SOME CANONICAL CONFORMAL MAPS AND REPRESENTATIONS

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A. Mori [1] proved the existence of the function $\Omega(z; z_0, \zeta)$, satisfying the following conditions. Let D be an n-ply connected domain on z-plane, whose boundary Γ consists of n-Jordan curves Γ_j ($j = 1, 2, \ldots, n$). For a fixed point ζ on Γ and z_0 in D

(i) $\Omega(z_0, z_0, \zeta) = 0$, $\Omega'(z_0; z_0, \zeta) = 1$,

(ii) $\Omega(\zeta; z_0, \zeta) = \infty$;

(iii) $\Omega(z; z_0, \zeta)$ maps D conformally onto the whole Ω-plane cut along an "infinite radial slit" and $(n-1)$ "finite radial slits".

These terminologies are defined as follows:

(a) infinite radial slit: it arrives at the infinity from the finite point along a radial rectilinear line. (b) zero radial slit: it arrives at the origin from a finite point along a radial rectilinear line. (c) infinite-zero radial slit: it arrives at the origin from the infinity along a radial rectilinear line. (d) finite radial slit: it is a radial slit, different from the ones defined above.

Recently in the theory of functions, especially, in the theory of conformal mapping the so-called "kernel function" $K(z, t^*)$, which has been introduced by Bergman and made use of by many authors, namely, Garabedian, Nehari Schiffer, etc., to the research of conformal mapping, plays an important and fruitful role. They attempted to express the several domain functions by kernel function. Following these programs, we shall here attempt to construct an expression of Ω and of related functions by means of kernel function. We omit the precise definitions and terminologies, but shall follow to the Nehari's paper [1].

Now we assume that in the neighborhood of the point $z = \zeta$, Γ has the tangent everywhere.

Case (a). In the first place we treat the case $\alpha\pi \neq 0$, where $\alpha\pi$ is defined in the following manner. At the boundary element $z = \zeta$ which corresponds to $\Omega = \infty$, two boundary curves (one starts from $z = \zeta$ and the other ends to $z = \zeta$) make the angle $\alpha\pi$.

In this case, in the neighboring point of $z = \zeta$, $\Omega(z; z_0, \zeta)$ is almost equal to $1/(z-\zeta)^{2/\alpha}$, and hence $\Omega'(z; z_0, \zeta)/\Omega(z; z_0, \zeta) = -2/\alpha(z-\zeta)$ is asymptotically true at $z = \zeta$ in D. Putting

$$(1)\quad R(z; z_0, \zeta) = \frac{\Omega'(z; z_0, \zeta)}{\Omega(z; z_0, \zeta)} - \frac{1}{z - z_0} + \frac{2}{\alpha}\frac{1}{z-\zeta},$$

then $R(z; z_0, \zeta)$ is regular in D, and has a single-valued integral in D, since the total variation of $\arg \Omega(z, z_0, \zeta)$ along any closed boundary curve vanishes and the periodicity moduli of $-\lg(z - z_0)$ and $\lg(z-\zeta)^{2/\alpha}$ cancel each other.

On the other hand, if we put

$$T(z, t^*) = \int_t^z K(z, t^*)\, dz, \quad z, t \in D,$$

then from the reproducing property of the kernel function we have

$$(2)\quad \frac{1}{2i}\int_\Gamma (T(z, t^*))^* f(z)\, dz = f(t)$$

for any function $f(z)$ regular and of class L^2 in D and possessing there a single-valued integral.

We may therefore apply (2), obtaining

$$\frac{\Omega'(t)}{\Omega(t)} - \frac{1}{t-z_0} + \frac{2}{\alpha}\frac{1}{t-\zeta} = \frac{1}{2i}\int_\Gamma (T(z,t^*))^* \left(\frac{\Omega'}{\Omega} - \frac{1}{z-z_0} + \frac{2}{\alpha}\frac{1}{z-\zeta}\right) dz.$$

From the definition of Ω, if $z \in \Gamma$,

$$\frac{\Omega'}{\Omega}\, dz = \text{real}.$$

Hence we attain

$$(3)\quad \frac{1}{2i}\int_\Gamma (T(z, t^*))^* \frac{\Omega'}{\Omega}\, dz = -\left(\frac{1}{2i}\int_\Gamma T(z, t^*)\frac{dz}{z - z_0} - \frac{1}{\alpha i}\int_\Gamma T(z, t^*)\frac{dz}{z-\zeta} + \frac{1}{2i}\int_\Gamma T(z, t^*)\, R\, dz\right)^*.$$

For a sufficiently small positive number ε, we denote by $\Gamma_{\varepsilon 1}$ a path of integration which belongs to the circular disc $|z - \zeta| < \varepsilon$ and ends to the point ζ, and by $\Gamma_{\varepsilon 2}$ that which belongs to the circular disc $|z-\zeta| < \varepsilon$ and starts from the point ζ. Moreover we denote $D \cap [|z-\zeta| = \varepsilon]$ by Γ_ε.

Then we have

$$\int_\Gamma \frac{T(z, t^*)}{z-\zeta}\, dz = \int_{\Gamma - \Gamma_{\varepsilon 1} - \Gamma_{\varepsilon 2}} + \int_{\Gamma_\varepsilon} - \int_{\Gamma_\varepsilon} + \int_{\Gamma_{\varepsilon 1} + \Gamma_{\varepsilon 2}} = \int_{\Gamma_{\varepsilon 1}} + \int_{\Gamma_{\varepsilon 2}} - \int_{\Gamma_\varepsilon}$$

Now, we can easily verify

$$\left|\frac{1}{\alpha i}\int_{\Gamma_{\varepsilon 1} + \Gamma_{\varepsilon 2}} \frac{T(z, t^*)}{z-\zeta}\, dz\right| \leqq \frac{2\sin\frac{\alpha\pi}{2}}{\alpha}\left(|K(\zeta, t^*)| + o(\varepsilon)\right)\varepsilon$$

Hence the left-hand side of this inequality tends to zero with ε. On the other hand, a similar discussion as the above-mentioned one leads us to the relation

$$\lim_{\varepsilon \to 0} \frac{1}{\alpha i}\int_{\Gamma_\varepsilon} \frac{T(z, t^*)}{z-\zeta}\, dz = -\pi\, T(\zeta, t^*),$$

and hence we have

$$\frac{1}{2i}\int_\Gamma \frac{T(z,t^*)}{z-\zeta}\,dz = \pi\, T(\zeta,t^*) .$$

In the expression (3) the last term of the right-hand side is equal to 0 by the Cauchy's integral theorem. Therefore we obtain

$$(4)\quad \frac{1}{2i}\int_\Gamma (T(z,t^*))^* \frac{\Omega'}{\Omega}\,dz = -\pi\,(T(z_0,t^*) - T(\zeta,t^*))^* = -\pi\int_\zeta^{z_0} K(t,z^*)\,(dz)^* .$$

Next, the integration by parts gives us the relation

$$\frac{1}{2i}\int_\Gamma (T(z,t^*))^*\left(\frac{1}{z-z_0} - \frac{2}{\alpha}\frac{1}{z-\zeta}\right)dz = \left[(T(z,t^*))^* \lg \frac{z-z_0}{(z-\zeta)^{2/\alpha}}\right]_\Gamma - \frac{1}{2i}\int_\Gamma \lg\frac{z-z_0}{(z-\zeta)^{2/\alpha}}(K(z,t^*)dz)^* .$$

Here we have to remark that the periodicity modulus of $\lg(z-z_0)/(z-\zeta)^{2/\alpha}$ by the rounding of z once along Γ vanishes. Therefore, the following expression also remains true:

$$(5)\quad \frac{1}{2i}\int_\Gamma (T(z,t^*))^*\left(\frac{1}{z-z_0} - \frac{2}{\alpha}\frac{1}{z-\zeta}\right)dz = -\frac{1}{2i}\int_\Gamma \lg\frac{z-z_0}{(z-\zeta)^{2/\alpha}}\,K(t,z^*)\,(dz)^* .$$

Thus we obtain, from (4) and (5), the desired result

$$\frac{\Omega'(t;z_0,\zeta)}{\Omega(t;z_0,\zeta)} = \frac{1}{t-z_0} - \frac{2}{\alpha}\frac{1}{t-\zeta} - \pi\int_\zeta^{z_0} K(t,z^*)(dz)^* + \frac{1}{2i}\int_\Gamma \lg\frac{z-z_0}{(z-\zeta)^{2/\alpha}}\,K(t,z^*)\,(dz)^* .$$

Case (b). We have now to consider the case $\alpha\pi = 0$. In this case, at the neighboring points of $z=\zeta$, the asymptotic relation $\Omega \sim c\exp(-\frac{1}{z-\zeta})$ holds good, and hence

$$\frac{\Omega'}{\Omega} \sim \frac{1}{(z-\zeta)^2} .$$

If we put

$$R(z;z_0,\zeta) = \frac{\Omega'(z;z_0,\zeta)}{\Omega(z;z_0,\zeta)} - \frac{1}{z-z_0} + \frac{1}{(z-\zeta)^2} ,$$

$R(z;z_0,\zeta)$ has the same property as in the case (a). Therefore we may discuss by the similar method as in the above case (a). But no precise results shall be able to obtain in this case, because the circumstances of the contact of two curves $\Gamma_{\xi 1}$ and $\Gamma_{\xi 2}$, defined as before, may strongly effect to the discussion. Therefore we shall only represent in the following form:

$$\frac{\Omega'(t;z_0,\zeta)}{\Omega(t;z_0,\zeta)} = \frac{1}{t-z_0} - \frac{1}{(t-\zeta)^2} - \pi(T(z_0,t^*))^* + \left(\frac{1}{2i}\int_\Gamma T(z,t^*)\frac{dz}{(z-\zeta)^2}\right)^* - \frac{1}{2i}\int_\Gamma (T(z,t^*))^*\frac{dz}{z-z_0} + \frac{1}{2i}\int_\Gamma (T(z,t^*))^*\frac{dz}{(z-\zeta)^2} .$$

Remark. The function $\Omega(z,z_0,\zeta)$ is able to express by the other domain functions; Green function, Neumann function and harmonic measure

We can establish the representation of the following three canonical types of conformal mapping by making use of the Bergman kernel function.

(A) $\Omega(z;\xi,\eta)$ maps D onto a schlicht full plane cut along $n-2$ finite radial slits, an infinite radial slit and a zero radial slit.

(B) $\Omega(z;\xi,\eta)$ maps D onto a schlicht full plane cut along $n-1$ finite radial slits and an infinite-zero radial slit, that is, $\lg\Omega(z)$ maps D onto a schlicht parallel strip having $n-1$ parallel slits.

In each case $\Omega(\xi;\xi,\eta) = 0$ and $\Omega(\eta;\xi,\eta) = \infty$, and in (A) $\xi \in \Gamma_1$, $\eta \in \Gamma_2$ in (B) $\xi, \eta \in \Gamma_1$. Putting

$$\Omega(z;\xi,\eta) = k\,\frac{\Omega(z;z_0,\eta)}{\Omega(z;z_0,\xi)} ,$$

the proof of existence and uniqueness is easily obtained from the Mori's function.

Next, if we put

$$\Omega(z;z_0,z_\infty) = k\,\frac{\Omega(z;z_0,\xi)}{\Omega(z;z_\infty,\xi)} ,$$

then this function maps D onto a schlicht full plane cut along n finite radial slits, where $\Omega(z_0) = 0$, $\Omega(z_\infty) = \infty$ and $z_0, z_\infty \in D$.

(C) Schwarz-Christoffel's formula for an n-ply connected case.

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