ON THE EXISTENCE OF ANALYTIC MAPPINGS

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1. Let R and S be two ultrahyperelliptic surfaces defined by the equations $y^2=G(z)$ and $u^2=g(w)$, respectively, where G and g are two entire functions having no zero other than an infinite number of simple zeros. Let $\mathfrak{P}_S$ and $\mathfrak{P}_R$ be the projection maps $(w, u)\to w$ and $(z, y)\to z$, respectively. Let φ be an analytic mapping from R into S. Let Φ be the sifted mapping $\mathfrak{P}_S\circ\varphi$, then Φ is an entire function on R.

DEFINITION 1. *If φ satisfies $\mathfrak{P}_S\circ\varphi(p)=\mathfrak{P}_S\circ\varphi(q)$ for $p\neq q$, $\mathfrak{P}_R p=\mathfrak{P}_R q$, then we say that φ satisfies the rigidity of projection map or φ is a rigid analytic mapping from R into S. Similarly we can define the rigidity of projection map for an analytic function on R. If it is not the case, then we say that Φ is a non-rigid analytic mapping or function.*

Is there any non-rigid analytic mapping from R into S?

This is one of the most important problems in the analytic mapping theory in our case and we shall give here a negative answer for this problem. Although we conjectured that this problem would be solved by Sario's result [5] in our previous paper [4], we shall adopt a quite different way of proof. We shall make use of Nevanlinna-Selberg's theory on algebroid functions [6]. If S is a hyperelliptic surface of lower genus, then there are pairs of R and S which admit non-rigid analytic mappings. We shall show this by two examples.

2. We shall prove the following non-existence theorem of non-rigid analytic mapping.

THEOREM 1. *There is no non-rigid analytic mapping from an ultrahyperelliptic surface R into another such surface S.*

In order to prove this theorem we need a lemma on algebroid functions.

LEMMA 1. *There is no solution of an equation of the following form*

$$g\circ(h_1(z)+h_2(z)\sqrt{G(z)})=(L_1(z)+L_2(z)\sqrt{G(z)})^2$$

for any two entire functions L_1 and L_2 of z, where h_1 and h_2 are two entire functions of z and $h_2(z)\not\equiv 0$.

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Proof. Since g has only simple zeros $\{w_\mu\}$, the equations

$$h(z)\equiv h_1(z)+h_2(z)\sqrt{G(z)}=w_\mu$$

must have an infinite number of zeros $\{z_{\mu\nu}\}$ excepting at most four values w_1, w_2, w_3 and w_4. All the $z_{\mu\nu}$ are at least double w_μ-points of $h(z)$ and there is no other zero of $g\circ h(z)$. Thus $h(z)$ has an infinite number of perfectly branched values w_μ. On the other hand by Nevanlinna-Selberg's second fundamental theorem and by the ramification theorem, which remains true in our case,

$$\sum \Theta(w_\mu)\leqq 4,$$

where

$$\Theta(w)=1-\varlimsup_{r\to\infty}\frac{\bar{N}(r;w,h)}{T(r,h)}.$$

If w is a perfectly branched value of h, then $\Theta(w)\geqq 1/2$. Thus there are at most eight perfectly branched values $\{w_\mu\}$ of h. This is a contradiction. Thus we have the desired result.

Proof of Theorem 1. Every ultrahyperelliptic surface admits two univalent conformal mappings onto itself, that is, the identity map and its sheet-exchanged map. Correspondingly there is another analytic function or mapping $\bar{\alpha}$ when there is an analytic function or mapping α. Evidently φ is rigid if and only if $\Phi=\bar{\Phi}$. Φ can be represented by a two-valued algebroid entire function h of z in the following manner:

$$\begin{aligned}\Phi\circ\mathfrak{P}_R^{-1}(z)&=h_1(z)+h_2(z)\sqrt{G(z)},\\ \bar{\Phi}\circ\mathfrak{P}_R^{-1}(z)&=h_1(z)-h_2(z)\sqrt{G(z)},\end{aligned}\qquad h(z)=h_1(z)+h_2(z)\sqrt{G(z)},$$

or their sheet-exchanged forms with two entire functions h_1 and h_2. Let $\sqrt{g}^*$ be an analytic function on S corresponding to the two-valued algebroid entire function $\sqrt{g(w)}$ of w, that is, $\sqrt{g}^*=\sqrt{g\circ\mathfrak{P}_S}$. Then $\sqrt{g}^*\circ\varphi$ is an analytic function on R. Therefore it must be represented in the following manner:

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\mathfrak{P}_R"'] \arrow[dr, "\Phi"] & S \arrow[d, "\mathfrak{P}_S"'] \arrow[dr, "\sqrt{g}^*"] & \\ z \arrow[r, "h"] & w \arrow[r, "\sqrt{g}"] & {} \end{tikzcd}$$

$$\sqrt{g}^*\circ\varphi\circ\mathfrak{P}_R^{-1}(z)=L_1(z)+L_2(z)\sqrt{G(z)}$$

with two entire functions of z. However we have

$$\begin{aligned}\sqrt{g}^*\circ\varphi\circ\mathfrak{P}_R^{-1}(z)&=\sqrt{g\circ\mathfrak{P}_S}\circ\varphi\circ\mathfrak{P}_R^{-1}(z)\\ &=\sqrt{g\circ\mathfrak{P}_S\circ\mathfrak{P}_S^{-1}\circ h\circ\mathfrak{P}_R\circ\mathfrak{P}_R^{-1}(z)}=\sqrt{g\circ h(z)}.\end{aligned}$$

It should be remarked that some modifications should be made in the above discussions. The choices of analytic branches of $\mathfrak{P}_R^{-1}$ and $\mathfrak{P}_S^{-1}$ are free unless they are fixed suitably. However, then, it is sufficient to consider some suitable sheet-exchanged functions or mappings as the cases may be and we arrive at the same conclusion

$$g\circ(h_1(z)\pm h_2(z)\sqrt{G(z)})=(L_1(z)\pm L_2(z)\sqrt{G(z)})^2.$$

Since φ is not rigid, $h_2 \not\equiv 0$. By Lemma 1 the above equation has no solution, which is untenable. This completes the proof of theorem 1.

3. By theorem 1 and by our previous result in [4], we can conclude the following perfect condition for the existence of analytic mapping from R into S.

THEOREM 2. *If there exists an analytic mapping φ from R into S, then there exists a pair of two entire functions $h(z)$ and $f(z)$ of z satisfying an equation*

$$f(z)^2 G(z) = g \circ h(z)$$

and vice versa.

Sufficiency part can be proved in the following manner. Let φ be an analytic mapping defined by

$$\varphi = \mathfrak{P}_S^{-1} \circ h \circ \mathfrak{P}_R.$$

Then φ is evidently rigid analytic mapping from R into S, since every branch point of R corresponds to a branch point of S by $\sqrt{g \circ h} = \pm f\sqrt{G}$.

This theorem gives a powerful tool in order to investigate the existence problem and the growth problem of analytic mappings in our case. However to solve the equation

$$f(z)^2 G(z) = g \circ h(z)$$

is quite difficult.

4. We shall solve the problem 2) in [4].

THEOREM 3. *Let R and S be two ultrahyperelliptic surfaces. Suppose that there exists an analytic mapping φ from R into S. Then φ is a semi-degenerate analytic mapping, that is,*

$$\varlimsup_{r\to\infty} \frac{N(r, R)}{T(r, \Phi)} > 2,$$

where $N(r, R)$ is the quantity $N(r, \mathfrak{X})$ defined in [6]. *Further it satisfies*

$$\varlimsup_{r\to\infty} \frac{N(r, R)}{T(r, \Phi)} = \infty.$$

Proof. By theorem 2 we may consider the possibility of an equation of the form $F^2 G = g \circ h$ with two suitable entire functions F and h of z. Since g has no zero other than an infinite number of simple zeros, we have

$$N(r; 0, g \circ h) = N_2(r; 0, g \circ h) + N_1(r; 0, g \circ h) + \bar{N}_1(r; 0, g \circ h),$$

$$\bar{N}_1(r; 0, g \circ h) \leqq N_1(r; 0, g \circ h) \leqq N(r; 0, h') \leqq m(r, h')$$

$$\leqq m\left(r, \frac{h'}{h}\right) + m(r, h),$$

$$m\left(r, \frac{h'}{h}\right) = O(\log r m(r, h)),$$

where $N_2(r; 0, f)$ is the N-function of simple zeros of f. Therefore we have

$$\begin{aligned}&N(r; 0, g\circ h)-2m(r, h)+O(\log rm(r, h))\\ &\leqq N_2(r; 0, g\circ h)=N_2(r; 0, F^2G)\leqq N(r; 0, G)\\ &=2N(r, R)+O(\log r).\end{aligned}$$

Let $\{w_\mu\}$ be a set of an arbitrary finite number p of zeros of g. Then we have

$$N(r; 0, g\circ h)\geqq \sum N(r; w_\mu, h),$$

where the summation is taken over all p zeros $\{w_\mu\}$ of g and r is a sufficiently large number. By the second fundamental theorem for h

$$\sum N(r; w_\mu, h)\geqq(p-2)m(r, h)+O(\log rm(r, h)).$$

Therefore

$$\begin{aligned}&2N(r, R)+O(\log r)\\ &\geqq(p-2)m(r, h)-2m(r, \mathrm{h})+O(\log rm(r, h)).\end{aligned}$$

If h is a polynomial, then $m(r, h)=O(\log r)$. Hence

$$\varlimsup_{r\to\infty}\frac{2N(r, R)}{m(r, h)}\geqq p-k.$$

If h is a transcendental entire function, then

$$\varlimsup_{r\to\infty}\frac{2N(r, R)}{m(r, h)}\geqq p-4.$$

Since p may be an arbitrary finite number by the assumption on the number of simple zeros of g, the desired result

$$\varlimsup_{r\to\infty}\frac{N(r, R)}{m(r, h)}=\infty$$

holds as $T(r, \Phi)=m(r, h)$.

5. Let $h(z)$ be again a two-valued meromorphic function of the following form

$$h_1(z)+h_2(z)\sqrt{G(z)}, \qquad h_2(z)\not\equiv 0,$$

where h_1 and h_2 are two meromorphic functions of z. Let $g(z)$ be a polynomial

$$\prod_{\nu=1}^{2p}(z-a_\nu).$$

LEMMA 2. *There is no solution of an equation*

$$g\circ h(z)=(L_1(z)+L_2(z)\sqrt{G(z)})^2$$

for any meromorphic functions L_1 and L_2, when $p\geqq 5$.

Proof. g has $2p(\geqq 10)$ simple zeros $a_1, \cdots, a_{2p}$. This implies that every root of $h(z)=a$ must be of even multiplicity whenever it exists. Thus

$$\Theta(a_\nu)\geqq\frac{1}{2}$$

holds for every a_ν, where $\Theta(w)$ has already been defined. On the other hand by the second fundamental theorem on algebroid functions

$$\sum_{1}^{2p} \Theta(a_\nu) \leqq 4.$$

This is untenable for $2p>8$.

THEOREM 4. *Let R be an ultrahyperelliptic surface and let S be a hyperelliptic surface of genus $p-1$, $p \geqq 5$. Then there is no non-rigid analytic mapping from R into S. If there is an analytic mapping φ from R into S, then it is rigid and satisfies*

$$\varlimsup_{r\to\infty} \frac{N(r, R)}{T(r, \Phi)} \leqq p$$

and there exist two meromorphic functions $h(z)$ and $F(z)$ of z in such a manner that a functional equation of the following form

$$F(z)^2 G(z) = g \circ h(z)$$

remains true.

The following fact is another interpretation of the above theorem. *For every hyperelliptic curve $y^2=P(x)$ of genus greater than* 3, *there is no pair of two-valued algebroid functions $x(t)$, $y(t)$ on an ultrahyperelliptic surface in such a manner that $y(t)^2 \equiv P(x(t))$.*

If the genus of S is less than four, then the first part of the above theorem does not hold in general. Here we shall give two examples. Let $\wp(z)$ be Weierstrass' elliptic function. Then it is well known that

$$\wp'(z)^2 = 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3).$$

Let R be the proper existence domain of $\sqrt{\wp(z)}$, then it is an ultrahyperelliptic surface. Let S be a hyperelliptic surface defined by

$$y^2 = 4(x-\sqrt{e_1})(x-\sqrt{e_2})(x-\sqrt{e_3})(x+\sqrt{e_1})(x+\sqrt{e_2})(x+\sqrt{e_3}).$$

Evidently we have

$$\wp'(z)^2 = 4(\sqrt{\wp(z)}-\sqrt{e_1})(\sqrt{\wp(z)}-\sqrt{e_2})(\sqrt{\wp(z)}-\sqrt{e_3})$$
$$(\sqrt{\wp(z)}+\sqrt{e_1})(\sqrt{\wp(z)}+\sqrt{e_2})(\sqrt{\wp(z)}+\sqrt{e_3}),$$

which shows that there is a non-rigid analytic mapping φ from R into S, which corresponds to $\sqrt{\wp(z)}$, that is, $\mathfrak{P}_S \circ \varphi \circ \mathfrak{P}_R^{-1}(z) = \sqrt{\wp(z)}$. Thus the first part of theorem 4 does not hold when the genus of S is equal to 2.

Let $\operatorname{sn} z$ be Jacobi's elliptic function. Then it is well known that

$$\left(\frac{d}{dz} \operatorname{sn} z\right)^2 = (1-\operatorname{sn}^2 z)(1-k^2 \operatorname{sn}^2 z).$$

Let R be the proper existence domain of $\sqrt{\operatorname{sn} z}$, then it is an ultrahyperelliptic surface. Let S be a hyperelliptic surface defined by

$$y^2=(1-x^4)(1-k^2x^4).$$

Then

$$\left(\frac{d}{dz}\operatorname{sn} z\right)^2=(1-\sqrt{\operatorname{sn} z})(1+\sqrt{\operatorname{sn} z})(1-i\sqrt{\operatorname{sn} z})(1+i\sqrt{\operatorname{sn} z})$$
$$(1-\sqrt{k}\sqrt{\operatorname{sn} z})(1+\sqrt{k}\sqrt{\operatorname{sn} z})(1-i\sqrt{k}\sqrt{\operatorname{sn} z})(1+i\sqrt{k}\sqrt{\operatorname{sn} z}).$$

This equation shows that there is a non-rigid analytic mapping φ from R into S which corresponds to $\sqrt{\operatorname{sn} z}$, that is, $\mathfrak{P}_S\circ\varphi\circ\mathfrak{P}_R^{-1}(z)=\sqrt{\operatorname{sn} z}$. Thus the first part of theorem 4 does not hold, when the genus of S is equal to 3. For these two examples we have

$$\overline{\lim_{r\to\infty}}\frac{N(r, R)}{T(r, \Phi)}=2.$$

This fact is easy to prove.

6. In the theory of analytic mappings between two-sheeted surfaces there remain some unsettled problems. Here we shall list them.

Let $\mathfrak{M}(R)$ be the class of non-constant meromorphic functions on R. Let $P(f)$ be the number of exceptional values, where we say a value α an exceptional value when it is never taken by f in R. Let $P(R)$ be the supremum of $P(f)$, that is,

$$\sup_{f\in\mathfrak{m}(R)} P(f).$$

1) Determine all the ultrahyperelliptic surfaces with $P(R)=3$.

It is very plausible to conjecture that the surface R defined by

$$y^2=1-2\beta_1e^{H_1}-2\beta_2e^{H_2}+\beta_1^2e^{2H_1}-2\beta_1\beta_2e^{H_1+H_2}+\beta_2^2e^{2H_2},$$
$$\beta_1\beta_2\neq 0,\qquad H_1(0)=H_2(0)=0$$

with non-constant entire functions H_1 and H_2 has $P(R)=3$ unless $H_1=H_2$ or $H_1=2H_2$ and $\beta_1=\beta_2^2/16$ or $H_2=2H_1$ and $\beta_2=\beta_1^2/16$. It is not difficult to prove that the surface $P(R)=3$ must have the above form. If it reduces to the following form

$$f^2(e^H-\gamma)(e^H-\delta),\qquad \gamma\delta\neq 0,\quad \gamma\neq\delta$$

with an entire function H and a meromorphic function f and two constants γ and δ, then $P(R)=4$ and vice versa [3].

2) Is there any example of analytic mapping from an ultrahyperelliptic surface R with $P(R)=4$ into another S with $P(S)=3$?

This would be negative and depend on the problem 1).

3) Is there any example of analytic mapping from an ultrahyperelliptic surface R with $P(R)=3$ into another S with $P(S)=2$?

This would be again negative.

4) Is there any rigid analytic mapping together with a non-rigid analytic mapping from an ultrahyperelliptic surface R into a hyperelliptic surface S?

This would be also negative.

5) Determine all the hyperelliptic surfaces S into which a non-rigid analytic mapping is possible from a suitable ultrahyperelliptic surface.

This would depend on the ramification relation in the Nevanlinna theory and its generalization to the algebroid functions. There would rarely exist such examples among all the hyperelliptic surfaces of genus less than 4. Our two examples of non-rigid analytic mappings show that the non-rigidity of analytic mapping φ is eliminated by composing to the defining function of the surface S and then the rigidity holds, that is,

$$\sqrt{g^*}\circ\varphi$$

satisfies the rigidity of projection map. This would be worth while to notice.

All of our problems would demand the several different considerations in the classical value distribution theory. Especially it is necessary to study the value distribution of a composite entire function. This is the problem 3) in our previous paper [4].

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