ON DISTORTION IN SCHLICHT MAPPINGS

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0. Several distortion theorems have been derived, in various ways, for functions regular and schlicht in a circle. In the present Note we shall attempt certain estimations about their spherical derivative. The aim is to obtain estimates of spherical derivative, depending only on $r = |z|$, for family of functions regular, schlicht in the unit circle $|z| < 1$ and normalized at the origin. The results which will be obtained in the following lines are only partially precise. In fact, although the best possible bounds together with extremal functions can be found for points z comparatively near the origin, the precise bounds for remaining points are yet unknown. But it will be noteworthy to remark that the precise bounds are not analytic in the whole range of z.

On the other hand, the concept of spherical derivative is really rather useful for meromorphic functions than merely for regular functions. But, in comparison with rich results in the theory of schlicht functions regular in a circle, those referring to schlicht functions meromorphic in a circle are still poor. Making use of invariant character of spherical derivative with respect to any rotation of Riemann sphere, distortion inequalities will be derived for spherical derivative of certain schlicht functions meromorphic in a circle.

1. The spherical derivative of an analytic function $w(z)$ is defined as

$$(1.1) \qquad Dw(z) \equiv \frac{|w'(z)|}{1 + |w(z)|^2},$$

if z is a pole of the first order with residue c or of higher order, we put $Dw = 1/|c|$ or $Dw = 0$, respectively.

Consider first the family of functions $\{w(z)\}$ regular and schlicht in $|z| < 1$ and normalized at the origin such as

$$(1.2) \qquad w(0) = 0, \quad w'(0) = 1.$$

We shall attempt to estimate the spherical derivative of such functions from both sides. Now, as is well-known, the classical distortion theorems

$$(1.3) \qquad \frac{r}{(1+r)^2} \leqq |w(z)| \leqq \frac{r}{(1-r)^2},$$

$$(1.4) \qquad \frac{1-r}{1+r} \leqq \left|\frac{z\,w'(z)}{w(z)}\right| \leqq \frac{1+r}{1-r}.$$

due to Koebe-Bieberbach and to R. Nevanlinna respectively, hold good for any functions of the family. Moreover, for any z with $0 < r = |z| < 1$, the equality sign of left and right side is, in each case, realized only by Koebe's extremal function

$$(1.5) \qquad w = \frac{z}{(1+\varepsilon z)^2} \qquad (|\varepsilon| = 1),$$

and, in fact, merely at $z = \bar{\varepsilon}|z|$ and $z = -\bar{\varepsilon}|z|$, respectively.

Denoting now, for brevity, by

$$(1.6) \qquad r^* = \frac{\sqrt{5}-1}{2} = 0.618\ldots$$

the positive root of the quadratic equation $1 - r^2 = r$, we get, with regard to both bounds contained in Koebe-Bieberbach's distortion theorem (1.3), the relations

$$(1.7) \qquad \frac{r}{(1+r)^2} \leqq \frac{r}{(1-r)^2} \leqq 1\Big/\frac{r}{(1+r)^2} \qquad (r \leqq r^*),$$

$$(1.8) \qquad 1\Big/\frac{r}{(1-r)^2} < \frac{r}{(1+r)^2} < \frac{r}{(1-r)^2} \qquad (r^* < r < 1).$$

Hence, if $r = |z| \leqq r^*$, we have

$$\frac{r}{(1-r)^2} + \frac{(1-r)^2}{r} \leqq |w| + \frac{1}{|w|} \leqq \frac{r}{(1+r)^2} + \frac{(1+r)^2}{r},$$

or

$$(1.9) \qquad \frac{r^2+(1-r)^4}{r(1-r)^2} \leqq \frac{1+|w|^2}{|w|} \leqq \frac{r^2+(1+r)^4}{r(1+r)^2} \quad (r \leqq r^*).$$

Combining both relations (1.4) and (1.9), we obtain for spherical derivative which may be written in the form

$$Dw(z) = \left|\frac{w'}{w}\right| \frac{|w|}{1+|w|^2},$$

the following estimation:

$$(1.10) \qquad \frac{1-r^2}{r^2+(1+r)^4} \leqq Dw(z) \leqq \frac{1-r^2}{r^2+(1-r)^4} \qquad (r \leqq r^*)$$

The extremal functions for this distortion inequality must, as readily seen from the above argument, be of the form (1.5). For such a function the actual calculation shows that

$$w = \frac{z}{(1+\varepsilon z)^2}, \quad w' = \frac{1-\varepsilon z}{(1+\varepsilon z)^3};$$

$$(1.11) \qquad Dw = \frac{|w'|}{1+|w|^2} = \frac{|1-\varepsilon^2 z^2|}{|z|^2+|1+\varepsilon z|^4},$$

and hence the left and right bound in (1.10) is indeed attained at $z = r\bar{\varepsilon}$ and $z = -r\bar{\varepsilon}$, and only at these points, respectively.

We note here, in passing, that the same is valid for distortion inequality

$$(1.12)\qquad \frac{r(1+r)^2}{r^2+(1+r)^4} \le \frac{|w|}{1+|w|^2} \le \frac{r(1-r)^2}{r^2+(1-r)^4} \quad (r \le r^*)$$

which is equivalent to (1.9).

We have seen that the estimation (1.10) (and also (1.12)) is valid for $r = |z| \le r^*$ and is the best possible one so far as it depends only on r. Next, if z lies in the remaining range $r^* < r = |z| < 1$, a similar argument, by using (1.8) instead of (1.7), shows that in this case the inequality (1.9) must be replaced by

$$\frac{r}{(1+r)^2} + \frac{(1+r)^2}{r} \le |w| + \frac{1}{|w|} \le \frac{r}{(1-r)^2} + \frac{(1-r)^2}{r},$$

or

$$(1.13)\qquad \frac{r^2+(1+r)^4}{r(1+r)^2} \le \frac{1+|w|^2}{|w|} \le \frac{r^2+(1-r)^4}{r(1-r)^2} \quad (r^* < r < 1).$$

Combining the last relation with (1.4), we have a distortion inequality

$$(1.14)\qquad \frac{(1-r)^3}{1+r}\,\frac{1}{r^2+(1-r)^4} \le Dw(z) \le \frac{(1+r)^3}{1-r}\,\frac{1}{r^2+(1+r)^4} \quad (r^* < r < 1),$$

similar to (1.10). But the last estimation is not the most precise one. In fact, the only extremal functions for Nevanlinna's distortion theorem (1.4) are of the form (1.5) for which the equality sign of left and right side appears only at $z = r\bar{\varepsilon}$ and $z = -r\bar{\varepsilon}$, respectively. On the other hand, although the only extremal functions for the distortion inequality

$$(1.15)\qquad \frac{r(1-r)^2}{r^2+(1-r)^4} \le \frac{|w|}{1+|w|^2} \le \frac{r(1+r)^2}{r^2+(1+r)^4} \quad (r^* < r < 1)$$

equivalent to (1.13) are also of the same form (1.5), the equality sign of left and right side holds here only at $z = -r\bar{\varepsilon}$ and $z = r\bar{\varepsilon}$, respectively. Hence the estimation (1.14), obtained by combining both inequalities for which in spite of the community of extremal functions the arguments of the extremal points are different, cannot be the most precise one.

Well, the existence of the best possible bounds for $Dw(z)$ depending only on r and realized by functions of the family, also for the range $r^* < r = |z| < 1$, is evident from the fact that the family of functions in question is a normal one. The determination of these exact bounds is left open for future considerations. It is, however, at any rate interesting that the precise bounds are given by very simple rational functions of r for $r \le r^*$ but not analytic in r for the whole range

2. We consider next the Riemann sphere Σ with diameter unity touching the complex w-plane at its origin, and denote by $\delta(w_1, w_2)$ the spherical distance between the points corresponding to w_1 and w_2 by stereographic projection. The line element on Σ is given by the expression

$$(2.1)\qquad ds = \delta(w, w+dw) = \frac{|dw|}{1+|w|^2}.$$

On the other hand, a linear transformation corresponding to any rotation of Σ is represented in the form

$$(2.2)\qquad W = \eta\frac{w+w_0}{1-\bar{w}_0 w}, \quad w = \bar{\eta}\frac{W-\eta w_0}{1+\bar{\eta}\bar{w}_0 W} \quad (|\eta| = 1),$$

w_0 being a parametric point; but in case $w_0 = \infty$ (2.2) has to be replaced by

$$(2.3)\qquad W = \eta_1\frac{1}{w}, \quad w = \eta_1\frac{1}{W} \quad (|\eta_1| = 1).$$

Let now $w(z)$ be a function regular and schlicht in $|z| < 1$ and normalized at the origin such as in (1.2). There corresponds then, by the transformation (2.2), a function

$$(2.4)\qquad W(z) = \eta\frac{w(z)+w_0}{1-\bar{w}_0 w(z)} = \eta w_0 + \eta(1+|w_0|^2)z + \cdots \quad (|z| < \rho \le 1)$$

also schlicht in $|z| < 1$. If the point $1/\bar{w}_0$ does not belong to the range-domain of $w(z)$, then the transformed function (2.4) is also regular throughout in $|z| < \rho = 1$. But, if on the contrary the point $1/\bar{w}_0$ belongs to the range-domain of $w(z)$, the function (2.4) possesses a pole of the first order at the $1/\bar{w}_0$-point ζ ($\rho = |\zeta|$) of $w(z)$ with residue equal to

$$(2.5)\qquad \eta\frac{1/\bar{w}_0 + w_0}{-\bar{w}_0 w'(\zeta)} = -\frac{\eta}{w'(\zeta)}\left(w(\zeta)^2 + \frac{w(\zeta)}{|w(\zeta)|^2}\right)$$
$$\left(w(\zeta) \equiv \frac{1}{\bar{w}_0}\right).$$

As is well-known, the line element (2.1) remains invariant for any rotation (2.2) of Σ, that is

$$(2.6)\qquad \frac{|dW|}{1+|W|^2} = \frac{|dw|}{1+|w|^2}.$$

Hence, all the estimations obtained with respect to the spherical derivative of $w(z)$ remain valid also for schlicht functions $W(z)$ which are represented in the form (2.4).

In particular, the precise estimation of the spherical distance

$$(2.7)\qquad \delta(W(0), W(z)) = \int_\Gamma \frac{|dW|}{1+|W|^2}$$

may be obtained for any z with $r = |z| \le r^* = (\sqrt{5}-1)/2$, Γ denoting here the circular arc from $W(0)$ to $W(z)$ which corresponds to a minor arc of great circle on Σ. In fact, integrating along the curve γ on the z-plane which corresponds to Γ, we have, by (1.10),

$$\delta(W(0), W(z)) = \int_\gamma DW(z)\,|dz| \ge \int_0^{|z|} DW(z)\,d|z|$$

$$\ge \int_0^r \frac{1-r^2}{r^2+(1+r)^4}\,dr = \arctan\frac{r}{(1+r)^2}.$$

On the other hand, denoting by $\hat{\Gamma}$ the curve on the W-plane corresponding to the radial segment from 0 to z, we have, again by (1.10),

$$\delta(W(0), W(z)) \leqq \int_{\hat{\Gamma}} \frac{|dW|}{1+|W|^2} = \int_0^{|z|} DW(z)\, d|z|$$

$$\leqq \int_0^r \frac{1-r^2}{r^2+(1-r)^4}\, dr = \arctan \frac{r}{(1-r)^2}.$$

Hence, we conclude finally that

$$(2.8) \quad \arctan \frac{r}{(1+r)^2} \leqq \delta(W(0), W(z)) \leqq \arctan \frac{r}{(1-r)^2} \quad (r \leqq r^*).$$

The only extremal functions for the last estimation are those obtained from (1.5) by linear transformations of the form (2.2) (or (2.3)).

More generally, with regard to the spherical distance between two points $W(z_j)$ for any z_j $(j = 1, 2)$ with $r_j = |z_j| \leqq r^*$, we get

$$\left| \arctan \frac{r_2}{(1+r_2)^2} - \arctan \frac{r_1}{(1+r_1)^2} \right| \leqq \delta(W(z_1), W(z_2))$$

$$\leqq \left| \arctan \frac{r_2}{(1-r_2)^2} - \arctan \frac{r_1}{(1-r_1)^2} \right|,$$

or

$$(2.9) \quad \arctan \frac{(1-r_1 r_2)|r_2 - r_1|}{r_1 r_2 + (1+r_1)^2(1+r_2)^2} \leqq \delta(W(z_1), W(z_2))$$

$$\leqq \arctan \frac{(1-r_1 r_2)|r_2 - r_1|}{r_1 r_2 + (1-r_1)^2(1-r_2)^2} \quad \left(\begin{matrix} r_j = |z_j| \leqq r^*; \\ j = 1, 2 \end{matrix} \right).$$

The extremal cases can be discussed similarly as above.

We may notice here that the distortion inequality for $DW(z)$ is nothing but the one obtained from (2.9) by putting $\arg z_1 = \arg z_2$, dividing each member by $|r_2 - r_1| = |z_2 - z_1|$ and then letting z_j $(j = 1, 2)$ both tend to z. In fact, we have then

$$\delta(W(z_1), W(z_2)) / |z_2 - z_1| \longrightarrow DW(z),$$

and hence the relation (2.9) yields, by this limiting process,

$$\frac{1-r^2}{r^2+(1+r)^4} \leqq DW(z) \leqq \frac{1-r^2}{r^2+(1-r)^4} \quad (r \leqq r^*).$$

3. We have hitherto considered quite generally the whole family of schlicht functions normalized in respective ways. But, if we restrict ourselves to particular sub-families, then the results will be correspondingly ameliorated. For instance, if we consider the family consisting of functions which are regular and schlicht in $|z| < 1$, normalized at the origin such as in (1.2) and moreover map $|z| < 1$ onto convex domains, then the distortion theorems (1.3), (1.4) are improved in the following manner:

$$(3.1) \quad \frac{r}{1+r} \leqq |w(z)| \leqq \frac{r}{1-r},$$

$$(3.2) \quad \frac{1}{1+r} \leqq \left| \frac{z w'(z)}{w(z)} \right| \leqq \frac{1}{1-r}$$

If we now put, instead of (1.6),

$$(3.3) \quad r_c^* = \frac{\sqrt{2}}{2} = 0.707\cdots,$$

we get, corresponding to (1.7),

$$(3.4) \quad \frac{r}{1+r} \leqq \frac{r}{1-r} \leqq 1 \Big/ \frac{r}{1+r} \quad (r \leqq r_c^*)$$

and hence obtain, in place of (1.12),

$$(3.5) \quad \frac{r(1+r)}{r^2+(1+r)^2} \leqq \frac{|w|}{1+|w|^2} \leqq \frac{r(1-r)}{r^2+(1-r)^2} \quad (r \leqq r_c^*).$$

Combining (3.2) with (3.5), the estimation corresponding to (1.10) is, in our case of functions possessing convex images, obtained in the form

$$(3.6) \quad \frac{1}{r^2+(1+r)^2} \leqq Dw(z) \leqq \frac{1}{r^2+(1-r)^2} \quad (r \leqq r_c^*).$$

The only extremal functions are of the form

$$(3.7) \quad w = \frac{z}{1+\varepsilon z} \quad (|\varepsilon| = 1),$$

for which we have

$$w' = \frac{1}{(1+\varepsilon z)^2}$$

and hence

$$(3.8) \quad Dw(z) = \frac{1}{|z|^2 + |1+\varepsilon z|^2}$$

Though we may further assert, for remaining range of z, the validity of distortion inequality

$$(3.9) \quad \frac{1-r}{1+r} \frac{1}{r^2+(1-r)^2} \leqq Dw(z) \leqq \frac{1+r}{1-r} \frac{1}{r^2+(1-r)^2} \quad (r_c^* < r < 1),$$

this is not the most precise one as is shown by the similar argument as above. The determination of the exact bounds are here also left open.

The relation which corresponds to (2.8) becomes, in our present case,

$$(3.10) \quad \arctan \frac{r}{1+r} \leqq \delta(W(0), W(z)) \leqq \arctan \frac{r}{1-r} \quad (r \leqq r_c^*)$$

More generally, corresponding to (2.9), we now obtain

$$(3.11) \quad \arctan \frac{|r_2 - r_1|}{r_1 r_2 + (1+r_1)(1+r_2)} \leqq \delta(W(z_1), W(z_2))$$

$$\leqq \arctan \frac{|r_2 - r_1|}{r_1 r_2 + (1-r_1)(1-r_2)} \quad \left(\begin{matrix} r_j = |z_j| \leqq r_c^*; \\ j = 1, 2 \end{matrix} \right).$$

Each bound in (3.10) and also in (3.11) is the best possible one.

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