ON THE EXISTENCE OF ANALYTIC MAPPINGS

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§1. Let R and S be two Riemann surfaces. When are there any analytic mappings from R into S ? This would be one of the most important problems. However it is very difficult to establish a perfect condition for the existence of analytic mappings between two arbitrary Riemann surfaces.

Let R and S be two ultrahyperelliptic surfaces defined by two equations $y^2=G(z)$ and $w^2=g(w)$, respectively, where G and g are two entire functions having no zero other than an infinite number of simple zeros. Then Ozawa [1], [2], [3], [4] obtained a perfect condition for the existence of analytic mappings from R into S and several other interesting results.

In the first place (§2~§5) we shall discuss the Riemann surfaces defined in the following manner: Let R_3 and S_3 be two regularly branched three-sheeted covering Riemann surfaces defined by two equations $y^3=G_2(z)$ and $w^3=g_2(w)$, respectively, where G_2 and g_2 are two entire functions having no zero other than an infinite number of simple or double zeros.

Next (§6~§7) we shall discuss on the existence of analytic mappings between ultrahyperelliptic surface and our Riemann surface. Then we can conclude that there is no analytic mapping.

Finally (§8~§10) we shall treat the case where S is a closed and regularly branched three-sheeted (or two-sheeted) covering Riemann surface.

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§2. Let p_{R_3} and p_{S_3} be the projection maps $(z, y) \rightarrow z, (w, u) \rightarrow w$, respectively. Let φ be an analytic mapping from R_3 into S_3 .

DEFINITION. If φ satisfies $p_{S_3} \circ \varphi(p_1) = p_{S_3} \circ \varphi(p_2) = p_{S_3} \circ \varphi(p_3)$ for every distinct p_i ($i=1, 2, 3$) with $p_{R_3}(p_1)=p_{R_3}(p_2)=p_{R_3}(p_3)$, then we say that φ satisfies the *rigidity of projection map* or φ is a *rigid* analytic mapping from R_3 into S_3 . Similar definition for the rigidity of projection map may be given for an arbitrary analytic function on R_3 (or S_3). If the condition is not satisfied, then we say that φ is a non-rigid analytic mapping or function.

Here we shall prove the following theorem:

THEOREM 1. *There is no non-rigid analytic mapping from R_3 into S_3 .*

In order to prove this theorem we need the following lemma on algebroid functions.

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LEMMA 1. Let $h(z) \equiv h_1(z) + h_2(z) \sqrt[3]{G_2(z)} + h_3(z) \sqrt[3]{G_2(z)^2}$ be a three-valued algebroid entire function of z , where h_1 and h_2 are entire functions of z and h_3 a single-valued regular function of z excepting at most all the double zeros of G_2 at which it might have simple poles. Then the functional equation of the form

$$g_2 \circ h(z) = \{\lambda_1(z) + \lambda_2(z) \sqrt[3]{G_2(z)} + \lambda_3(z) \sqrt[3]{G_2(z)^2}\}^3$$

has no solution for any three-valued algebroid entire function $\lambda(z) \equiv \lambda_1(z) + \lambda_2(z) \sqrt[3]{G_2(z)} + \lambda_3(z) \sqrt[3]{G_2(z)^2}$ of z , where λ_1 and λ_2 are entire functions of z and λ_3 a single-valued regular function of z excepting at most all the double zeros of G_2 at which it might have simple poles.

Proof. Let $\{w_k\}$ be the set of zero points of $g_2(w)$. The equations

$$h(z) \equiv h_1(z) + h_2(z) \sqrt[3]{G_2(z)} + h_3(z) \sqrt[3]{G_2(z)^2} = w_k$$

must have an infinite number of roots $\{z_{kl}\}$ excepting at most six values. Since g_2 has only simple or double zeros, these roots $\{z_{kl}\}$ are taken with multiplicity at least 3. Hence there exist infinitely many values $\{w_k\}$ which have multiplicity at least 3 for $h(z)$.¹⁾ On the other hand by the ramification theorem and Nevanlinna-Selberg's second fundamental theorem,

$$\sum \Theta(w_k) \leq 6,$$

where

$$\Theta(w) = 1 - \lim_{r \rightarrow \infty} \frac{\bar{N}(r; w, h)}{T(r, h)}.$$

If w has multiplicity at least 3 for $h(z)$, then $\Theta(w) \geq 2/3$. Thus there are at most 9 values $\{w_k\}$ which have multiplicity at least 3 for $h(z)$. This is a contradiction. Thus we have the desired conclusion.

Proof of theorem 1. If φ is a non-rigid analytic mapping from R_3 into S_3 , then the shifted mapping $\Phi = p_{S_3} \circ \varphi$, which is an entire function on R_3 , is represented as a three-valued algebroid entire function in the following manner:

$$h(z) \equiv \Phi \circ p_{R_3}^{-1}(z) = h_1(z) + h_2(z) \sqrt[3]{G_2(z)} + h_3(z) \sqrt[3]{G_2(z)^2},$$

where h_1 and h_2 are entire functions of z and h_3 a single-valued regular function of z excepting at most all the double zeros of G_2 at which it might have simple poles. Let $\sqrt[3]{g_2^*}$ be an analytic function on S_3 defined by $\sqrt[3]{g_2^*} = \sqrt[3]{g_2 \circ p_{S_3}}$. Then $\sqrt[3]{g_2^*} \circ \varphi$ is an analytic function on R_3 . Hence it can be represented in the following manner:

$$\sqrt[3]{g_2^*} \circ \varphi \circ p_{R_3}^{-1}(z) = \lambda_1(z) + \lambda_2(z) \sqrt[3]{G_2(z)} + \lambda_3(z) \sqrt[3]{G_2(z)^2},$$

where λ_1 and λ_2 are entire functions of z and λ_3 a single-valued regular function of z excepting at most all the double zeros of G_2 at which it might have simple poles. On the other hand we have

1) We say that a value w has multiplicity at least m for $f(z)$ if all roots of the equation $f(z) = w$ have multiplicity at least m .

$$\sqrt[3]{g_2^* \circ \varphi \circ p_{R_3}^{-1}}(z) = \sqrt[3]{g_2 \circ p_{S_3} \circ \varphi \circ p_{R_3}^{-1}}(z) = \sqrt[3]{g_2 \circ p_{S_3} \circ p_{S_3}^{-1} \circ h \circ p_{R_3} \circ p_{R_3}^{-1}}(z) = \sqrt[3]{g_2 \circ h}(z).$$

Hence

$$g_2 \circ h(z) = \{\lambda_1(z) + \lambda_2(z) \sqrt[3]{G_2(z)} + \lambda_3(z) \sqrt[3]{G_2(z)^2}\}^3$$

must hold. However by lemma 1 it is untenable. In the above discussions the choice of three analytic branches of $p_{R_3}^{-1}$ and $p_{S_3}^{-1}$ are free unless they are fixed suitably. However in these cases it is sufficient to consider some sheet-exchanged mappings or functions according to circumstances and we can arrive at the same conclusion. This completes the proof of theorem 1.

§3. We shall prove a perfect condition for the existence of analytic mappings from R_3 into S_3 .

THEOREM 2. *If there exists an analytic mapping φ from R_3 into S_3 , then there exists an entire function $h(z)$ of z such that either $\nu(z)^3 G_2(z) = g_2 \circ h(z)$ or $\mu(z)^3 G_2(z)^2 = g_2 \circ h(z)$ remains true, where $\nu(z)$ is an entire function of z and $\mu(z)$ a single-valued regular function of z excepting at most all the double zeros of $G_2(z)$ at which it might have simple poles. The converse holds good.*

Proof. In our discussions it is necessary to consider the effects occurring from the choice of three analytic branches of $p_{R_3}^{-1}$ and $p_{S_3}^{-1}$ and to modify suitably as the cases may be. In these cases we may adopt suitable sheet-exchanged analytic mappings. Then we can arrive at the same conclusion.

Let φ be an analytic mapping from R_3 into S_3 . Then the shifted mapping $\Phi = p_{S_3} \circ \varphi$ is represented in the form:

$$h(z) \equiv \Phi \circ p_{R_3}^{-1}(z) = h_1(z) + h_2(z) \sqrt[3]{G_2(z)} + h_3(z) \sqrt[3]{G_2(z)^2},$$

where h_1 and h_2 are entire functions of z and h_3 a single-valued regular function of z excepting at most all the double zeros of G_2 at which it might have simple poles. By theorem 1 φ is rigid, we have then $h_2 \equiv h_3 \equiv 0$, which is equivalent to the rigidity of φ . Thus $h(z) \equiv p_{S_3} \circ \varphi \circ p_{R_3}^{-1}(z)$ is an entire function of z .

Let F be an analytic mapping from S_3 into the F -plane, then $f \equiv F \circ \varphi$ is an analytic mapping from R_3 into the F -plane. Let F^* and f^* be two three-valued functions corresponding to F and f , that is, $F^*(w) = F \circ p_{S_3}^{-1}(w)$ and $f^* = f \circ p_{R_3}^{-1}(z)$, respectively. Then we have the representations:

$$F^*(w) = F_1(w) + F_2(w) \sqrt[3]{g_2(w)} + F_3(w) \sqrt[3]{g_2(w)^2},$$

where F_1 and F_2 are entire functions of w and F_3 a single-valued regular function of w excepting at most all the double zeros of g_2 at which it might have simple poles;

$$f^*(z) = f_1(z) + f_2(z) \sqrt[3]{G_2(z)} + f_3(z) \sqrt[3]{G_2(z)^2},$$

where f_1 and f_2 are entire functions of z and f_3 a single-valued regular function of z excepting at most all the double zeros of G_2 at which it might have simple poles. Hence we have

$$\begin{aligned}
& f_1(z) + f_2(z) \sqrt[3]{G_2(z)} + f_3(z) \sqrt[3]{\overline{G_2(z)}} = f \circ \mathfrak{p}_{R_3}^{-1}(z) = F \circ \varphi \circ \mathfrak{p}_{R_3}^{-1}(z) \\
& = F \circ \mathfrak{p}_{S_3}^{-1} \circ h \circ \mathfrak{p}_{R_3} \circ \mathfrak{p}_{R_3}^{-1}(z) = F \circ \mathfrak{p}_{S_3}^{-1} \circ h(z) = F^* \circ h(z) \\
& = F_1 \circ h(z) + F_2 \circ h(z) \sqrt[3]{g_2 \circ h(z)} + F_3 \circ h(z) \sqrt[3]{\overline{g_2 \circ h(z)}}.
\end{aligned}$$

If F is the analytic mapping from S_3 into the F -plane which is defined by $F \circ \mathfrak{p}_{S_3}^{-1}(w) = \sqrt[3]{g_2(w)}$, then we have

$$f_1(z) + f_2(z) \sqrt[3]{G_2(z)} + f_3(z) \sqrt[3]{\overline{G_2(z)}} = \sqrt[3]{g_2 \circ h(z)}.$$

If $\sqrt[3]{g_2 \circ h(z)}$ reduces to a single-valued function, then all zeros $\{w_k\}$ of $g_2(w)$, which are not Picard's values, have multiplicity at least 3 for $h(z)$, since $g_2(w)$ has only simple or double zeros. On the other hand by Nevanlinna's second fundamental theorem, which remains true in this case, $\Sigma \theta(w_k) \leq 2$, where $\theta(w)$ has already been defined. If w has multiplicity at least 3, then $\theta(w) \geq 2/3$. Thus there exist at most 3 values $\{w_k\}$ which have multiplicity at least 3 for $h(z)$. This is a contradiction. Therefore $\sqrt[3]{g_2 \circ h(z)}$ is a three-valued function. We have then

$$\begin{aligned}
& f_1^3 + 6f_1f_2f_3G_2 + f_2^3G_2 + f_3^3G_2^2 \\
& + 3\sqrt[3]{G_2}(f_1^2f_2 + f_1f_3^2G_2 + f_2^2f_3G_2) + 3\sqrt[3]{\overline{G_2}}(f_1^2f_3 + f_1f_2^2 + f_2f_3^2G_2) = g_2 \circ h.
\end{aligned}$$

By the single-valuedness of $g_2 \circ h$ we have

$$f_1^2f_2 + f_1f_3^2G_2 + f_2^2f_3G_2 \equiv 0, \quad f_1^2f_3 + f_1f_2^2 + f_2f_3^2G_2 \equiv 0.$$

Therefore we have either $f_3 \equiv 0$ or $f_1^3 - f_2^3G_2 \equiv 0$. On the other hand by the three-valuedness of $\sqrt[3]{g_2 \circ h}$, either f_2 or f_3 does not vanish identically. If $f_3 \equiv 0$ holds, then we have $f_1f_2 \equiv 0$. Hence $f_1 \equiv f_3 \equiv 0$, $f_2 \not\equiv 0$. Thus $f_2^3G_2 = g_2 \circ h$ holds in this case. If $f_1^3 - f_2^3G_2 \equiv 0$ holds, then we can conclude that $f_1 \equiv f_2 \equiv 0$, $f_3 \not\equiv 0$, because G_2 has no zero other than an infinite number of simple or double zeros. Hence $f_3^3G_2^2 = g_2 \circ h$. This completes the proof of necessity.

Suppose that either $\nu^3G_2 = g_2 \circ h$ or $\mu^3G_2^2 = g_2 \circ h$ holds. Then it is clear that the mapping $\varphi = \mathfrak{p}_{S_3}^{-1} \circ h \circ \mathfrak{p}_{R_3}$ is an analytic mapping from R_3 into S_3 , since every branch point of R_3 corresponds to a branch point of S_3 , by $\nu^3G_2 = g_2 \circ h$ or $\mu^3G_2^2 = g_2 \circ h$. This completes the proof of sufficiency.

§4. Let φ be an analytic mapping from R_3 into S_3 , then the shifted mapping $\Phi = \mathfrak{p}_{S_3} \circ \varphi$ is an entire function on R_3 . Let $T(r, \Phi)$ be the Nevanlinna-Selberg characteristic function of Φ . Let $N(r, R_3)$ be the quantity $N(r, \mathfrak{R})$ defined by Selberg [5]. We shall now prove

THEOREM 3. *Suppose that there exists an analytic mapping φ from R_3 into S_3 . Then φ satisfies*

$$\lim_{r \rightarrow \infty} \frac{N(r, R_3)}{T(r, \Phi)} = \infty.$$

Proof. By theorem 2 we may assume that either $\nu(z)^3G_2(z) = g_2 \circ h(z)$ or $\mu(z)^3G_2(z)^2 = g_2 \circ h(z)$ holds good.

In the first place we shall consider the case where the equation $\nu(z)^3 G_2(z) = g_2 \circ h(z)$ holds for two suitable entire functions h and ν . Since g_2 has only simple or double zeros, we have

$$\begin{aligned} N(r; 0, g_2 \circ h) &= N_2(r; 0, g_2 \circ h) + N_1(r; 0, g_2 \circ h) + \bar{N}_1(r; 0, g_2 \circ h), \\ \bar{N}_1(r; 0, g_2 \circ h) &\leq N_1(r; 0, g_2 \circ h) \leq N(r; 0, h') + N_3(r; 0, g_2 \circ h) \\ &\leq m(r, h'/h) + m(r, h) + N_3(r; 0, g_2 \circ h), \\ m(r, h'/h) &\leq O(\log(rm(r, h))), \end{aligned}$$

where N_2 denotes the N -function of simple zeros and N_3 that of double zeros. Hence we have

$$\begin{aligned} &N(r; 0, g_2 \circ h) - 2m(r, h) + O(\log(rm(r, h))) \\ &\leq N_2(r; 0, g_2 \circ h) + 2N_3(r; 0, g_2 \circ h) \leq N_2(r; 0, G_2) + 2N_3(r; 0, G_2) \\ &\leq 2N(r; 0, G_2) \leq 6N(r, R_3) + O(\log r). \end{aligned}$$

Let $\{w_1, w_2, \dots, w_p\}$ be a set of an arbitrary finite number of zeros of $g_2(w)$. If h is a transcendental entire function, then by the second fundamental theorem applied to h and a slight discussion we have

$$6N(r, R_3) + O(\log r) \geq (p-4)m(r, h) + O(\log(rm(r, h))).$$

Therefore we have

$$\overline{\lim}_{r \rightarrow \infty} \frac{6N(r, R_3)}{m(r, h)} \geq p-4.$$

If h is a polynomial, then by a direct calculation we have

$$\overline{\lim}_{r \rightarrow \infty} \frac{N(r, R_3)}{m(r, h)} \geq p-K,$$

where K is a constant. By the arbitrariness of the number p and $T(r, \Phi) = m(r, h)$, we have the desired result:

$$\overline{\lim}_{r \rightarrow \infty} \frac{N(r, R_3)}{T(r, \Phi)} = \infty.$$

If $\mu(z)^3 G_2(z)^2 = g_2 \circ h(z)$ holds, then by means of a method similar as above we can arrive at the same conclusion.

§5. Let G_2^c and g_2^c be two canonical products having the same zeros with the same multiplicities as those of G_2 and g_2 , respectively. Let $\rho_{G_2^c}$ and $\rho_{g_2^c}$ be the orders of G_2^c and g_2^c , respectively. Then by theorem 2 we can prove the following theorem:

THEOREM 4. Assume that $\rho_{G_2^c} < \infty$ and $0 < \rho_{g_2^c} < \infty$ and that there exists an analytic mapping φ from R_3 into S_3 . Then $\rho_{G_2^c}$ is an integral multiple of $\rho_{g_2^c}$.

The proof may be omitted, since the same method as in the proof of theorem 1 in [4] remains valid.

§6. Let S_2 be an ultrahyperelliptic surface defined by $u^2 = g_1(w)$, where g_1 is an entire function having no zero other than an infinite number of simple zeros. Let R_3 be a Riemann surface defined in §1.

In the subsequent discussions of the paper we define the rigidity of an analytic mapping similarly as in the cases already discussed.

We state a lemma which can be proved by the same reasoning as in the proof of lemma 1.

LEMMA 2. *Let $h(z) \equiv h_1(z) + h_2(z)\sqrt[3]{G_2(z)} + h_3(z)\sqrt[3]{G_2(z)^2}$ be a three-valued algebroid entire function of z , where h_1 and h_2 are entire functions of z and h_3 a single-valued regular function of z excepting at most all the double zeros of G_2 at which it might have simple poles. Then the functional equation of the form*

$$g_1 \circ h(z) = \{\lambda_1(z) + \lambda_2(z)\sqrt[3]{G_2(z)} + \lambda_3(z)\sqrt[3]{G_2(z)^2}\}^2$$

has no solution for any three-valued algebroid entire function $\lambda(z) \equiv \lambda_1(z) + \lambda_2(z)\sqrt[3]{G_2(z)} + \lambda_3(z)\sqrt[3]{G_2(z)^2}$, where λ_1 and λ_2 are entire functions of z and λ_3 a single-valued regular function of z excepting at most all the double zeros of G_2 at which it might have simple poles.

We shall now prove the following theorem:

THEOREM 5. *There is no analytic mapping from R_3 into S_2 .*

Proof. If there exists a non-rigid analytic mapping φ from R_3 into S_2 , then by the same reasoning as in the proof of theorem 1, which can be applied in this case, we have an equation of the form

$$g_1 \circ \{h_1(z) + h_2(z)\sqrt[3]{G_2(z)} + h_3(z)\sqrt[3]{G_2(z)^2}\} = \{\lambda_1(z) + \lambda_2(z)\sqrt[3]{G_2(z)} + \lambda_3(z)\sqrt[3]{G_2(z)^2}\}^2,$$

where h_1, h_2, λ_1 and λ_2 are entire functions of z and h_3 and λ_3 are single-valued regular functions of z excepting at most all the double zeros of G_2 at which they might have simple poles. By lemma 2 it is untenable. Thus every analytic mapping φ from R_3 into S_2 is rigid whenever it exists. Hence $h(z) \equiv p_{S_2} \circ \varphi \circ p_{R_3}^{-1}(z)$ is an entire function of z .

Let F be the analytic mapping from S_2 into the F -plane defined by $F \circ p_{S_2}^{-1}(w) = \sqrt{g_1(w)}$, then we have

$$f_1(z) + f_2(z)\sqrt[3]{G_2(z)} + f_3(z)\sqrt[3]{G_2(z)^2} = \sqrt{g_1 \circ h(z)},$$

where f_1 and f_2 are entire functions of z and f_3 a single-valued regular function of z excepting at most all the double zeros of G_2 at which it might have simple poles. Therefore

$$f_1^2 + 2f_2f_3G_2 + \sqrt[3]{G_2}(2f_1f_2 + f_3^2G_2) + \sqrt[3]{G_2^2}(2f_1f_3 + f_2^2) = g_1 \circ h.$$

By the single-valuedness of $g_1 \circ h$ we have

$$2f_1f_2 + f_3^2G_2 \equiv 0, \quad 2f_1f_3 + f_2^2 \equiv 0.$$

Thus $f_2^3 - f_3^3 G_2 \equiv 0$ holds. Then we have $f_2 \equiv f_3 \equiv 0$. Hence $f_1^3 = g_1 \circ h$. Since g_1 has only simple zeros $\{w_k\}$ which are infinite in number, every w_k , which is not a Picard's value, has multiplicity at least 2 for $h(z)$. Hence $\theta(w_k) \geq 1/2$. On the other hand by Nevanlinna's second fundamental theorem, $\sum \theta(w_k) \leq 2$. It is a contradiction. Thus there is no analytic mapping from R_3 into S_2 .

§7. Let R_2 be an ultrahyperelliptic surface defined by $y^2 = G_1(z)$, where G_1 is an entire function having no zero other than an infinite number of simple zeros. Let S_3 be a Riemann surface defined in §1. We state a lemma which can be proved by the same reasoning as in the proof of lemma 1.

LEMMA 3. *Let $h(z) \equiv h_1(z) + h_2(z)\sqrt{G_1(z)}$ be a two-valued algebroid entire function of z , where h_1 and h_2 are entire functions of z . Then the functional equation of the form*

$$g_2 \circ h(z) = \{\lambda_1(z) + \lambda_2(z)\sqrt{G_1(z)}\}^3$$

has no solution for any two-valued algebroid entire function $\lambda(z) \equiv \lambda_1(z) + \lambda_2(z)\sqrt{G_1(z)}$ of z , where λ_1 and λ_2 are entire functions of z .

We shall now prove the following theorem:

THEOREM 6. *There is no analytic mapping from R_2 into S_3 .*

Proof. If there exists a non-rigid analytic mapping φ from R_2 into S_3 , then we have

$$g_2 \circ \{h_1(z) + h_2(z)\sqrt{G_1(z)}\} = \{\lambda_1(z) + \lambda_2(z)\sqrt{G_1(z)}\}^3,$$

where h_1, h_2, λ_1 and λ_2 are entire functions of z . By lemma 3 it is untenable. Thus there is no non-rigid analytic mapping from R_2 into S_3 . Hence for every analytic mapping φ from R_2 into S_3 , whenever it exists, $h(z) \equiv p_{S_3} \circ \varphi \circ p_{R_2}^{-1}(z)$ is a single-valued regular function of z .

Let F be the analytic mapping from S_3 into the F -plane defined by $F \circ p_{S_3}^{-1}(w) = \sqrt[3]{g_2(w)}$, then we have

$$f_1(z) + f_2(z)\sqrt{G_1(z)} = \sqrt[3]{g_2 \circ h(z)}.$$

Therefore

$$f_1^3 + 3f_1 f_2^2 G_1 + \sqrt{G_1}(3f_1^2 f_2 + f_2^3 G_1) = g_2 \circ h.$$

By the single-valuedness of $g_2 \circ h$ we have

$$3f_1^2 f_2 + f_2^3 G_1 \equiv 0.$$

If $f_2 \not\equiv 0$, then G_1 has double zeros. It is a contradiction. Therefore $f_2 \equiv 0$. Hence we have $f_1^3 = g_2 \circ h$. Then, by Nevanlinna's second fundamental theorem, we can say that this relation does not hold. Thus there is no analytic mapping from R_2 into S_3 .

§8. In this section we shall discuss the case where S is a closed and regularly branched three-sheeted covering Riemann surface.

Let R_3 be a regularly branched three-sheeted covering Riemann surface defined by $y^3 = G_2(z)$, where G_2 is an entire function of z having no zero other than an infinite number of simple or double zeros. Let S_3^* be a closed and regularly branched three-sheeted covering Riemann surface defined by $u^3 = P_2(w)$, where P_2 is a polynomial of the form

$$\prod_{k=1}^n (w - w_k)^{c_k},$$

where $c_k \leq 2$ are positive integers.

LEMMA 4. *Let $h(z) \equiv h_1(z) + h_2(z)\sqrt[3]{G_2(z)} + h_3(z)\sqrt[3]{G_2(z)^2}$ be a three-valued meromorphic function of z , where h_1, h_2 and h_3 are meromorphic functions of z . Then the functional equation of the form*

$$P_2 \circ h(z) = \{\lambda_1(z) + \lambda_2(z)\sqrt[3]{G_2(z)} + \lambda_3(z)\sqrt[3]{G_2(z)^2}\}^3$$

has no solution for any three-valued meromorphic function $\lambda(z) \equiv \lambda_1(z) + \lambda_2(z)\sqrt[3]{G_2(z)} + \lambda_3(z)\sqrt[3]{G_2(z)^2}$, where λ_1, λ_2 and λ_3 are meromorphic functions of z , when $n \geq 10$.

Proof. Since P_2 has $n (\geq 10)$ simple or double zeros $\{w_k\}$, every root of the equation $h(z) = w_k$ has multiplicity at least 3 whenever it exists. Hence for every w_k , $\Theta(w_k) \geq 2/3$, where $\Theta(w)$ has already been defined. On the other hand by Nevanlinna-Selberg's second fundamental theorem on algebroid functions $\sum_{k=1}^n \Theta(w_k) \leq 6$. This is untenable provided $n \geq 10$.

By lemma 4 and the same method used in the proof of theorem 1, we can say that every analytic mapping φ from R_3 into S_3^* is rigid, when the genus of S_3^* is not less than 8. Then every branch point of R_3 has its φ -image on a branch point of S_3^* . Thus there exists at least one branch point of S_3^* which is covered by $\varphi(R_3)$ infinitely many times. Therefore in this case $h(z) \equiv p_{S_3^*} \circ \varphi \circ p_{R_3}^{-1}(z)$ should be a transcendental meromorphic function. Then we can conclude that either $\nu(z)^3 G_2(z) = P_2 \circ h(z)$ or $\mu(z)^3 G_2(z)^2 = P_2 \circ h(z)$ holds, where ν and μ are meromorphic functions of z . Since G_2 has only simple or double zeros, we have

$$3N(r, R_3) + O(\log r) \leq 2N(r; 0, G_2) \leq 2N(r; 0, P_2 \circ h)$$

$$\leq 2 \sum_{k=1}^n c_k T(r, h) + O(1) = 2 \sum_{k=1}^n c_k T(r, \Phi) + O(1),$$

where Φ is the shifted mapping $p_{S_3^*} \circ \varphi$. Since h is transcendental in this case, we have

$$\overline{\lim}_{r \rightarrow \infty} \frac{N(r, R_3)}{T(r, \Phi)} \leq \frac{2}{3} \sum_{k=1}^n c_k.$$

Consequently we have the following theorem:

THEOREM 7. *There is no non-rigid analytic mapping from R_3 into S_3^* , when the genus of S_3^* is not less than 8. If there exists an analytic mapping φ from R_3 into S_3^* , then there exists a meromorphic function $h(z)$ of z such that either $\nu(z)^3 G_2(z) = P_2 \circ h(z)$ or $\mu(z)^3 G_2(z)^2 = P_2 \circ h(z)$ remains true, where ν and μ are meromorphic functions of z and P_2 a polynomial of the form*

$$\prod_{k=1}^n (w - w_k)^{c_k},$$

where $c_k \leq 2$ and $n \geq 10$. Further it holds

$$\lim_{r \rightarrow \infty} \frac{N(r, R_3)}{T(r, \Phi)} \leq \frac{2}{3} \sum_{k=1}^n c_k.$$

If the genus of S_3^* is less than 8, then the assertion of the first part of theorem 7 does not hold in general. Here we shall give an example. Set

$$z = \phi(w) = \int_0^w \frac{dt}{\sqrt[3]{\{(t-a_1)(t-a_2)(t-a_3)\}^2}},$$

where a_1, a_2 and a_3 are non-zero distinct complex numbers. The inverse function can be analytically continued over the whole plane as a single-valued meromorphic function by Schwarz's reflection and the resulting function $w = f(z)$ is doubly periodic. It satisfies

$$f'(z)^3 = \{(f(z) - a_1)(f(z) - a_2)(f(z) - a_3)\}^2.$$

Let R_3 be the proper existence domain of $\sqrt[3]{f(z)}$, then it is a regularly branched three-sheeted covering Riemann surface. Let S_3^* be the closed and regularly branched three-sheeted covering Riemann surface defined by

$$u^3 = \{(w^3 - a_1)(w^3 - a_2)(w^3 - a_3)\}^2.$$

Evidently we have

$$\begin{aligned} f'(z)^3 &= \{(\sqrt[3]{f(z)} - \sqrt[3]{a_1})(\sqrt[3]{f(z)} - \sqrt[3]{a_1}\omega)(\sqrt[3]{f(z)} - \sqrt[3]{a_1}\omega^2) \\ &\quad \cdot (\sqrt[3]{f(z)} - \sqrt[3]{a_2})(\sqrt[3]{f(z)} - \sqrt[3]{a_2}\omega)(\sqrt[3]{f(z)} - \sqrt[3]{a_2}\omega^2) \\ &\quad \cdot (\sqrt[3]{f(z)} - \sqrt[3]{a_3})(\sqrt[3]{f(z)} - \sqrt[3]{a_3}\omega)(\sqrt[3]{f(z)} - \sqrt[3]{a_3}\omega^2)\}^2, \end{aligned}$$

where $\omega = \exp(2\pi i/3)$. This shows that there is a non-rigid analytic mapping φ from R_3 into S_3^* , which is induced by $\sqrt[3]{f(z)}$, that is, $h(z) = \sqrt[3]{f(z)}$, $\varphi = \mathfrak{p}_{S_3^*}^{-1} \circ h \circ \mathfrak{p}_{R_3}$. Thus the assertion of the first part of theorem 7 does not hold when the genus of S_3^* is equal to 7.

§ 9. Let $P_1(w)$ be a polynomial of the form

$$\prod_{k=1}^{2n} (w - w_k) \quad \text{or} \quad \prod_{k=1}^{2n-1} (w - w_k).$$

Let S_2^* be a hyperelliptic surface defined by $u^2 = P_1(w)$. Let R_3 be a Riemann surface which has already been defined. Then we can state the following lemma, the proof of which may be omitted since it can be readily proved by the ramification relation.

LEMMA 5. *Let $h(z) \equiv h_1(z) + h_2(z)\sqrt[3]{G_2(z)} + h_3(z)\sqrt[3]{G_2(z)^2}$ be a three-valued meromorphic function of z , where h_1 , h_2 and h_3 are meromorphic functions of z . Then the functional equation of the form*

$$P_1 \circ h(z) = \{\lambda_1(z) + \lambda_2(z)\sqrt[3]{G_2(z)} + \lambda_3(z)\sqrt[3]{G_2(z)^2}\}^2$$

has no solution for any three-valued meromorphic function $\lambda(z) \equiv \lambda_1(z) + \lambda_2(z)\sqrt[3]{G_2(z)} + \lambda_3(z)\sqrt[3]{G_2(z)^2}$, where λ_1 , λ_2 and λ_3 are meromorphic functions of z , when $n \geq 7$.

By this lemma and the same consideration as before we can see that every analytic mapping from R_3 into S_2^* is rigid whenever it exists, when the genus of S_2^* is not less than 6. On the other hand by a method quite similar as in the theorem 5, we can say that there is no rigid analytic mapping from R_3 into S_2^* , when the genus of S_2^* is greater than 1. Hence we have

THEOREM 8. *There is no analytic mapping from R_3 into S_2^* , when the genus of S_2^* is not less than 6.*

If the genus of S_2^* is less than 6, then the assertion of theorem 8 does not hold in general. Here we shall give two examples.²⁾

Let R_3 be the proper existence domain of $\sqrt[3]{\text{sn } z}$ with Jacobi's sn-function. Let S_2^* be the hyperelliptic surface defined by

$$u^2 = (1 - w^6)(1 - k^2 w^6).$$

It is well-known that

$$\begin{aligned} (\text{sn}' z)^2 &= (1 - \text{sn}^2 z)(1 - k^2 \text{sn}^2 z) \\ &= (1 - \sqrt[3]{\text{sn } z})(\omega - \sqrt[3]{\text{sn } z})(\omega^2 - \sqrt[3]{\text{sn } z}) \\ &\quad \cdot (-1 - \sqrt[3]{\text{sn } z})(-\omega - \sqrt[3]{\text{sn } z})(-\omega^2 - \sqrt[3]{\text{sn } z}) \\ &\quad \cdot (1 - \sqrt[3]{k \text{sn } z})(\omega - \sqrt[3]{k \text{sn } z})(\omega^2 - \sqrt[3]{k \text{sn } z}) \\ &\quad \cdot (-1 - \sqrt[3]{k \text{sn } z})(-\omega - \sqrt[3]{k \text{sn } z})(-\omega^2 - \sqrt[3]{k \text{sn } z}), \end{aligned}$$

where $\omega = \exp(2\pi i/3)$. This shows that there is an analytic mapping φ from R_3 into S_2^* , which is induced by $\sqrt[3]{\text{sn } z}$, that is, $h(z) = \sqrt[3]{\text{sn } z}$, $\varphi = p_{S_2^*}^{-1} \circ h \circ p_{R_3}$. This example shows that the assertion of theorem 8 does not hold when the genus of S_2^* is equal to 5.

Let R_3 be the proper existence domain of $\sqrt[3]{\wp(z)}$ with Weierstrass' $\wp$ -function

2) The author is indebted to Professor M. Ozawa for pointing out these examples.

with the primitive periods $2\omega_1$ and $2\omega_2$. Let S_2^* be the hyperelliptic surface defined by

$$u^2 = 4(w^3 - e_1)(w^3 - e_2)(w^3 - e_3),$$

where $e_1 = \wp(\omega_1)$, $e_2 = \wp(\omega_2)$ and $e_3 = \wp(\omega_1 + \omega_2)$. Evidently S_2^* is of genus 4. It is well-known that

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$

Thus we have

$$\begin{aligned} \wp'(z)^2 &= 4(\sqrt[3]{\wp(z)} - \sqrt[3]{e_1})(\sqrt[3]{\wp(z)} - \sqrt[3]{e_1}\omega)(\sqrt[3]{\wp(z)} - \sqrt[3]{e_1}\omega^2) \\ &\quad \cdot (\sqrt[3]{\wp(z)} - \sqrt[3]{e_2})(\sqrt[3]{\wp(z)} - \sqrt[3]{e_2}\omega)(\sqrt[3]{\wp(z)} - \sqrt[3]{e_2}\omega^2) \\ &\quad \cdot (\sqrt[3]{\wp(z)} - \sqrt[3]{e_3})(\sqrt[3]{\wp(z)} - \sqrt[3]{e_3}\omega)(\sqrt[3]{\wp(z)} - \sqrt[3]{e_3}\omega^2), \end{aligned}$$

where $\omega = \exp(2\pi i/3)$. This shows that there is an analytic mapping φ from R_3 into S_2^* , which is induced by $\sqrt[3]{\wp(z)}$, that is, $h(z) = \sqrt[3]{\wp(z)}$, $\varphi = p_{S_2^*}^{-1} \circ h \circ p_{R_3}$. Thus the assertion of theorem 8 does not hold when the genus of S_2^* is equal to 4.

§ 10. Let $P_2(w)$ be again a polynomial of the form

$$\prod_{k=1}^n (w - w_k)^{c_k},$$

where $c_k \leq 2$ are positive integers. Let $G_1(z)$ be an entire function of z having no zero other than an infinite number of simple zeros. Let S_3^* be a regularly branched three-sheeted covering Riemann surface defined by $u^3 = P_2(w)$. Let R_2 be an ultra-hyperelliptic surface defined by $y^2 = G_1(z)$. Then we can state the following lemma, the proof of which may be omitted since it can be readily proved by the ramification relation.

LEMMA 6. *Let $h(z) \equiv h_1(z) + h_2(z)\sqrt{G_1(z)}$ be a two-valued meromorphic function of z , where h_1 and h_2 are meromorphic functions of z . Then the functional equation of the form*

$$P_2 \circ h(z) = \{\lambda_1(z) + \lambda_2(z)\sqrt{G_1(z)}\}^3$$

has no solution for any two-valued meromorphic function of z , where λ_1 and λ_2 are meromorphic functions of z , when $n \geq 7$.

By this lemma and a quite similar discussion as above we can conclude that there is no analytic mapping from R_2 into S_3^* , when the genus of S_3^* is not less than 5. Furthermore we can say that there is no rigid analytic mapping from R_2 into S_3^* , when the genus of S_3^* is greater than 1. This is proved as in the theorem 6. Hence we have the following theorem:

THEOREM 9. *There is no analytic mapping from R_2 into S_3^* , when the genus*

of S_3^* is not less than 5.

If the genus of S_3^* is less than 5, then the assertion of theorem 9 does not hold in general. We shall give an example. Let $f(z)$ be the inverse function of

$$z = \phi(w) = \int_0^w \frac{dt}{\sqrt[3]{\{(t-a_1)(t-a_2)(t-a_3)\}^2}},$$

where a_1, a_2 and a_3 are non-zero distinct complex numbers. Then it satisfies

$$f'(z)^3 = \{(f(z)-a_1)(f(z)-a_2)(f(z)-a_3)\}^2.$$

Let R_2 be the proper existence domain of $\sqrt{f(z)}$, then it is an ultrahyperelliptic surface. Let S_3^* be the closed and regularly branched three-sheeted covering Riemann surface, which is defined by

$$u^3 = \{(w^2-a_1)(w^2-a_2)(w^2-a_3)\}^2.$$

Evidently we have

$$\begin{aligned} f'(z)^3 = & \{(\sqrt{f(z)} - \sqrt{a_1})(\sqrt{f(z)} + \sqrt{a_1})(\sqrt{f(z)} - \sqrt{a_2}) \\ & \cdot (\sqrt{f(z)} + \sqrt{a_2})(\sqrt{f(z)} - \sqrt{a_3})(\sqrt{f(z)} + \sqrt{a_3})\}^2. \end{aligned}$$

This shows that there is an analytic mapping φ from R_2 into S_3^* , which is induced by $\sqrt{f(z)}$, that is, $h(z) = \sqrt{f(z)}$, $\varphi = \mathfrak{p}_{S_3^*}^{-1} \circ h \circ \mathfrak{p}_{R_2}$. This shows that the assertion of theorem 9 does not hold when the genus of S_3^* is equal to 4.

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