paper, submanifolds, especially minimal ones, in a Kählerian manifold. The notations and terminologies are found in [1], but we state some of them at the beginning of the next section for the later use.

2. Let M be a Kählerian manifold of real dimension $2n$ and S[1] a connected orientable submanifold of M whose real dimension is $2n-2$. It is well known that a Riemannian metric g on S can be induced from the Riemannian metric G of M. We denote by $\langle\ ,\ \rangle_M$ the inner product with respect to G and by $\langle\ ,\ \rangle_S$ the inner product with respect to g. We now put, for a tangent vector X on S,

$$F(\xi_* X) = T(X) + N(X), \tag{2.1}$$

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1) We use here an identification of a differentiable manifold S with $\xi(S)$, where ξ is a differentiable immersion from S into M, whose differential ξ_*: $T_p(S) \to T_{\xi(p)}(M)$ is injective. Manifolds, mappings and geometric objects considered in this paper are all assumed to be of differentiability class C^∞.

where F is the almost complex structure of M, $T(X)$ denotes the tangential part and $N(X)$ the normal part, both of $F(\xi_* X)$. Since $T(X)$ is tangent to S, we may put

$$\langle T(X), \xi_* Y\rangle_M = \langle AX, Y\rangle_S,$$

where A is a tensor on S of type (1, 1) and Y is an arbitrary vector on S. If we define a 2-form $\tilde{A}$ by

$$\tilde{A}(X, Y) = \langle AX, Y\rangle_S$$

for any pair of vector fields X and Y on S, then we have, denoting by $\tilde{F}$ the fundamental 2-form of M,

$$\tilde{A}(X, Y) = \tilde{F}(\xi_* X, \xi_* Y). \tag{2. 2}$$

(2. 2) shows that $\tilde{A}$ is a skew-symmetric bilinear form.

Now, we restrict ourselves to a sufficiently small neighborhood $\mathcal{U}$ in which there exist two fields of unit normal vectors to S. First, we fix in $\mathcal{U}$ two unit normal vector fields C and D to S which are mutually orthogonal. Then $N(X)$ defined by (2. 1) can be expressed in $\mathcal{U}$ as

$$N(X) = \tilde{\alpha}(X)C + \tilde{\beta}(X)D, \tag{2. 3}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are 1-forms on S. We have

$$\tilde{\alpha}(X) = \tilde{F}(\xi_* X, C) \tag{2. 4}$$

and

$$\tilde{\beta}(X) = \tilde{F}(\xi_* X, D) \tag{2. 5}$$

for any vector fields X on S. We define $\|\alpha\|$ and $\|\beta\|$ respectively by

$$\|\alpha\| = \sqrt{\langle\alpha, \alpha\rangle_S} \qquad \text{and} \qquad \|\beta\| = \sqrt{\langle\beta, \beta\rangle_S},$$

where α and β are contravariant tensors of degree 1 defined by $\langle\alpha, X\rangle_S = \tilde{\alpha}(X)$ and $\langle\beta, X\rangle_S = \tilde{\beta}(X)$ respectively. Then we have, by a direct computation,

$$\|\alpha\|^2 = \|\beta\|^2 = 1 - \varphi^2, \tag{2. 6}$$

where

$$\varphi = \tilde{F}(C, D). \tag{2. 7}$$

$\tilde{F}(C, D)$ seems to depend upon the choice of the pair of unit normal vector fields C and D in $\mathcal{U}$, but it is not hard to show that $\tilde{F}(C, D)$ is independent of the choice of C and D. That is to say, $\tilde{F}(C, D)$ is left invariant under any orthogonal transformation applied to C and D, since S is assumed to be orientable. Thus φ is a globally defined function on S. On the other hand, (2. 6) implies that if $\tilde{\alpha}(X)=0$ at a point p for any vector X on S, then $\tilde{\beta}(X)=0$ at p and vice versa. Straightforward computation shows that

(2.8) $$\langle\alpha, \beta\rangle_S=0$$

and

(2.9) $$A^2=-I+\tilde{\alpha}\otimes\alpha+\tilde{\beta}\otimes\beta,$$

where I is the unit tensor.

The *maximal holomorphic subspace* H_p of the tangent space $T_p(S)$ to S at p is defined by

$$H_p=\{V\in T_p(S)|F(\xi_* V)\in T_p(S)\}$$

and the *anti-holomorphic subspace* H'_p of $T_p(S)$ is defined by

$$H'_p=\{W\in T_p(S)|F(\xi_* W)\in N_p(S)\},$$

where $N_p(S)$ denotes the normal space to S at p. These definitions show that

$$H_p\oplus H'_p=T_p(S) \qquad \text{(direct sum)}$$

and H_p and H'_p are mutually orthogonal. In fact

$$\langle V, W\rangle_S=\langle\xi_* V, \xi_* W\rangle_M=\langle F(\xi_* V), F(\xi_* W)\rangle_M=0,$$

if $V\in H_p$ and $W\in H'_p$. If we restrict ourselves to $\mathcal{U}$ and if we take account of (2.3), then we see that a necessary and sufficient condition that V belongs to H_p is expressed as

(2.10) $$\langle V, \alpha\rangle_S=0 \qquad \text{and} \qquad \langle V, \beta\rangle_S=0.$$

The identity (2.9) and the equations (2.10) show that A restricted to H_p is an almost complex structure. We note again from (2.10) that H'_p is spanned by α and β at p at which $||\alpha||\neq 0$. Thus we have

$$\dim H_p\geqq \dim S-2$$

and the equality holds at p at which we have $||\alpha||=0$.

The next proposition is a result of a direct computation.

PROPOSITION 2.1. *If S is a totally geodesic submanifold of M, then the function φ defined by* (2.7) *is constant and therefore* $\dim H_p$ *is constant on S.*

We shall assume, from now on, that there is at least one point p at which $||\alpha||$ does not vanish.

An assignment of H_p to each p of S defines a distribution D, if $\dim H_p$ is constant. Let X and Y be any local vector fields which belong to H_p in a sufficiently small neighborhood $\mathcal{V}$. It is well known that the distribution D is completely integrable if and only if $[X, Y]$ is also a local vector field belonging to H_p in $\mathcal{V}$. This condition is equivalent to

$$\langle [X, Y], \alpha\rangle_S = 0 \quad \text{and} \quad \langle [X, Y], \beta\rangle_S = 0.^{2)} \tag{2.11}$$

The equations (2.11) can be written as

$$\begin{aligned} &\langle \nabla_X \alpha, Y\rangle_S - \langle \nabla_Y \alpha, X\rangle_S = 0, \\ &\langle \nabla_X \beta, Y\rangle_S - \langle \nabla_Y \beta, X\rangle_S = 0 \end{aligned} \tag{2.12}$$

by virtue of

$$\langle X, \alpha\rangle_S = \langle X, \beta\rangle_S = \langle Y, \alpha\rangle_S = \langle Y, \beta\rangle_S = 0,$$

where ∇ denotes the covariant differentiation along S with respect to the connection induced on S from the Riemannian connection of M. On the other hand we have, from (2.4) and (2.5),

$$\langle \nabla_X \alpha, Y\rangle_S = -\varphi\tilde{k}(X, Y) - \tilde{h}(AY, X) + \tilde{l}(X)\tilde{\beta}(Y) \tag{2.13}$$

and

$$\langle \nabla_X \beta, Y\rangle_S = \varphi\tilde{h}(X, Y) - \tilde{k}(AY, X) - \tilde{l}(X)\tilde{\alpha}(Y), \tag{2.14}$$

where $\tilde{h}$ and $\tilde{k}$ are the second fundamental forms of S and $\tilde{l}$ the third fundamental form of S, X and Y being arbitrary vector fields on S. Thus the equations (2.12) become

$$\begin{cases} \langle (Ah + hA)X, Y\rangle_S = 0, \\ \langle (Ak + kA)X, Y\rangle_S = 0, \end{cases} \tag{2.15}$$

h and k being tensors on S of type (1, 1) defined respectively by $\langle hX, Y\rangle_S = \tilde{h}(X, Y)$ and $\langle kX, Y\rangle_S = \tilde{k}(X, Y)$. Thus we have

PROPOSITION 2.2. *Suppose that* $\dim H_p$ *is constant on* S. *In order that the distribution* D: $p \to H_p$ *is completely integrable, it is necessary and sufficient that the equations* (2.15) *are valid for arbitrary vectors* X *and* Y *on* H_p.

This proposition, together with Proposition 2.1, gives

COROLLARY 2.1. *For a totally geodesic submanifold* S *of* M, *the distribution* D: $p \to H_p$ *is always completely integrable.*

In the case in which the distribution D: $p \to H_p$ is completely integrable, the integral manifold H of the distribution D is a minimal submanifold of M (Theorem A). We denote submanifold maps by η: $H \to S$ and ζ: $H \to M$ and their differentials by η_* and ζ_* respectively. We shall use, in the neighborhood $\mathcal{U}$, C, D, $\xi_*\alpha/||\alpha||$ and $\xi_*\beta/||\beta||$ as unit normal vector fields to H and denote them by C_1, C_2, C_3 and C_4 respectively. Then we have

2) Vectors X and Y on H_p are regarded as vectors on $T_p(S)$ by the identification map.

$$\langle C_x, C_y\rangle_M=\delta_{xy} \qquad (x, y=1, 2, 3, 4).$$

Now we introduce van der Waerden-Bortolotti derivative along a submanifold of M. Let M' be a submanifold of M and σ a submanifold map from M' into M whose differential is denoted by σ_*. We denote by $T^r_s(M)$ (resp. $T^r_s(M')$) the space of all tensor fields of type (r, s) and let $T(M)=\sum_{r,s} T^r_s(M)$ (resp. $T(M')=\sum_{r,s} T^r_s(M')$). Given an element $X\in T^1_0(M')$, we define a derivation ∇^σ_X in the formal tensor product $T(M)\# T(M')$ by the following properties:

1) $$\nabla^\sigma_X V=\nabla_{\sigma_* X} V \qquad \text{for} \qquad V\in T(M),$$

where ∇ denotes the covariant derivation with respect to an affine connection of M;

2) $$\nabla^\sigma_X W=(\text{the tangential part of } \nabla_{\sigma_* X}(\sigma_* W)),$$

for $W\in T(M')$ and

3) $$\nabla^\sigma_X(V\# W)=(\nabla^\sigma_X V)\# W+V\#(\nabla^\sigma_X W),$$

for $V\in T(M)$ and $W\in T(M')$. *Van der Waerden-Bortolotti derivative* ∇^σ *along* M' is defined as the assignment: $(X, W^*)\to\nabla^\sigma_X W^*$ for $X\in T^1_0(M')$ and $W^*\in T(M)\# T(M')$. For detail, see Yano-Ishihara [2].

Van der Waerden-Bortolotti derivative ∇^ζ along H as a submanifold of M gives

$$\langle \nabla^\zeta_V C_x, \zeta_* W\rangle_M=-\langle h^{(x)} V, W\rangle_H, \tag{2.16}$$

where V and W are tangent to H, each $h^{(x)}$ is a tensor on H of type $(1,1)$ and $\langle\ ,\ \rangle_H$ denotes the inner product on H with respect to the metric induced from that of M. C_1 and C_2 are respectively transformed under F as follows:

$$FC_1=\varphi C_2-\|\alpha\| C_3 \tag{2.17}$$

and

$$FC_2=-\varphi C_1-\|\alpha\| C_4. \tag{2.18}$$

Substituting (2.17) into (2.16) we have

$$\langle F\nabla^\zeta_V C_1, \zeta_* W\rangle_M=-\varphi\langle h^{(2)} V, W\rangle_H+\|\alpha\|\langle h^{(3)} V, W\rangle_H,$$

because of the fact that F is covariant constant. Since $\zeta_* W$ is tangent to the holomorphic submanifold H so is also $F(\zeta_* W)$ and therefore we can put

$$F(\zeta_* W)=\zeta_*(fW), \tag{2.19}$$

where f is a tensor of type $(1,1)$ on H. We can easily show that

$$f^2=-I,$$

I being the unit tensor. Thus we have

$$\langle h^{(1)} V, fW\rangle_H=-\varphi\langle h^{(2)} V, W\rangle_H+\|\alpha\|\langle h^{(3)} V, W\rangle_H, \tag{2.20}$$

by virtue of the relation

$$\langle F\nabla^{\zeta}_{V}C_1, \zeta_* W\rangle_M = -\langle \nabla^{\zeta}_{V}C_1, F(\zeta_* W)\rangle_M.$$

A similar method gives

$$\langle h^{(2)}V, fW\rangle_H = \varphi\langle h^{(1)}V, W\rangle_H + ||\alpha||\langle h^{(4)}V, W\rangle_H, \tag{2.21}$$

because of (2.18).

On the other hand, if we consider H as a submanifold of S and we choose $\alpha/||\alpha||$ and $\beta/||\beta||$ as fields of unit normals to H, then we have

$$\langle \nabla^{\eta}_{V}(\alpha/||\alpha||), \eta_* W\rangle_S = -\langle h' V, W\rangle_H, \tag{2.22}$$

and

$$\langle \nabla^{\eta}_{V}(\beta/||\beta||), \eta_* W\rangle_S = -\langle k' V, W\rangle_H, \tag{2.22}$$

where V and W are tangent to H and ∇^{η} denotes van der Waerden-Bortolotti covariant derivation along H as a submanifold of S. h' and k' are the so-called second fundamental tensors of H in S. We can easily verify, by the definition of van der Waerden-Bortolotti covariant derivation that

$$h' = h^{(3)} \qquad \text{and} \qquad k' = h^{(4)}, \tag{2.24}$$

if we take account of (2.16). By a similar argument we have

$$\langle h^{(1)}W, V\rangle_H = \langle h(\eta_* W), \eta_* V\rangle_S \tag{2.25}$$

and

$$\langle h^{(2)}W, V\rangle_H = \langle k(\eta_* W), \eta_* V\rangle_S \tag{2.26}$$

where V and W are vector fields on H. Since H is holomorphic submanifold of M, we have

$$\mathrm{Tr}\, h^{(x)} = 0 \qquad (x = 1, 2, 3, 4)$$

and therefore

$$\mathrm{Tr}\, h' = 0 \qquad \text{and} \qquad \mathrm{Tr}\, k' = 0.$$

These equations imply

PROPOSITION 2.3. *The integral manifold H of the distribution D: $p \to H_p$ is a minimal submanifold of S.*

We also have

$$\mathrm{Tr}\, h = (\langle h\alpha, \alpha\rangle_S + \langle h\beta, \beta\rangle_S)/||\alpha||^2$$

and

$$\mathrm{Tr}\, k = (\langle k\alpha, \alpha\rangle_S + \langle k\beta, \beta\rangle_S)/||\alpha||^2$$

by virtue of (2.25) and (2.26). Thus we have

PROPOSITION 2.4. *If S is a minimal submanifold of M and the integrability condition* (2.15) *of the distribution D is satisfied, then we have*

$$\langle h\alpha, \alpha\rangle_S+\langle h\beta, \beta\rangle_S=0 \tag{2.27}$$

and

$$\langle k\alpha, \alpha\rangle_S+\langle k\beta, \beta\rangle_S=0. \tag{2.28}$$

We can write the equations (2.15) as

$$\langle h^{(1)}X, fY\rangle_H-\langle h^{(1)}Y, fX\rangle_H=0$$

and

$$\langle h^{(2)}X, fY\rangle_H-\langle h^{(2)}Y, fX\rangle_H=0,$$

if we take account of (2.19), (2.25) and (2.26).

A tensor T of type (1,1) is said to be *hybrid with respect to f*, if it satisfies

$$fT+Tf=0,$$

where f is a tensor of type (1,1). (See, e.g. Yano [3].)

Thus we have, taking account of (2.20),

PROPOSITION 2.5. *Under the integrability condition* (2.15) *of the distribution D: $p\rightarrow H_p$, each $h^{(x)}$ is hybrid tensor with respect to the almost complex structure f on H induced from the almost complex structure F of M.*

Let us define $\tilde{h}^{(x)}$ by

$$\tilde{h}^{(x)}(V, W)=\langle h^{(x)}V, W\rangle_H \qquad (x=1, 2, 3, 4)$$

for any pair of vectors V and W on H. Then we can see, from (2.25) and (2.26), that $\tilde{h}^{(1)}$ and $\tilde{h}^{(2)}$ are both symmetric bilinear form. As consequences of (2.15), we see that $\tilde{h}^{(3)}$ and $\tilde{h}^{(4)}$ become symmetric when the distribution D is integrable.[3)]

3. We study, in this section, the integrability condition of the distribution D' which assigns H'_p to $p\in S$. If we use α and β as a local basis of D' in a sufficiently small neighborhood of p, then the integrability condition of D': $p\rightarrow H'_p$ is written as

$$\langle X_\mu, [\alpha, \beta]\rangle_S=0, \tag{3.1}$$

where X_μ is the local basis of the distribution D: $p\rightarrow H_p$. The equation (3.1) is written as

$$\langle X_\mu, (hA-Ah)\beta-(kA-Ak)\alpha\rangle_S=0 \tag{3.2}$$

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3) If we define, in a sufficiently small neighborhood $\mathcal{V}$, van der Waerden-Bortolotti covariant derivative along a distribution D and introduce tensors $L^{(x)}$ of type (1, 1) in a similar way as we did for $h^{(x)}$, i.e. by (normal part of $\nabla_X Y)=\tilde{L}^{(x)}(X, Y)C_x$ $(x=1, 2, 3, 4)$, for local vector fields $X, Y\in D$ then the integrability condition of the distribution D is given by the hybridness of $L^{(1)}$ and $L^{(2)}$ or equivalently by the symmetry of $\tilde{L}^{(3)}$ and $\tilde{L}^{(4)}$.

or

$$\varphi(A_\mu+\bar{B}_\mu)+f_\mu{}^\nu(B_\nu-\bar{A}_\nu)=0, \tag{3.3}$$

because of $A\alpha=-\varphi\beta$ and $A\beta=\varphi\alpha$, where $A_\mu=\langle X_\mu, h\alpha\rangle_S$, $B_\mu=\langle X_\mu, h\beta\rangle_S$, $\bar{A}_\mu=\langle X_\mu, k\alpha\rangle_S$, $\bar{B}_\mu=\langle X_\mu, k\beta\rangle_S$ and $(f_\mu{}^\nu)$ are components of the tensor f on H. Thus we have

PROPOSITION 3.1. *Suppose that* $\dim H'_p=\text{const}$. *In order that the distribution* D': $p\to H'_p$ *is completely integrable, it is necessary and sufficient that the equation* (3.2) *or* (3.3) *holds.*

COROLLARY 3.1. *If S is a totally geodesic submanifold of M, then the distribution D' is completely integrable.*

On the other hand, we have

$$\langle X_\mu, \operatorname{grad}\varphi\rangle_S=-B_\mu+\bar{A}_\mu$$

and thus we have

COROLLARY 3.2. *For a submanifold S on which $\varphi=0$, the distribution D' is completely integrable.*

The number of equations (3.3) is $m-2$ and that of unknown variables A_μ, B_μ, $\bar{A}_\mu$ and $\bar{B}_\mu$ is $4(m-2)$. Therefore, it seems that a submanifold which satisfies the integrability condition (3.3) of the distribution D' is a very special one. We shall show, at the end of this section, an example of such submanifolds. In that example, the second fundamental tensors h and k which are considered as linear transformations on $T_p(S)$ leave invariant the holomorphic subspace H_p of $T_p(S)$.

When the distribution D': $p\to H'_p$ is integrable, we denote by H' the integral manifold of the distribution D' and by ζ' a submanifold map ζ': $H'\to M$. We can choose $\xi_* X_\lambda$, C, D as unit normal vector fields to H'. By using van der Waerden-Bortolotti covariant derivation $\nabla^{\zeta'}$ along H' we have

$$\langle \nabla^{\zeta'}_X C, \zeta'_* Y\rangle_M=-\langle 'h^{(m-1)}X, Y\rangle_{H'},$$

$$\langle \nabla^{\zeta'}_X D, \zeta'_* Y\rangle_M=-\langle 'h^{(m)}X, Y\rangle_{H'}$$

and

$$\langle \nabla^{\zeta'}_X \zeta_* X_\lambda, \zeta'_* Y\rangle_M=-\langle 'h^{(\lambda)}X, Y\rangle_{H'},$$

where X and Y are arbitrary vector fields on H' and $'h^{(\lambda)}$, $'h^{(m-1)}$ and $'h^{(m)}$ are the second fundamental tensors of H' as a submanifold of M. Since we have chosen α and β as a local basis of H', we have

$$\|\alpha\|^2 \operatorname{Tr} 'h^{(m-1)}=(\langle h\alpha, \alpha\rangle_S+\langle h\beta, \beta\rangle_S)/\|\alpha\|^2 \tag{3.4}$$

and

$$\|\alpha\|^2 \operatorname{Tr} 'h^{(m)}=(\langle k\alpha, \alpha\rangle_S+\langle k\beta, \beta\rangle_S)/\|\alpha\|^2. \tag{3.5}$$

On the other hand, we have

$$\|\alpha\|^2 \operatorname{Tr} 'h^{(\lambda)}=\varphi(B_\lambda-\bar{A}_\lambda)-f_\lambda{}^\mu(A_\mu+\bar{B}_\mu),$$

from which we obtain

$$\mathrm{Tr}\,'h^{(\lambda)}=-f_{\lambda}{}^{\mu}(A_{\mu}+\bar{B}_{\mu}),$$

by virtue of (3. 3). On the other hand, we have, by a straightforward computation,

$$\langle X_{\mu},\ \mathrm{div}\ A\rangle_{S}=A_{\mu}+\bar{B}_{\mu},$$

where A is the tensor defined in § 2. This proves

PROPOSITION 3. 2. *A necessary condition that H' is a minimal submanifold of M is*

1) *The second fundamental tensors h and k of S satisfy* (3. 4) *and* (3. 5) *respectively and*

2) $\mathrm{div}\ A\in H'_p$.

Conversely, if we assume 1) *and* 2) *mentioned above and we assume further* $\mathrm{grad}\ \varphi\in H'_p$ *and* $\dim H'_p=\mathrm{const.}$, *then the distribution D': $p\to H'_p$ is completely integrable and the integral manifold H' of the distribution D' is a minimal submanifold of M.*

We shall now discuss a sufficient condition under which the distributions D: $p\to H_p$ and D': $p\to H'_p$ are both integrable. We assume that the following equations are valid for any local vector field X of the distribution D:

$$\begin{cases}(Ah+hA)X=0,\\ (Ak+kA)X=0.\end{cases}\tag{3. 6}$$

The equation (2. 15) shows that (3. 6) is one of sufficient conditions under which the distribution D is completely integrable. The next lemma is a result of a direct computation

LEMMA 3. 1. *Under the condition* (3. 6), *we have*

$$A_{\mu}=B_{\mu}=\bar{A}_{\mu}=\bar{B}_{\mu}=0\tag{3. 7}$$

and therefore $\mathrm{grad}\ \varphi$ *and* $\mathrm{div}\ A$ *belong to* H'_p.

(3. 7) proves

PROPOSITION 3. 3. *If we assume* (3. 6), *then the holomorphic subspace H_p of $T_p(S)$ is left invariant under the linear transformations induced from the second fundamental tensors h and k of S.*

PROPOSITION 3. 4. *The distributions D and D' are both completely integrable, if we assume* $\dim H_p=\mathrm{const.}$ *and the equations* (3. 6).

LEMMA 3. 2. *Let S be a minimal submanifold of M. If the equations* (3. 6) *are valid for any $X \in H_p$, then we have*

$$\begin{cases} Ah+hA=0, \\ Ak+kA=0 \end{cases} \tag{3.8}$$

on $T_p(S)$.

Conversely, a submanifold S whose second fundamental tensors h and k satisfy the equations (3.8), *then S is a minimal submanifold of M, if φ does not vanish.*

Proof.

Straightforward computations show that

$$\begin{aligned}\langle (Ah+hA)\alpha, \alpha\rangle_S &= \langle Ah\alpha, \alpha\rangle_S - \varphi\langle h\beta, \alpha\rangle_S \\ &= \varphi\langle h\alpha, \beta\rangle_S - \varphi\langle h\beta, \alpha\rangle_S = 0; \\ \langle (Ah+hA)\alpha, \beta\rangle_S &= \langle Ah\alpha, \beta\rangle_S - \varphi\langle h\beta, \beta\rangle_S \\ &= -\varphi\langle h\alpha, \alpha\rangle_S - \varphi\langle h\beta, \beta\rangle_S \\ &= 0, \qquad \text{by } (2.27); \\ \langle (Ah+hA)\beta, \alpha\rangle_S &= -\langle (Ah+hA)\alpha, \beta\rangle_S = 0;\end{aligned}$$

and

$$\langle (Ah+hA)\beta, \beta\rangle_S = -\varphi\langle h\beta, \alpha\rangle_S + \varphi\langle h\alpha, \beta\rangle_S = 0.$$

We have, from (3.6),

$$\langle (Ah+hA)\alpha, X\rangle_S = \langle (Ah+hA)\beta, X\rangle_S = 0$$

for any $X \in H_p$. These equations give

$$(Ah+hA)\alpha = (Ah+hA)\beta = 0. \tag{3.9}$$

Similar computations show that

$$(Ak+kA)\alpha = (Ak+kA)\beta = 0. \tag{3.10}$$

The equations (3.8) follow from (3.6), (3.9) and (3.10).

The converse is now obvious by a straightforward computation. q.e.d.

COROLLARY 3.3. *Let S be a minimal submanifold of M. We assume that the equations* (3.6) *are valid and further the function φ is constant. Then $\tilde{A}$ defined by* (2.2) *is harmonic form.*

Proof. From the definition of $\tilde{A}$ and the equation

$$\begin{aligned}(\nabla_X \tilde{A})(Y, Z) = &-\tilde{h}(X, Y)\tilde{\alpha}(Z) + \tilde{h}(X, Z)\tilde{\alpha}(Y) \\ &-\tilde{k}(X, Y)\tilde{\beta}(Z) + \tilde{k}(X, Z)\tilde{\beta}(Y),\end{aligned}$$

it is obvious that $\tilde{A}$ is skew-symmetric and closed. We can easily see that

$$\langle \text{div } A, \alpha\rangle_S = \langle h\alpha, \alpha\rangle_S + \langle k\beta, \alpha\rangle_S$$

and

$$\langle \text{div } A, \beta\rangle_S = \langle h\alpha, \beta\rangle_S + \langle k\beta, \beta\rangle_S.$$

On the other hand, we have

$$\langle \text{grad } \varphi, \alpha\rangle_S = \langle k\alpha, \alpha\rangle_S - \langle h\beta, \alpha\rangle_S$$

and

$$\langle \text{grad } \varphi, \beta\rangle_S = \langle k\alpha, \beta\rangle_S - \langle h\beta, \beta\rangle_S,$$

from which we have

$$\langle \text{div } A, \alpha\rangle_S = \langle h\alpha, \alpha\rangle_S + \langle h\beta, \beta\rangle_S$$

and

$$\langle \text{div } A, \beta\rangle_S = \langle k\alpha, \alpha\rangle_S + \langle k\beta, \beta\rangle_S.$$

The right hand side of each equation above must be zero because of (2. 27) and (2. 28). Since div A belongs to H_p' (Lemma 3. 1), we have div $A=0$ which implies, together with $d\tilde{A}=0$, that $\tilde{A}$ is a harmonic form. q.e.d.

Summing up the results, we have

THEOREM 3. 1. *Let S be a minimal submanifold of a Kählerian manifold M whose codimension is* 2.[4] *We assume that* $\dim H_p=$const. *and the second fundamental tensors h and k of S satisfy the condition* (3. 6). *Then S is locally decomposed into two submanifolds one of which is holomorphic in M and the other is anti-holomorphic in M both being minimal submanifolds of S and at the same time of M. The dimension of the anti-holomorphic submanifold equals to the codimension of S.*

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4) We have assumed, throughout this paper, that codimension of S is 2, but we can also discuss in the same way the case in which the condimension of S is even and smaller than or equals to the half of the dimension of M.