# SOME METRICAL THEOREMS ON FUCHSIAN GROUPS

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**1.** Let $G$ be a Fuchsian group of linear transformations $S_n(z)$ $(n=0,1,2,\dots)$, which make $|z|<1$ invariant and $D_0$ be its fundamental domain, which contains $z=0$ and is bounded by orthogonal circles to $|z|=1$ and a closed set $e_0$ on $|z|=1$. We remark that $D_0$ can be so constructed that the equivalent points on the boundary of $D_0$ are equidistant from $z=0$[1]. Let $z_n$, $D_n$, $e_n$ be equivalents of $z_0=0$, $D_0$, $e_0$ respectively.

Theorem 1. If $me_0>0$, then $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$. The converse is not true in general.

Proof. Since $me_0>0$, we have $me_n>0$ $(n=0,1,2,\dots)$. Let

$$u_n(z)=\int_{e_n}\frac{1-|z|^2}{|z-e^{i\theta}|^2}\,d\theta ,$$

then $u_n(z_n)=u(0)=me_0$, so that

$$me_0=u_n(z_n)\leqq\frac{2\,me_n}{1-|z_n|},$$

hence

$$\sum_{n=0}^{\infty}(1-|z_n|)\leqq\frac{2\sum_{n=0}^{\infty}me_n}{me_0}\leqq\frac{4\pi}{me_0}<\infty .$$

Let $K_1,\dots,K_n$ $(n\geqq 3)$ be $n$ circles on the $w$-plane, which lie outside each other. We invert them on any one of them indefinitely, then we obtain infinitely many circles clustering to a non-dense perfect set $E$. As Myrberg[2] proved, $E$ is of positive logarithmic capacity, so that if we map the outside of $E$ on $|z|<1$ by $w=f(z)$, then $f(z)$ is automorphic to a certain Fuchsian group $G$, such that $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$. On the other hand, as I have proved in a former paper,[3] $me_0=0$. Hence the converse is not true in general.

**2.** Let $z$ be any point in $|z|<1$ and $(z)$ be its equivalent in $D_0$. Let $z=re^{i\theta}$ $(0\leqq r<1)$ be a radius through $e^{i\theta}$. We denote the set $(re^{i\theta})(0\leqq r<1)$ by $E(\theta)$. Then

Theorem 2. (i) If $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$, then $\lim_{r\to 1}|(re^{i\theta})|=1$ for almost all $e^{i\theta}$. (ii) If $\sum_{n=0}^{\infty}(1-|z_n|)=\infty$, then $E(\theta)$ is everywhere dense in $D_0$ for almost all $e^{i\theta}$.

I have proved this theorem in a former paper,[4] but my proof depends on a theorem, which is false. A proof is given by Yûjôbô.[5] I will give the following proof, which is somewhat simpler than his. In the proof, we use the following lemma.[6] Let $E_0$ be a closed set in $D_0$, which is of positive logarithmic capacity and $E_n$ be its equivalents. We take off $\sum_{n=0}^{\infty}E_n$ from $|z|<1$ and let $\Delta$ be the remaining domain. We map $\Delta$ on $|\zeta|<1$ and let $\sum_{n=0}^{\infty}E_n$ be mapped on a set $e$ on $|\zeta|=1$. Then

Lemma. (i) If $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$, then $0<me<2\pi$. (ii) If $\sum_{n=0}^{\infty}(1-|z_n|)=\infty$, then $me=2\pi$.

Proof of Theorem 2.

(i) Suppose that $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$, then the Green's function

$$G(z)=\sum_{n=0}^{\infty}\log\left|\frac{1-\bar{z}_n z}{z-z_n}\right| \tag{1}$$

exists and $\lim_{z\to e^{i\theta}}G(z)=0$ almost everywhere on $|z|=1$, when $z\to e^{i\theta}$ non-tangentially to $|z|=1$. From this, we see that $\lim_{r\to 1}|(re^{i\theta})|=1$ for almost all $e^{i\theta}$.

(ii) Next suppose that $\sum_{n=0}^{\infty}(1-|z_n|)=\infty$. Let $K_0$ be a disc contained in $D_0$ and $K_n$ be its equivalents and $C_n$ be its boundary. We take off $\sum_{\nu=0}^{\infty}K_\nu$ from $|z|<1$ and $\Delta$ be the remaining domain and we take off $\sum_{\nu=0}^{\infty}K_\nu$ from $|z|<1$ and $\Delta_n$ be the remaining domain.

Let $u_n(z)$ be a bounded harmonic function in $\Delta_n$, such that

$$\left.\begin{aligned} u_n(z)&=0 \quad \text{on} \quad \sum_{\nu=0}^{n}C_\nu ,\\ u_n(z)&=1 \quad \text{on} \quad |z|=1 . \end{aligned}\right\} \tag{2}$$

First we will prove that

$$\lim_{n\to\infty}u_n(z)\equiv 0 \quad \text{in } \Delta \tag{3}$$

We map $\Delta$ on $|\zeta|<1$, then by the lemma, $|z|=1$ is mapped on a null set on $|\zeta|=1$. Let $\sum_{\nu=n+1}^{\infty}C_\nu$ be mapped on a set $e_n$ on $|\zeta|=1$, then $\lim_{n\to\infty}me_n=0$. Let $u_n(z)$ become a harmonic function $v_n(\zeta)$ in $|\zeta|<1$, then, since $|z|=1$ is mapped on a null set, we have

$$v_n(\zeta)=\frac{1}{2\pi}\int_{e_n} v_n(e^{i\theta})\frac{1-|\zeta|^2}{|\zeta-e^{i\theta}|^2}\,d\theta .$$

Since $0<v_n(e^{i\theta})<1$ and $\lim_{n\to\infty} m e_n=0$, we have $\lim_{n\to\infty} v_n(\zeta)=0$, or $\lim_{n\to\infty} u_n(z)=0$ in $\Delta$, q.e.d.

Let $K: |z-a_0|<r_0$ be a disc contained in $D_0$. For any $0<\rho<1$, let $M(\rho)$ be the set of $e^{i\theta}$, such that the part of the radius $z=re^{i\theta}$ $(1-\rho<r<1)$ does not meet the equivalents of $K$. Then $M(\rho)$ is a closed set.

Let $K': |z-a_0|<r_0/2$. Since the segment $z=re^{i\theta}(1-\rho<r<1)$ does not meet the equivalents of $K$ and the non-euclidean distance is invariant by $S_n$, it is easily seen that for any $e^{i\theta}\in M(\rho)$, its neighbourhood:

$$\Delta(e^{i\theta},\delta): |z-e^{i\theta}|<\rho,\ |\arg(1-ze^{-i\theta})|<\delta$$

does not meet the equivalents of $K'$, if $\delta$ is sufficiently small, where $\delta$ depends on $a_0$ and $r_0$ only.

Then by the well known way, we can construct a rectifiable Jordan curve $\Lambda$ in $|z|<1$, such that $\Lambda$ meets $|z|=1$ in $M(\rho)$ and $\Delta(e^{i\theta},\delta)$ is contained in $\Lambda$ for any $e^{i\theta}\in M(\rho)$. We may assume that the equivalents of $K'$ lie outside $\Lambda$. If $mM(\rho)>0$, then there exists a bounded harmonic function $v(z)$ in $\Lambda$, such that $0<v(z)<1$ in $\Lambda$, $v(z)=0$ on $\Lambda-M(\rho)$ and $v(z)=1$ almost everywhere on $M(\rho)$. Let $u_n(z)$ be defined as before with respect to $K'$, then $0<v(z)\leqq u_n(z)$ on $\Lambda$, so that $0<v(z)\leqq u_n(z)$ in $\Lambda$. Since by (3), $\lim_{n\to\infty}u_n(z)=0$, we have $v(z)=0$, which is absurd. Hence $mM(\rho)=0$.

Let $\rho_1>\rho_2>\cdots>\rho_n\to 0$ and $M=\sum_{n=1}^{\infty}M(\rho_n)$, then $M$ is a null set and if $e^{i\theta}$ does not belong to $M$, then the radius through $e^{i\theta}$ meets the equivalents of $K$ infinitely often. Let $a_n$ $(n=1,2,\cdots)$ be rational points in $D_0$ and $r_1>r_2>\cdots>r_n\to 0$ and $K_{m,n}$: $|z-a_n|<r_n$, then there exists a null set $M_{mn}$ on $|z|=1$, such that if $e^{i\theta}$ does not belong to $M_{mn}$, then the radius through $e^{i\theta}$ meets the equivalents of $K_{mn}$ infinitely often. Hence if we put $M=\Sigma_{m,n}M_{mn}$, then $M$ is a null set and if $e^{i\theta}$ does not belong to $M$, then $E(\theta)$ is everywhere dense in $D_0$, q.e.d.

By modifying slightly the proof, we can prove

<u>Theorem 3.</u> If $\Sigma_{n=0}^{\infty}(1-|z_n|)=\infty$, then there exists a null set $M$ on $|z|=1$, such that if $e^{i\theta}$ does not belong to $M$, then for any segement through $e^{i\theta}$, its equivalent in $D_0$ is everywhere dense in $D_0$.

3. Suppose that $D_0$ has a boundary point on $|z|=1$ and for any $0<\rho<1$, let $D_0'(\rho)$ be the part of $D_0$, which lies in $1-\rho<|z|<1$. $D_0'(\rho)$ consists of a finite number of connected closed domains. We consider only such connected ones, which have boundary points on $|z|=1$ and let $D_0(\rho)$ be the sum of such domains and $D_n(\rho)$ be its equivalents and put

$$\Delta(\rho)=\sum_{n=0}^{\infty}D_n(\rho).$$

Then $\Delta(\rho)$ consists of a countable number of disjoint continua $\Delta_n(\rho)$, such that $\Delta(\rho)=\Sigma_{n=0}^{\infty}\Delta_n(\rho)$.

Since as remarked in § 1, equivalent points on the boundary of $D_0$ are equidistant from $z=0$, $\Delta_n(\rho)$ is bounded by Jordan arcs $\lambda_n^k$ $(k=0,1,2,\cdots)$ and a closed set $E_n$ on $|z|=1$. We put

$$E(\rho)=\sum_{n=0}^{\infty}E_n . \qquad (1)$$

$\lambda_n^k$ ends at two points $\xi_n^k$, $\eta_n^k$ on $|z|=1$, which are fixed points of some $S_m$. If $\xi_n^k=\eta_n^k$ for one $k$, then $\Delta_n(\rho)$ is bounded by a single Jordan curve, which touches $|z|=1$ at $\xi_n^k=\eta_n^k$. It is easily seen that if $\xi_n^k\neq\eta_n^k$, then $\lambda_n^k$ is contained between two circular arcs $C_n^k$, $C_n'^k$ through $\xi_n^k$, $\eta_n^k$, which meets $|z|=1$ with an angle $\alpha_n^k$, $\alpha_n'^k$ respectively. Since $\lambda_n^k(n,k=0,1,2,\cdots)$ can be grouped into a finite number of equivalent classes, there exists $\alpha,\beta$ $(0<\alpha<\beta<\pi)$, such that for any $\lambda_n^k$, for which $\xi_n^k\neq\eta_n^k$,

$$\alpha\leqq\alpha_n^k\leqq\beta,\quad \alpha\leqq\alpha_n'^k\leqq\beta \qquad (2)$$
$$(n,k=0,1,2,\cdots).$$

We will prove

<u>Theorem 4.</u> (i) If $\Sigma_{n=0}^{\infty}(1-|z_n|)<\infty$, then $mE(\rho)=2\pi$. (ii) if $\Sigma_{n=0}^{\infty}(1-|z_n|)=\infty$, then $mE(\rho)=0$. Hence if we put $E=\lim_{\rho\to 0}E(\rho)$, then

$$\left.\begin{aligned} &mE=2\pi,\ \text{if}\ \sum_{n=0}^{\infty}(1-|z_n|)<\infty,\\ &\text{and}\\ &mE=0,\ \text{if}\ \sum_{n=0}^{\infty}(1-|z_n|)=\infty.\end{aligned}\right\}(3)$$

For the proof, we use the following lemma.

<u>Lemma.</u> Let $K$ be a circle on the -plane, which meets a half-line $L$: $\arg z=\eta$ $(0<\eta\leqq\frac{\pi}{2})$ and the real axis at two points $A$, $B$ with an angle $\alpha>0$. We suppose that $A$ lies to the right of the origin $O$ and $B$ lies to the right of $A$. Let $M$ be

the middle point of $AB$. Then

$$AM \geqq OM \sin\alpha \tan\frac{\eta}{2}$$

Proof. First, denoting by $r$ the radius of $K$, we have $AM = r \cdot \sin\alpha$. Next, while we change the position of $K$ with fixed $r$, we see easily that $OM$ attains its maximum, when $K$ is inscribed in the angle between the positive real axis and $L$ and the maximum value is $\cot\frac{\eta}{2}$. Hence $r \geqq OM \cdot \tan\frac{\eta}{2}$, so that $AM \geqq OM \cdot \sin\alpha \cdot \tan\frac{\eta}{2}$. q.e.d.

4. Proof of Theorem 4.

(i) First suppose that $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$. Then $D_0$ has a boundary point on $|z|=1$, since, otherwise, $D_0$ lies in $|z| \leqq r_0 < 1$, so that we can prove easily that $\sum_{n=0}^{\infty}(1-|z_n|)=\infty$, which contradicts the hypothesis. Let $G(z)$ be the Green's function defined in the proof of Theorem 2, then $\lim_{r\to 1} G(re^{i\theta})=0$ for almost all $e^{i\theta}$. Let $e^{i\theta}$ be such a point, then, since $G(z) \geqq \alpha(\rho) > 0$ outside $\Delta(\rho)$, the segment $L(\theta,\delta)$: $z = re^{i\theta}$ $(1-\delta < r < 1)$ belongs to $\Delta(\rho)$ for a sufficiently small $\delta > 0$. Since $\Delta_n(\rho)$ are disjoint from each others, $L(\theta,\delta)$ belongs to a certain $\Delta_{n_0}(\rho)$, so that $e^{i\theta} \in E_{n_0}$. Since the set of $e^{i\theta}$ is of measure $2\pi$, we have $mE(\rho) = 2\pi$.

(ii) Next suppose that $\sum_{n=0}^{\infty}(1-|z_n|) = \infty$. We will prove $mE_n = 0$ $(n=0,1,2,\cdots)$. We may assume that $\xi_n^k \neq \eta_n^k$ for any $k$, since otherwise, $E_n$ reduces to one point as remarked above. Let $e^{i\theta}$ be a point of $E_n$. We divide $E_n$ into three parts $E_n^{(1)}$, $E_n^{(2)}$, $E_n^{(3)}$, according to whether (i) $L(\theta,\delta)$ meets $\sum_{k=0}^{\infty}\lambda_n^k$ for any $\delta > 0$, (ii) $L(\theta,\delta)$ belongs to $\Delta_n(\rho)$ for a value $\delta > 0$, or (iii) $L(\theta,\delta)$ lies outside $\Delta_n(\rho)$ for a value $\delta > 0$.

If $e^{i\theta} \in E_n^{(1)}$ and $\xi_n^k \neq \eta_n^k$, then by mapping $|z|<1$ on the upper half $w$-plane, such that $z=0$, $z=e^{i\theta}$ become $w=i$, $w=0$ and applying the lemma with $\eta = \frac{\pi}{2}$, we see that the lower density of $E_n$ at $e^{i\theta}$ is $\leqq 1-\sin\alpha < 1$, where $\alpha$ is defined by (2). Since by Lebesgue's theorem, the density is 1 almost everywhere on $E_n$, we have $mE_n^{(1)} = 0$.

If $e^{i\theta} \in E_n^{(2)}$, then the equivalent of $L(\theta,\delta)$ in $D_0$ lies in $1-\rho<|z|<1$, so that is not everywhere dense in $D_0$. Hence by Theorem 2, $mE_n^{(2)} = 0$.

If $e^{i\theta} \in E_n^{(3)}$, then it must coincide with one of $\xi_n^k, \eta_n^k$, since the complementary set of $\Delta_n(\rho)$ with respect to $|z|<1$ consists of a countable number of domains, each of which is bounded by a single $\lambda_n^k$ and an arc on $|z|=1$. Hence $mE_n^{(3)} = 0$. Thus we have proved $mE_n = mE_n^{(1)} + mE_n^{(2)} + mE_n^{(3)} = 0$, hence $mE(\rho) = \sum_{n=0}^{\infty} mE_n = 0$, q.e.d.

Remark. By means of the lemma, we can prove similarly as the above proof, the following theorem:

Let $D$ be a domain in $|z|<1$, which is bounded by orthogonal circles to $|z|=1$ and a closed set $e$ on $|z|=1$. Then for almost all $e^{i\theta}$ of $e$, its sufficiently small neighbourhood: $|z-e^{i\theta}| < \delta = \delta(\eta)$, $|\arg(1-ze^{-i\theta})| < \frac{\pi}{2}-\eta$ is contained in $D$ for any $\eta > 0$.

5. Let $F$ be an open Riemann surface of hyperbolic type spread over the $w$-plane and we map $F$ on $|z|<1$ by $w = w(z)$, then $w(z)$ is automorphic to a Fuchsian group $G$. We approximate $F$ by a sequence of Riemann surfaces $F_n$: $F_1 \subset F_2 \subset \cdots \subset F_n \nearrow F$, where $F_n$ consists of only inner points and consists of a finite number of sheets and is bounded by a finite number of analytic Jordan curves.

Let $\Delta_n$ be the image of $F - F_n$ in $|z|<1$, then $\Delta_n$ consists of a countable number of connected domains $\Delta_n^k$. Let $E_n^k$ be the part of the boundary of $\Delta_n^k$, which lies on $|z|=1$ and let $E_n = \sum_{k=0}^{\infty} E_n^k$. Then we call $E = \lim_{n\to\infty} E_n$ the image of the ideal boundary of $F$. It is easily seen that $E$ is independent of the approximating sequence $F_n$ and coincides with the set $E$ defined in Theorem 4. $E$ is the set of all $e^{i\theta}$, such that there exists a certain curve in $|z|<1$ ending at $e^{i\theta}$, whose image curve on $F$ tends to the ideal boundary of $F$. Since as Myrberg[7] proved, the Green's function of $F$ exists or not, according as $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$, or $\sum_{n=0}^{\infty}(1-|z_n|)=\infty$, we have

Theorem 5. Let $F$ be a Riemann surface of hyperbolic type spread over the $w$-plane. We map $F$ on $|z|<1$ and let the ideal boundary of $F$ be mapped on a set $E$ on $|z|=1$. Then $mE = 2\pi$ or $mE = 0$, according as the Green's function of $F$ exists or not.

In the case that the Green's function of $F$ does not exist, we have $mE_n = 0$ by Theorem 4, so that to a curve on $F$, which tends to the ideal boundary of $F$, there corresponds in $|z|<1$ a curve ending at a point of $E$, where $mE = 0$.

6. We will prove

Theorem 6. Let $F$ be a Riemann surface spread over the $w$-plane, on which the Green's function does not exist. Let $K$: $|w-a|<\rho$ be a disc and $F_\rho$ be a connected piece of $F$, which lies above $K$. Then $F_\rho$ covers any point of $K$, except a set of logarithmic capacity zero.[8)]

For the proof, we use the following lemma[9)].

**Lemma.** Let $w=f(z)$ be regular and $|f(z)|<1$ in $|z|<1$, $f(0)=0$. Let $E$ be the set of $e^{i\theta}$, such that $\lim_{r\to 1} f(re^{i\theta})=f(e^{i\theta})$ exists and $|f(e^{i\theta})|=1$, such that $f(e^{i\theta})=e^{i\psi}$ and $E_*$ be the set of $e^{i\psi}$ on $|w|=1$. Then $E$ and $E_*$ are measurable and

$$mE \leqq mE_* .$$

If $0<mE<2\pi$, then $mE<mE_*$.

**7. Proof of Theorem 6.**

(i) First suppose that $F$ is of parabolic type. Then $F$ is a Riemann surface of an inverse function $z=z(w)$ of a transcendental meromorphic function $w=w(z)$ $(|z|<\infty)$. We map $F_\rho$ on $|\zeta|<1$ by $w=\varphi(\zeta)$, then by Fatou's theorem, $\lim_{\zeta\to e^{i\theta}}\varphi(\zeta)=\varphi(e^{i\theta})$ exists almost everywhere on $|\zeta|=1$, when $\zeta\to e^{i\theta}$ non-tangentially to $|\zeta|=1$. Let $E$ be the set of $e^{i\theta}$, such that $|\varphi(e^{i\theta})-a|<\rho$, then $\varphi(e^{i\theta})$ belongs to the boundary of $F$, so that $z(\varphi(\zeta))\to\infty$, when $\zeta\to e^{i\theta}$ non-tangentially to $|\zeta|=1$. Hence by Lusin-Priwaloff's theorem, $mE=0$, so that almost all points of $|\zeta|=1$ are mapped on $|w-a|=\rho$, hence $\varphi(\zeta)$ belongs to the $U$-class in Seidel's sense, so that by a Frostman's theorem,[10)] $w=\varphi(\zeta)$ takes any value in $K$, except a set of logarithmic capacity zero.

(ii) Next suppose that $F$ is of hyperbolic type. We map $F$ on $|z|<1$, then by Theorem 4, the ideal boundary of $F$ is mapped on a null set $M$ on $|z|=1$. We map $F_\rho$ on $|\zeta|<1$ by $w=\varphi(\zeta)$, then by a Fatou's theorem, $\lim_{\zeta\to e^{i\theta}}\varphi(\zeta)=\varphi(e^{i\theta})$ exists almost everywhere on $|\zeta|=1$, when $\zeta\to e^{i\theta}$ non-tangentially to $|\zeta|=1$. Let $E$ be the set of $e^{i\theta}$, such that $|\varphi(e^{i\theta})-a|<\rho$. We will prove that $mE=0$. Suppose that $mE>0$, then $E$ contains a closed sub-set $E_0$, such that $mE_0>0$ and $\lim_{\zeta\to e^{i\theta}}\varphi(\zeta)=\varphi(e^{i\theta})$ uniformly, when $\zeta\to e^{i\theta}$ in an angular domain $\Delta(e^{i\theta})$ : $|\arg(1-ze^{-i\theta})|<\pi/4$. We construct a rectifiable Jordan curve $\Lambda$ in $|\zeta|<1$, such that $\Lambda$ meets $|\zeta|=1$ in $E_0$ and for any $e^{i\theta}\in E_0$, its sufficiently small neighbourhood in $\Delta(e^{i\theta})$ is contained in $\Lambda$. Let the inside of $\Lambda$ be mapped on $F'_\rho\subset F_\rho$. Then $F'_\rho$ is mapped on a countable number of equivalent domains $\{\Delta_n\}$ in $|z|<1$. We consider one $\Delta_0$ of them and let $M_0$ be the part of the boundary of $\Delta_0$, which lies on $|z|=1$. Then $M_0$ is a sub-set of $M$, so that $mM_0=0$. We map the inside of $\Lambda$ on $|x|<1$ and let $E_0$ be mapped on a set $e_0$ on $|x|=1$. Then by F.Riesz' theorem, $me_0>0$. Then $|x|<1$ is mapped on $\Delta_0$ and $e_0$ corresponds to $M_0$. We may suppose that $z=0$ lies in $\Delta_0$ and $z=0$ corresponds to $x=0$. Then by the lemma, $me_0\leqq mM_0=0$, so that $me_0=0$, which is absurd. Hence $mE=0$. From this, we proceed similarly as (i) and we conclude that $F$ covers any point of $K$, except a set of logarithmic capacity zero, q.e.d.

From Theorem 6, we have the following extension of Myrberg's theorem.[11)]

**Theorem 7.** Let $F$ be a Riemann surface spread over the $w$-plane and $F_\rho$ be a connected piece of $F$, which lies above a disc $K$ : $|w-a|<\rho$. If $F_\rho$ does not cover a set in $K$, which is of positive logarithmic capacity, then the Green's function of exists.

Myrberg assumed that the boundary of $F$ contains a sub-set, which lies in a schlicht disc and is of positive logarithmic capacity.

**8.** Let $F$ be a Riemann surface spread over the $w$-plane, which consists of a finite number of sheets and the projection of its boundary on the $w$-plane is a closed set of logarithmic capacity zero. We will call such a $F$ a quasi-closed Riemann surface. We can prove easily that:

On a quasi-closed Riemann surface, the Green's function does not exist.

**Proof.** Let $\Lambda$ be the boundary of $F$ and $\Lambda_w$ be its projection on the $w$-plane, then $\Lambda_w$ is of logarithmic capacity zero. We map $F$ on $|z|<1$ by $w=w(z)$ and let $\Lambda$ be mapped on a set $E$ on $|z|=1$. Since $\Lambda_w$ is the cluster set of $w=w(z)$ on $E$, if $mE>0$, then by a theorem proved[12)] by the author, $\Lambda_w$ is of logarithmic capacity positive, which is absurd. Hence $mE=0$, so that by Theorem 5, the Green's function of $F$ does not exist.

By means of Theorem 6, we can prove similarly as in the former paper[13)] the following theorem.

**Theorem 8.** Let $F$ be a Riemann surface spread over the $w$-plane, on which the Green's function does not exist. If $F$ is not quasi-closed, then $F$ covers any point of the $w$-plane infinitely often, except a set of logarithmic capacity zero.

It was proved formerly by K.Arima[14)] that $F$ covers any point of the $w$-plane, except a set of logarithmic capacity zero.

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