KŌDAI MATH. SEM. REP.
19 (1967), 507—512

ON COCOMPLEX STRUCTURES

BY KOICHI OGIUE AND MASAFUMI OKUMURA

Introduction.

Let Γ be a pseudogroup of differentiable transformations of a manifold V and let M be a differentiable manifold. A Γ-*atlas* on M is a collection of local diffeomorphisms $\{\lambda_i; U_i\}$ of M into V which satisfies $\cup U_i = M$ and $\lambda_i \circ \lambda_j^{-1} \in \Gamma$ for all i and j such that $U_i \cap U_j \neq \phi$. Two Γ-atlases are said to be equivalent if their union is a Γ-atlas. An equivalence class of Γ-atlases is called a Γ-*structure* on M.

By an *almost* Γ-*structure* on a manifold M we mean, roughly speaking, a structure on M which is identified with a Γ-structure up to a certain order of contact at each point. It is a G-structure of a certain order.

We consider the following correspondences between structures on an even dimensional manifold and those on an odd dimensional one:

(*) symplectic structure ——— (*) cosymplectic structure,
(♯) almost symplectic structure—(♯) almost cosymplectic structure,
(*) complex structure ——— (*) cocomplex structure,
(♯) almost complex structure—(♯) almost cocomplex structure.

The (*)ed structures are Γ-structures for some Γ and the (♯)ed structures are almost Γ-structures.

An almost cocomplex structure is defined by Sasaki [3] and called a (ϕ, ξ, η)-structure.

We shall give modern foundations for cocomplex structures and almost cocomplex structures.

§ 1. Preliminaries.

Let M be a differentiable manifold of dimension $2n+1$ and $F(M)$ the bundle of linear frames of M. Then $F(M)$ is a principal fibre boundle over M with structure group $GL(2n+1, R)$.

Let G be a subgroup of $GL(2n+1, R)$. A G-*structure* on M is a reduction of $F(M)$ to the group G.

Let $P_G(M)$ be a G-structure on M. We shall call a connection on $P_G(M)$ a G-*connection*.

Let $\mathfrak{g}$ be the Lie algebra of G. The cohomology class c in $\mathrm{Hom}(R^{2n+1} \wedge R^{2n+1}, R^{2n+1})/\partial\, \mathrm{Hom}(R^{2n+1}, \mathfrak{g})$ determined by the torsion form of a local G-connection is

Received June 19, 1967.

called the *first order structure tensor* of the G-structure $P_G(M)$.

§ 2. Cocomplex structures and almost cocomplex structures.

Suppose we are given in R^{2n+1} an involutive differential system of codimension one and a complex structure on its integral manifolds. To fix our notations, let $y^0, y^1, \cdots, y^{2n}$ be the natural coordinate system of R^{2n+1} and let

$$\alpha = dy^0$$

and

$$F = \sum_{i=1}^{n} \frac{\partial}{\partial y^i} \otimes dy^{i+n} - \sum_{i=1}^{n} \frac{\partial}{\partial y^{i+n}} \otimes dy^i.$$

Let $\mathcal{L}$ be the sheaf of germs of all vector fields X on R^{2n+1} which satisfy

$$L_X \alpha = 0$$

and

$$L_X F = 0,$$

where L_X denotes the Lie differentiation with respect to X.

Let $\mathcal{L}(0)$ be the stalk of $\mathcal{L}$ at the origin 0. Then $\mathcal{L}(0)$ is a *flat*, transitive filtered Lie algebra of infinite dimensions. The linear isotropy algebra $\mathfrak{g}$ of $\mathcal{L}(0)$ is the linear Lie algebra

$$\left\{ \left(\begin{array}{c|c} 0 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & A \\ 0 & \end{array} \right) \middle| \; A \in \mathfrak{gl}(n, C) \right\}.$$

The Lie algebra $\mathfrak{g}$ is involutive.

A diffeomorphism f: $U \to U'$, where U and U' are open subsets of R^{2n+1}, is called a *cocomplex transformation* if it satisfies

$$f^* \alpha = \alpha$$

and

$$f_* \circ F = F \circ f_*.$$

The collection, Γ, of all such cocomplex transformations forms an infinite, continuous pseudogroup.

Let M be a differentiable manifold of dimension $2n+1$. A Γ-structure on M is called a *cocomplex structure*.

A cocomplex structure is the same as a $2n$-dimensional involutive *complex* differential system. In other words, giving a cocomplex structure on M is the same as giving a closed 1-form ω and a tensor field J of type (1, 1) on M which satisfy

$$\omega\circ J=0,$$

$$J^2=-I+Z\otimes\omega$$

where Z is a unique vector field on M defined by

$$\omega(Z)=1 \quad \text{and} \quad J(Z)=0,$$

and

$$[JX, JY]-J[JX, Y]-J[X, JY]+J^2[X, Y]=0$$

for any vector fields X and Y which satisfy $\omega(X)=\omega(Y)=0$.

ω, J and Z can locally be written as

$$\omega=dx^0,$$

$$J=\sum_{i=1}^{n}\frac{\partial}{\partial x^i}\otimes dx^{i+n}-\sum_{i=1}^{n}\frac{\partial}{\partial x^{i+n}}\otimes dx^i,$$

$$Z=\frac{\partial}{\partial x^0}.$$

A local coordinate system in which ω, J and Z are written as above will be called an *admissible* coordinate.

Let Γ_0 be the subset of Γ consisting of the elements which leave the origin 0 invariant. Let j: $\Gamma_0\to GL(2n+1, \mathrm{R})$ be defined as follows: for $f\in\Gamma_0$, $j(f)$ is the 1-jet determined by f. Let $G=j(\Gamma_0)$. Then G is a subgroup of $GL(2n+1, \mathrm{R})$ whose Lie algebra is $\mathfrak{g}$, the linear isotropy algebra of $\mathcal{L}(0)$. G is isomorphic with $GL(n, \mathrm{C})$.

Let M be a differentiable manifold of dimension $2n+1$. An *almost cocomplex structure* on M is, by definition, a reduction of the bundle of linear frames $F(M)$ to G, that is, a G-structure $P_G(M)$ on M.

Given a G-structure $P_G(M)$ on M, we can define a 1-form η and a tensor field ϕ of type (1, 1) on M which satisfy

$$\eta\circ\phi=0 \tag{1}$$

and

$$\phi^2=-I+\xi\otimes\eta, \tag{2}$$

where ξ is a unique vector field on M defined by

$$\eta(\xi)=1 \quad \text{and} \quad \phi(\xi)=0.$$

In fact, for each $x\in M$, let u be a point of $P_G(M)$ with $\pi(u)=x$, where π: $P_G(M)\to M$ is the projection. For any tangent vector X at x, we set

$$\eta_x(X)=\alpha(u^{-1}X) \tag{3}$$

and

$$(4) \qquad \phi_x(X)=u(F(u^{-1}X)),$$

where we regard a frame u as a linear isomorphism of R^{2n+1} onto $T_x(M)$.

From the properties of G, this definition is independent of the choice of u.

Conversely, given a pair of a 1-form η and a tensor field ϕ of type (1, 1) on M, let $P_G(M)$ be the set of all linear frames u which satisfy (3) and (4) for any tangent vector X at $x=\pi(u)$. Then $P_G(M)$ is a G-structure on M.

Thus giving a G-structure on M is the same as giving a pair of a 1-form η and a tensor field ϕ of type (1, 1) which satisfy (1) and (2).

Let M_0 be a manifold with a cocomplex structure. Since every Γ-structure gives rise canonically to an almost Γ-structure, M_0 has an almost cocomplex structure.

THEOREM 2.1. *Let $P_G(M_0)$ be the almost cocomplex structure associated with a cocomplex structure on M_0. Then the first order structure tensor c vanishes.*

Proof. A representative of c is given by the torsion tensor of a G-connection. Let Π be a linear connection and ∇ the covariant differentiation with respect to Π. Let (ω, J) be the associated pair as before. Then Π is a G-connection if and only if

$$\nabla\omega=0 \quad \text{and} \quad \nabla J=0.$$

Using the local expressions

$$\omega=(1, 0, \cdots, 0)$$

and

$$J=\left(\begin{array}{c|cc} 0 & 0\cdots\cdots & 0 \\ \hline 0 & 0 & I_n \\ \vdots & & \\ 0 & -I_n & 0 \end{array}\right)$$

with respect to an admissible coordinate system $(x^0, x^1, \cdots, x^{2n})$, we can easily see that there exists a G-connection without torsion.

Since the first order structure tensor c is independent of the choice of a G-connection, our assertion is now clear. (Q.E.D.)

§ 3. The integrability problem for almost cocomplex structures.

Let M be a differentiable manifold of dimension $2n+1$ and $P_G(M)$ a G-structure, an almost cocomplex structure, on M.

$P_G(M)$ is said to be *integrable* if it determines a cocomplex structure on M.

Then the answer to the integrability problem for an almost cocomplex structure is the following

THEOREM 3. 1. *An almost cocomplex structure whose structure tensor of the first order vanishes is cocomplex.*

Proof. Let $P_G(M)$ be an almost cocomplex structure on M and (ϕ, η) the associated pair.

Let Π be a linear connection and ∇ the covariant differentiation with respect to Π. Then Π is a G-connection if and only if

$$\nabla\eta=0 \quad \text{and} \quad \nabla\phi=0.$$

Since the first order structure tensor of $P_G(M)$ vanishes, there exists a torsionfree G-connection.

In general, let Π be a torsionfree linear connection and α a differential form. Then

$$d\alpha=\mathcal{A}(\nabla\alpha),$$

where $\mathcal{A}$ is the alternation operator. Hence, let Π be a torsionfree G-connection. Then we have

$$d\eta=0.$$

Hence the differential system defined by η is involutive.

We have to prove that ϕ gives rise to a complex structure on each integral manifold of η.

The equation (2) implies that ϕ is an almost complex structure on each integral manifold of η. Let N be the *Nijenhuis torsion tensor field* of ϕ and let X and Y be vector fields on an integral manifold. Then

$$\begin{aligned} N(X, Y)&=[\phi X, \phi Y]-\phi[\phi X, Y]-\phi[X, \phi Y]+\phi^2[X, Y] \\ &=[\phi X, \phi Y]-\phi[\phi X, Y]-\phi[X, \phi Y]-[X, Y], \end{aligned}$$

since $\eta([X, Y])=0$.

On the other hand, since Π is a torsionfree G-connection, we have

$$\nabla\phi=0$$

and

$$[X, Y]=\nabla_X Y-\nabla_Y X$$

for any X and Y. Therefore

$$\begin{aligned} N(X, Y)&=\nabla_{\phi X}(\phi Y)-\nabla_{\phi Y}(\phi X)-\phi(\nabla_{\phi X} Y-\nabla_Y(\phi X))-\phi(\nabla_X(\phi Y)-\nabla_{\phi Y}X)-\nabla_X Y+\nabla_Y X \\ &=\phi^2(\nabla_Y X-\nabla_X Y)-(\nabla_X Y-\nabla_Y X) \\ &=-[Y, X]+\eta([Y, X])\cdot\xi-[X, Y] \\ &=0. \end{aligned}$$

This implies that ϕ defines a complex structure on each integral manifold of η. Hence (ϕ, η) determines a cocomplex structure. (Q.E.D.)

Let S be the *Sasakian torsion tensor* of ϕ [4]. Then

$$S = N + 2\xi \otimes d\eta.$$

Hence we have

COROLLARY. *If* $d\eta = 0$ *and* $S = 0$, *then the almost cocomplex structure is cocomplex.*

BIBLIOGRAPHY

[1] GUILLEMIN, V., AND S. STERNBERG, Deformation theory of pseudogroup structures. Memoirs of Amer. Math. Soc., No. 64 (1966).

[2] OGIUE, K., On almost contact structures. Kōdai Math. Sem. Rep. **19** (1967), 498–506.

[3] SASAKI, S., On differentiable manifolds with certain structures which are closely related to almost contact structure I. Tôhoku Math. J. **12** (1960), 459–476.

[4] SASAKI, S., AND Y. HATAKEYAMA, On differentiable manifolds with certain structures which are closely related to almost contact structure II. Tôhoku Math. J. **13** (1961), 281–294.

[5] SINGER, I. M., AND S. STERNBERG, The infinite groups of Lie and Cartan. J. d'Analyse Math. **15** (1965), 1–114.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.