ON CARTAN SUBALGEBRAS OF A LIE ALGEBRA

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(Communicated by Y. Komatu)

Let L be a Lie algebra over a ground field F and $x \to ad(x)$ be its adjoint representation. If the number ℓ_x of characteristic root 0 of $ad(x)$ attains the minimum for $x \in L$, x is called a regular element of L, and it is well known that the eigenspace P of $ad(x)$ for the root 0 is a nilpotent subalgebra of order ℓ_x of L. Such a subalgebra P is called a Cartan subalgebra of L.

Now if F is the complex number field, it is known[1] that, for any two Cartan subalgebras P_1 and P_2, there exists an element σ in the adjoint group of L, i.e. the linear group generated by $\exp(ad(x))$, $x \in L$, such that

$$\sigma(P_1) = P_2 .$$

However, for general ground field F, this does not hold. For example, if F does not contain $\alpha^{\frac{1}{2}}$ for some $\alpha \in F$, the simple Lie algebra of type A of order 3 over F, given by

$$L = uF + vF + wF$$

$$[u, v] = v, \quad [u, w] = -w, \quad [v, w] = u,$$

contains two Cartan subalgebras $P_1 = (\frac{v}{2} + \alpha w) F$ and $P_2 = (\frac{v}{2} + w) F$ corresponding to regular elements $\frac{v}{2} + \alpha w$ and $\frac{v}{2} + w$, respectively, such that for any automorphism τ of L, we have

$$\tau(P_1) \neq P_2 .$$

This shows that for the conjugateness of Cartan subalgebras some conditions on F and L will be necessary. In this note we shall study such conditions using the theory of algebraic Lie algebras of C. Chevalley[2].

§1. In this section, let F be an arbitrary field of characteristic 0.

Let L be an ℓ-algebraic[2] Lie algebra over F ($L \subset \mathfrak{gl}(F, n)$), R its radical, and let one of its Levi decompositions be $L = R + S$, then as is known,[3] there is an ideal N of L consisting of all nilpotent matrices in R and a subalgebra C of L consisting of only semi-simple matrices such that

$$R = N + C, \quad N \frown C = \{0\}, \quad [C, S] = \{0\}.$$

We call this decomposition $L = (N + C) + S$ a 'normal decomposition' of L.

Lemma 1. "Let L be a nilpotent Lie algebra and let M be its representation space, and let $x \to \rho(x)$ be the representation given by M. Let the following conditions be satisfied:

(i) If $\rho(x) \cdot m = 0$ for all $x \in L$ and for some $m \in M$, then necessarily $m = 0$;

(ii) The image $\bar{L} = \rho(L)$ consists of only semi-simple matrices.[4]

Then the first cohomology group[5] of L by M vanishes: $H^1(L, M) = 0$ "

Proof. First, we assume that all eigenvalues of $\rho(x)$, $x \in L$, belong to the ground field F. Then by (ii), M is a direct sum of $\rho(L)$-invariant subspaces of dimension one, so we may assume further that M is 1-dimensional. Now if $f(x) \in Z^1(L, M)$, then

$$0 = \delta f(x, y) = \rho(x) f(y) - \rho(y) f(x) + f([x, y])$$

and it follows from (i) that $f([x, y]) = 0$, and for some x_0, $\rho(x_0) \neq 0$. Hence if we put $f_0 = f(x_0)/\rho(x_0)$, we have

$$f(x) = \rho(x) f_0 = \delta f_0(x),$$

namely $f(x) \in B^1(L, M)$.

In the general case, let F^* be the finite Galois extension of F containing all eigenvalues of $\rho(x)$, $x \in L$, and let G be its Galois group. Since the same assumptions hold for the scalar extension L_{F^*} and M_{F^*}, we have $H^1(L_{F^*}, M_{F^*}) = 0$. Now if $f(x) \in Z^1(L, M) \subset Z^1(L_{F^*}, M_{F^*})$, then there exists $f_0^* \in M_{F^*}$ such that

$$f(x^*) = \rho(x^*) f_0^* \quad \text{for all } x^* \in L_{F^*}.$$

Let $f_0 = \frac{1}{r} \sum_{\sigma \in G} \sigma(f_0^*)$, $(r = [F^* : F])$ then for $x \in L$ we have

$$f(x) = \sigma(f(x)) = \rho(x) \sigma(f_0^*)$$

averaging over G, we have

$$f(x) = \rho(x) f_0 \quad \text{for } x \in L$$

where $f_0 \in M$, Q.E.D.

Lemma. 2. "Let L be a solvable ℓ-algebraic Lie algebra and $L = N + H_1 = N + H_2$ be two normal decompositions of L. Then there exists an element

$x \in [L, L]$ such that

$$\exp(ad(x))H_1 = H_2$$

where, as is well known,[6] $ad(x)$ is a nilpotent derivation on L and

$$\exp X = 1 + \frac{X}{1!} + \frac{X^2}{2!} + \cdot$$

$(X = ad(x))$, (finite series!)

gives an automorphism of L ."

Proof. i) The case where N is abelian: Put $N_0 = N \cap Z(H_1) = N \cap Z(H_2)$ (where $Z(H_i)$ denotes the centralizer of H_i in L, i.e. the set of all elements x in L such that $[x, H_i] = 0$), then by the complete reducibility of N as an $ad(H_1)$-modul[7] there exists a subspace N_1 of N such that

$$N = N_0 + N_1, \quad N_0 \cap N_1 = \{0\}, \quad [H_1, N_1] \subseteq N_1.$$

Now decompose $x \in H_1$:

$$x = h_2 + y_0 + y_1, \quad h_2 \in H_2, \; y_0 \in N_0, \; y_1 \in N_1,$$

then, as can be seen easily, the mapping $x \to y_1 = f(x)$ belongs to the first cycle group $Z'(H_1, N_1)$ so that by Lemma 1 there is an element $z \in N_1 \subset [L, L]$ such that $f(x) = [x, z]$ (for all $x \in H_1$) and hence

$$x + [z, x] = h_2 + y_0 = \exp(ad(z))x$$
$$= (\exp z)x(\exp(-z)) \;;$$

so h_2, y_0 is respectively the semi-simple and the nilpotent part of a semi-simple matrix $(\exp z)x(\exp(-z))$, whence $y_0 = 0$ which yields $\exp(ad(z))H_1 \subset H_2$. As $\exp(ad(z))$ is an automorphism of L, so comparing the dimensions, we have

$$\exp(ad(z))H_1 = H_2 \quad .$$

ii) General case: Induction on dim N gives the result without much difficulty.

Theorem 1. "Let L be a solvable Lie algebra over F , and let P_1, P_2 be two Cartan subalgebras of L . Then there exists an element $x \in [L, L]$ such that

$$(\exp X)P_1 = P_2 \qquad (X = ad(x))$$

where X is a nilpotent derivation as in Lemma 2.[6] "

Proof. By the use of the adjoint representation, it can be easily seen that we may assume without loss of generality that L is a linear Lie algebra.

Denoting the smallest ℓ-algebraic Lie algebra containing L by L^*, which is also solvable, we have[8]

$$L^* = L + P_1^*, \quad P_1 = L \cap P_1^* \; .$$

Let the normal decomposition of P_1^* be

$$P_1^* = N_1 + H_1$$

and let N be the set of all nilpotent matrices in L^*, then we have easily that

$$L^* = N + H_1$$

and that this is a normal decomposition of L^*. Now it holds that

$$Z(H_1) = P_1^* \quad .$$

In fact, as H_1 is in the center of P_1^* we have $Z(H_1) \supset P_1^*$. Conversely, let $x \in L^*$ be such that $[x, H_1] = 0$. Let us denote by M the sum of all the eigenspaces of $ad(a)$ which do not belong to eigenvalue 0 . As P_1 contains a regular element a , we have

$$L = P_1 + M, \quad P_1 \cap M = \{0\}, \quad ad(a)M = M.$$

Then we have

$$L^* = P_1^* + M, \quad P_1^* \cap M = \{0\} \; .$$

Put

$$x = x_0 + x_1, \quad x_0 \in P_1^*, \quad x_1 \in M \; .$$

Then the semi-simple part a' of a which belongs to H_1 has the following properties:

$$ad(a')M = M \;,$$
$$0 = ad(a')x = [a', x] = [a', x_0 + x_1]$$
$$= [a', x_1] \; .$$

Hence we have

$$x_1 = 0 \;, \quad x = x_0 \in P_1^* \quad .$$

Similarly, we have for P_2

$$P_2 = L \cap P_2^*, \quad P_2^* = Z(H_2),$$

and to a normal decomposition $P_2^* = N_2 + H_2$ corresponds the normal decomposition of L^* :

$$L^* = N + H_2 \quad .$$

Now we have by Lemma 2. there exists an $x \in [L^*, L^*] = [L, L]$ such that

$$\exp X(H_1) = H_2 \;, \quad X = ad(x),$$

as $\exp X$ is an automorphism of L^*, taking the centralizer of both sides, we have

$$\exp X(P_1^*) = P_2^* \quad .$$

Now since L is an ideal of L^* , we have

$$\exp X(L) = L \; .$$

So we have

$$\exp X(P_1) = \exp(P_1^* \cap L) = P_2^* \cap L = P_2,$$

Q. E. D.

Remark. From the uniqueness of Levi decomposition[4] and Lemma 2 we have easily the following L

Lemma 2'. "Let L be an ℓ-algebraic Lie algebra and

$$L=(N+C_1)+S_1=(N+C_2)+S_2$$

be two normal decompositions of L. Then there exists an element $x\in N\frown[L,L]$ such that

$$\exp X(C_1)=C_2,\quad \exp X(S_1)=S_2,$$

$$(X=ad(x))$$

(of course $\exp X(N)=N$)".

§2. Theorem 2. "Let F be the real number field, and let L be a Lie algebra over F, and R its radical. If any two Cartan subalgebras of L/R are conjugate to each other under the adjoint group of L/R, then the same holds for L. (So that the problem on conjugateness of Cartan subalgebras of a real Lie algebra reduces to the case of simple Lie algebras.)

We shall give only the outline of the proof of this Theorem, making use of the following two lemmas.

Lemma 3. "Let L be an ℓ-algebraic Lie algebra over a field of characteristic 0, R its radical and let P be one of its Cartan subalgebra belonging to regular element a. Let H be the set of all semi-simple matrices which are replicas of a, then

$$R=(H\frown R)+L^{\perp},$$

$$L^{\perp}=\{x;\ x\in L,\ tr(xy)=0\ \text{for all}\ y\in L\}$$

gives a normal decomposition of R and there exists a Levi decomposition of L such that

$$L=R+S,\quad H=(H\frown R)+(H\frown S)."$$

Proof. Decompose L into eigenspaces of a as follows:

$$L=M_0+M_1+\cdots+M_r,\quad M_0=P=H+N_0,$$

N_0 being the set of all nilpotent matrices in P, then

$$R=\Sigma_{i=0}^{r}(M_i\frown R),\quad M_0\frown L^{\perp}=N_0,$$

$$M_0\frown R=(H\frown R)+N_0,$$

$$M_i\frown R=M_i\frown L^{\perp}\quad(i\geq 1)$$

and hence we have

$$R=(H\frown R)+L^{\perp}.$$

Take the corresponding Levi decomposition of L:

$$L=R+S,\quad [H\frown R,\ S]=0$$

and let φ be the natural homomorphism of L on $S\cong L/R$ and put $\varphi(H)=B$, then B is abelian and since $N(H+R/R)=H+R/R$, the normalizer of B in S coincides with B. From this it can be shown without much difficulty that B is a Cartan subagebra of S and B is composed of only semi-simple matrices. Then since $[H\frown R,B]=0$, $(H\frown R)+B$ is composed of only semi-simple matrices, and we have

$$H+L^{\perp}=H+R=B+R=B+(H\frown R)+L^{\perp},$$

and this gives two normal decompositions of an ℓ-algebraic solvable Lie algebra $H+R$, so there is an element $x\in L^{\perp}$ such that

$$\exp X(H)=(H\frown R)+B,\quad X=ad(x).$$

Now $(H\frown R)+B=H'$ satisfies

$$H'=(H'\frown R)+(H'\frown S)$$

so the same holds for H, Q.E.D.

Lemma 4. "Let L be an ℓ-algebraic Lie algebra over the real number field F and let R be its radical. Suppose that any two Cartan subalgebras of L/R are conjugate to each other under the adjoint group of L/R. Then for any two regular elements a_1, a_2 of L the subalgebras H_1 and H_2 belonging to a_1 and a_2 respectively are conjugate under the adjoint group of L.

Proof. There exist by Lemma 3 two normal decompositions of L such that

$$L=(N+C_1)+S_1,$$

$$H_1=(H_1\frown R)+(H_1\frown S_1),\quad H_1\frown R=C_1,$$

$$L=(N+C_2)+S_2,$$

$$H_2=(H_2\frown R)+(H_2\frown S_2),\quad H_2\frown R=C_2.$$

Then, by Lemma 2', there exists an element $x\in N\frown[L,L]$ such that

$$\exp X(C_1)=C_2,\quad \exp X(S_1)=S_2,$$

then we have

$$\exp X(H_1)=C_2+(\exp X(H_1)\frown S_2).$$

Now by assumption there exist elements $y_1, y_2,\cdots,y_r$ in S_2 such that

$$\exp Y_1\cdots\exp Y_r\,(\exp X(H_1)\frown S_2)$$

$$=H_2\frown S_2.\quad (Y_i=ad(y_i),\ 1\leq i\leq r).$$

Now since $[C_2,S_2]=0$ we have $\exp Y_1\cdots\exp Y_r\,C_2=C_2$ whence we have

$$\exp Y_1\cdots\exp Y_r\,\exp X(H_1)=H_2,$$

Q. E. D.

Proof of Th. 2 comes now immediately from Lemma 4, note (10), and the similar reasoning as in Th. 1.

(*) Received July 8, 1950.

(1) C.Chevalley: An algebraic proof of a property of Lie groups, Amer. Journ. of Math. 63 (1941) (informed only by Math. Reviews).
Y.Matsushima: Zenkoku-Shijô-Danwa-Kai (2-5-50) (1946) (in Japanese).

(2) Cf. C.Chevalley: On algebraic Lie algebras, Ann. of Math. 48 (1947) (We shall cite this paper as C.), or M.Goto, On algebraic Lie algebras, Journ. Math. Soc. Japan, 1 (1948) (We shall cite this paper as G.), or Y.Matsushima, On algebraic Lie groups and algebras, as above.

(3) C. § 5. Th. 4.

(4) Then it can be easily seen by generalized Lie's Theorem (C. § 4. Th. 3) that $\bar{L}$ is abelian.

(5) For cohomology groups, cf. C.Chevalley and S.Eilenberg: Cohomology theory of Lie groups and Lie algebras, Trans. Amer. Math. 63 (1948), § 23.

(6) Cf. G. p.36, since $\exp X(y) = (\exp x)\, y\, (\exp(-x))$ holds, we have $\exp X(N) = N$.

(7) C. § 5. Lemma 1.

(8) G. § 4.

(9) G. Lemma 10.

(10) We call H the subalgebra belonging to a. H is abelian as H is composed of polynomials of a. It is shown easily that $M_0 = Z(H)$ and we shall denote H by $H(a)$

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