ON COMPLEX ANALYTIC MAPPINGS

By Mitsuru Ozawa

§1. Let R and S be two Riemann surfaces. When are there any analytic mappings from R into S? This would be one of the most important problem. Recently Sario [2], [3], [4], [5], [6], [7] established the general second fundamental theorem under an assumption of existence of analytic mappings from R into S. To establish the general defect relation and to guarantee its effectivity he made two additional assumptions for the growth of mappings in [6] when S is closed. One of them is his non-degeneracy condition and the other is the following condition: $(\mathbf{A}_1)$ The characteristic function must grow at least as rapidly as the Euler characteristic. Then he concluded the following curious but elegant fact: $(\mathbf{A}_2)$ The characteristic function cannot grow more rapidly than the Euler characteristic in order to the analytic mapping really exists, when the genus of S is not less than 2. Two conditions $(\mathbf{A}_1)$ and $(\mathbf{A}_2)$ act really into two oposite directions and hence the general defect relation has its proper sense only in a quite few cases.

In a case of algebroid functions the condition $(\mathbf{A}_1)$ is compatible to the well known Selberg's ramification theorem. The result $(\mathbf{A}_2)$ or its generalization to an open surface S of infinite genus: the affinity relation in [7] would be one of the most important contributions due to Sario.

In the present paper we shall offer two sufficient conditions for the non-existence of analytic mappings, by which several cases are decided as the non-existence cases of analytic mappings. We shall offer some examples in the present paper.

§2. We shall prove a sufficient condition for the non-existence of analytic mappings from a Riemann surface R into an open Riemann surface S. If $R \notin O_{AB}$, then there are many trivial analytic mappings. Thus we should put aside this case.

Let $\mathfrak{M}(R)$ be a family of non-constant meromorphic functions on R. Let $P(f)$ be the number of Picard's exceptional values, where we say α a Picard's value when it is not taken by f in R. Let $P(R)$ be a quantity

$$\sup_{f \in \mathfrak{M}(R)} P(f).$$

When R is open, we have always $P(R) \geqq 2$, since there exists a non constant regular function on R by the existence theorem due to Behnke-Stein and then it suffices to compose it to the exponential function. Let $P(S)$ be the corresponding quantity attached to S.

THEOREM 1. *If $P(R) < P(S)$, then there is no analytic mapping from R into S.*

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Proof. Let T be an analytic mapping from R into S. Let f be a member of $\mathfrak{M}(S)$ such that $P(f)=P(S)$. Then $f\circ T\in\mathfrak{M}(R)$. Thus $P(f\circ T)\leqq P(R)<P(S)$. This is a contradiction.

Since $P(S)>0$ by the assumption, the surface S should be open.

By its wide applicability of the proof we may adopt $P(R)$ as the maximal cardinal number of Picard's values or as the positivity or nullity of capacity or a sort of function-theoretic null-set or positive-set. For example we can conclude the following well-known fact: Let R be a plane region whose boundary is of capacity zero and S a plane region whose boundary is of positive capacity. Then there is no analytic mapping from R into S.

Further we can conclude the following fact. If there holds the Iversen (resp. Gross) property on R but there does not hold the Iversen (resp. Gross) property on S, then there is no analytic mapping from R into S.

§3. In general it is very difficult to calculate $P(R)$ of a given open Riemann surface R. It depends on the theory of value distributions on R. We shall give here two examples.

Let R be a proper existence domain of an algebroid function

$$\sqrt{\frac{e^z-1}{z(e^z+1)}}.$$

We shall prove that $P(R)=2$. To this end we assume that $P(R)\geqq 3$. We may assume that three Picard's exceptional values are 0, 1 and ∞. Then there is at least one entire function $f(p)$ on R whose Picard's values are 0, 1 and ∞. If the order of $f(p)$ is greater than one, then by Selberg's generalization [8] of Nevanlinna's theory and by a fact

$$\varlimsup_{r\to\infty}\frac{E(r,R)}{T(r,f)}=0,$$

where $E(r,R)$ is the integrated Euler characteristic of R, whose order is one in this case, and $T(r,f)$ is the Nevanlinna-Selberg characteristic function of f, the number of Picard's exceptional values is at most two. This is untenable. Thus $f(p)$ is at most of order one. Thus two coefficients $f_1(z)$, $A_2(z)$ of the defining equation

$$F(z,f)\equiv f^2-2f_1(z)f+A_2(z)=0$$

of $f(p)$ are entire functions of z and of order at most one. By the representation of f

$$f_1(z)+f_2(z)\sqrt{\frac{e^{2z}-1}{z}},$$

we have

$$A_2(z)=f_1(z)^2-f_2(z)^2\frac{e^{2z}-1}{z}.$$

Thus $f_2(z)$ is also an entire function of order at most one. By Rémoundos' method of proof of his celebrated generalization of Picard's theorem [1] pp. 25–27, the function $F(z,f)$ must satisfy the equations

$$\begin{pmatrix} F(z,0) \\ F(z,1) \end{pmatrix} = \begin{pmatrix} c \\ \beta_1 e^{\alpha_1 z} \end{pmatrix} \text{ or } \begin{pmatrix} \beta_1 e^{\alpha_1 z} \\ c \end{pmatrix} \text{ or } \begin{pmatrix} \beta_1 e^{\alpha_1 z} \\ \beta_2 e^{\alpha_2 z} \end{pmatrix},$$

where c, β_1 and β_2 are three constants and α_1, α_2 are two non-zero constants. Here we may assume that $\beta_1 \neq 0$, $\beta_2 \neq 0$ by Rémoundos' reasoning.

We shall consider the first case:

$$\begin{cases} f_1{}^2 - f_2{}^2 \dfrac{e^{2z}-1}{z} = c, \\ 1 - 2f_1 + f_1{}^2 - f_2{}^2 \dfrac{e^{2z}-1}{z} = \beta_1 e^{\alpha_1 z}. \end{cases}$$

Then we have

$$(1-c)^2 - 2\beta_1(1+c)e^{\alpha_1 z} + \beta_1{}^2 e^{\alpha_1 z} = 4 f_2{}^2 \frac{e^{2z}-1}{z}.$$

Let z be $n\pi i$ (n: a non-zero integer) and x denote $e^{\alpha_1 \pi i}$, then

$$(1-c)^2 - 2\beta_1(1+c)x^n + \beta_1{}^2 x^{2n} = 0.$$

Suppose that $x \neq 1$. Then by

$$\begin{cases} (1-c)^2 - 2\beta_1(1+c)x + \beta_1{}^2 x^2 = 0, \\ (1-c)^2 - 2\beta_1(1+c)x^2 + \beta_1{}^2 x^4 = 0, \\ (1-c)^2 - 2\beta_1(1+c)x^3 + \beta_1{}^2 x^6 = 0, \end{cases}$$

we have

$$\begin{cases} \beta_1 x(x+1) = 2(1+c), \\ \beta_1 x^2(x+1) = 2(1+c), \end{cases}$$

and hence we have $x=-1$, $c=-1$ and further $\beta_1{}^2=-4$. Thus $\alpha_1 = 2m+1$ with an integer m. Then

$$1 - e^{2(2m+1)z} = f_2{}^2 \frac{e^{2z}-1}{z}.$$

Putting $z=0$, then $f_2(0)^2=0$. Thus the right hand side term has at least a double zero at $z=0$. Thus the derivative of the left hand side term is equal to zero at $z=0$. On the other hand the derivative is

$$-2(2m+1)e^{2(2m+1)z},$$

which is never equal to zero. This is a contradiction. Therefore $x=1$, $\alpha_1=2m$ with a non-zero integer m. Thus

$$(1-c)^2-2\beta_1(1+c)+\beta_1^2=0.$$

Therefore we have

$$\beta_1^2(e^{4mz}-1)-2\beta_1(1+c)(e^{2mz}-1)=4f_2^2\frac{e^{2z}-1}{z}$$

and hence

$$\beta_1(e^{2mz}-1)(\beta_1 e^{2mz}+\beta_1-2(1+c))=4f_2^2\frac{e^{2z}-1}{z}.$$

If $\beta_1=2(1+c)$, then $c=1$ and $\beta_1=4$. Thus

$$4(e^{2mz}-1)e^{2mz}=f_2^2\frac{e^{2z}-1}{z}.$$

This equation leads to a contradiction, that is, $m=0$. If $\beta_1\neq 2(1+c)$, then we put $z=0$. Then $f_2(0)^2=0$ and hence the derivative of the right hand side term

$$2m\beta_1 e^{2mz}(\beta_1 e^{2mz}+\beta_1-2(1+c))+\beta_1(e^{2mz}-1)\beta_1 2me^{2mz}$$

must be equal to zero at $z=0$, that is,

$$2m\beta_1(\beta_1-1-c)=0.$$

Thus there must be $\beta_1=1+c$. Then we have

$$\beta_1^2(e^{2mz}-1)^2=4f_2^2\frac{e^{2z}-1}{z}.$$

At $z=p\pi i$ for every integer p the left hand side term has a double zero and hence the right hand side term has a triple zero. This is a contradiction.

Next we must consider the second case:

$$\begin{cases} f_1^2-f_2^2\dfrac{e^{2z}-1}{z}=\beta_1 e^{\alpha_1 z}, \\[2ex] 1-2f_1+f_1^2-f_2^2\dfrac{e^{2z}-1}{z}=c. \end{cases}$$

This case can be discussed quite similarly as in the first case and leads to the similar contradiction. Thus we may omit the details.

Finally we shall consider the last case:

$$\begin{cases} f_1^2-f_2^2\dfrac{e^{2z}-1}{z}=\beta_1 e^{\alpha_1 z}, \\[2ex] 1-2f_1+f_1^2-f_2^2\dfrac{e^{2z}-1}{z}=\beta_2 e^{\alpha_2 z}. \end{cases}$$

LEMMA. *Let (x_0, y_0) be a non-zero pair of two complex numbers. If every pair (x_0^n, y_0^n) for every non-zero integer n lies on a quadratic curve*

$$1-2\beta_1 x-2\beta_2 y+\beta_1^2x^2-2\beta_1\beta_2 xy+\beta_2^2y^2=0, \qquad \beta_1\beta_2\neq 0,$$

then $(x_0, y_0)=(1, 1)$ *and*

$$1-2\beta_1-2\beta_2+\beta_1^2-2\beta_1\beta_2+\beta_2^2=0.$$

Proof. For simplicity's sake we shall denote the left hand side term of the defining equation of the quadratic curve by $F(x, y)$. If $|x_0|<1$, $|y_0|<1$, then

$$F(x_0^n, y_0^n)\to 1 \qquad \text{for} \qquad n\to\infty.$$

If $|x_0|>1$, $|y_0|>1$, then

$$F(1/x_0^n, 1/y_0^n)\to 1 \qquad \text{for} \quad n\to\infty.$$

If $|x_0|<1$, $|y_0|>1$, $|x_0y_0|\geqq 1$, then

$$F(1/x_0^n, 1/y_0^n)=(1-\beta_1x_0^{-n})^2-2\beta_2y_0^{-n}-2\beta_1\beta_2x_0^{-n}y_0^{-n}+\beta_2^2y_0^{-2n}$$

tends to infinity, as n tends to infinity. If $|x_0|<1$, $|y_0|>1$, $|x_0y_0|\leqq 1$, then

$$F(x_0^n, y_0^n)=(1-\beta_2y_0^n)^2-2\beta_1x_0^n+\beta_1^2x_0^{2n}-2\beta_1\beta_2x_0^ny_0^n$$

tends to infinity, as n tends to infinity. Similarly $|x_0|>1$, $|y_0|<1$ lead to a contradiction. If $|x_0|<1$, $|y_0|=1$, then

$$(1-|\beta_2|)^2\leqq|1-\beta_2y_0^n|^2$$
$$\leqq 2|\beta_1||x_0|^n+|\beta_1|^2|x_0|^{2n}+2|\beta_1||\beta_2||x_0|^n|y_0|^n<\varepsilon^2$$

for every given positive ε and $n\geqq n_0$. Thus $|\beta_2|=1$ must hold. The inequality

$$|1-\beta_2y_0^n|<\varepsilon$$

leads to two equations $\beta_2=1$ and $y_0=1$. Then we have

$$-4\beta_1x_0+\beta_1^2x_0^2=0,$$
$$-4\beta_1x_0^2+\beta_1^2x_0^4=0.$$

Since $\beta_1x_0\neq 0$, we have finally $x_0=1$. This is absurd. Quite similarly there does not occur the case $|x_0|>1$, $|y_0|=1$. Similarly there does not occur the case $|x_0|=1$, $|y_0|\neq 1$. Thus there remains a case $|x_0|=|y_0|=1$. Then we have

$$\beta_2y_0^n=1+\beta_1x_0^n+2(\beta_1x_0^n)^{1/2}.$$

Let $x_0=e^{i\alpha\pi}$, $\beta_1=|\beta_1|e^{i\varepsilon\pi}$, then

$$|\beta_2|=1+|\beta_1|^{1/2}\cos\left(\frac{n\alpha+\varepsilon}{2}+\delta\right)\pi+|\beta_1|, \qquad \delta=0 \quad \text{or} \quad 1.$$

Thus α must be an even integer, that is, $x_0=1$. By an easy algebraic calculations we have $y_0=1$. Thus we have the desired fact.

Now we should return to our equations. Then we have

$$G(z;\ \alpha_1, \alpha_2)\equiv 1-2\beta_1e^{\alpha_1z}-2\beta_2e^{\alpha_2z}+\beta_1^2e^{2\alpha_1z}-2\beta_1\beta_2e^{(\alpha_1+\alpha_2)z}+\beta_2^2e^{2\alpha_2z}$$

$$=4f_2^2\frac{e^{2z}-1}{z}\equiv H(z).$$

Putting $z=n\pi i$ (n: a non-zero integer), we have

$$G(n\pi i;\ \alpha_1,\ \alpha_2)=F(e^{n\alpha_1 i\pi},\ e^{n\alpha_2\pi i})=0,$$

since $H(n\pi i)=0$ for $n\neq 0$. By the Lemma we have

$$\begin{cases} e^{\alpha_1\pi i}=1 \\ e^{\alpha_2\pi i}=1, \end{cases} \quad \text{that is,} \quad \begin{cases} \alpha_1=2p \\ \alpha_2=2q, \end{cases} \quad (p,\ q\text{: non-zero integers})$$

and

$$1-2\beta_1-2\beta_2+\beta_1^2-2\beta_1\beta_2+\beta_2^2=0.$$

Thus $G(0;\ 2p,\ 2q)=0$ and hence $H(0)=0$, which leads to a fact that f_2^2 has at least a double zero at $z=0$. Therefore $G'(z;\ 2p,\ 2q)=0$ at $z=0$. This leads to an equation

$$-4\beta_1 p-4\beta_2 q+4\beta_1^2 p-4\beta_1\beta_2(p+q)+4q\beta_2^2=0.$$

This is nothing but a relation

$$G'(n\pi i;\ 2p,\ 2q)=0, \qquad n\neq 0.$$

Thus $G(z;\ 2p,\ 2q)$ has at least a double zero at $z=n\pi i$ and hence $H(z)$ has at least a triple zero at $z=n\pi i$. Thus $H''(n\pi i)=0$. This leads to an equation

$$-8p^2\beta_1-8q^2\beta_2+16p^2\beta_1^2-8(p+q)^2\beta_1\beta_2+16q^2\beta_2^2=0.$$

This is nothing but $G''(0;\ 2p,\ 2q)=0$. This implies that $H(z)$ has a zero of order 4 at $z=0$. Thus $H'''(0)=0$. Repeating this process ad infinitum, we finally have

$$H^{(n)}(0)=0$$

for every n. Thus $H(z)\equiv 0$, that is, $f_2(z)\equiv 0$. Thus f reduces to a single-valued entire function of z. In this case $P(f)\leqq 2$, which is a contradiction. Thus we have finally the desired fact $P(R)=2$.

The same holds for a surface R, which is a proper existence domain of an algebroid function

$$\sqrt{\frac{(e^z-1)(z-a)}{(e^z+1)z}}, \qquad a\neq n\pi i,$$

that is, $P(R)=2$.

Next we shall proceed to the second example. Let R be an ultrahyperelliptic surface defined by an equation $y^2=g(x)$ with an entire function $g(x)$ of non-integral order $\rho_g(<\infty)$. Then $P(R)=2$. If this is not the case, then there exists an entire algebroid function f with three Picard's exceptional values 0, 1, ∞. Then the order of f is not greater than ρ_g by Selberg's theorem. Thus we have the defining equation

$$F(z,f)\equiv f^2-2f_1(z)f+f_1(z)^2-f_2(z)^2g(z)=0$$

of f, where f_1 and f_2 are two entire functions of order at most ρ_g. By Rémoundos' reasoning of his celebrated theorem we have equations

$$\begin{pmatrix} F(z,0) \\ F(z,1) \end{pmatrix} = \begin{pmatrix} c \\ \beta_1 e^{H_1} \end{pmatrix} \text{ or } \begin{pmatrix} \beta_1 e^{H_1} \\ c \end{pmatrix} \text{ or } \begin{pmatrix} \beta_1 e^{H_1} \\ \beta_2 e^{H_2} \end{pmatrix}.$$

Here β_1 and β_2 are non-zero constants and H_1 and H_2 are two polynomials of degrees at most $[\rho_g]$. In the first and the second cases we have

$$(1-c)^2-2(1+c)\beta_1 e^{H_1}+\beta_1{}^2 e^{2H_1}=4f_2{}^2 g.$$

In the last case we have

$$1-2\beta_1 e^{H_1}-2\beta_2 e^{H_2}+\beta_1{}^2 e^{2H_1}-2\beta_1\beta_2 e^{H_1+H_2}+\beta_2{}^2 e^{2H_2}=4f_2{}^2 g.$$

In every case the left hand side term is of order at most $[\rho_g]$ and hence the exponent of convergence of its zeros is not greater than $[\rho_g]$. However the right hand side term has zeros whose exponent of convergence is not less than ρ_g by Borel's theorem, when $f_2(z)\not\equiv 0$. This is a contradiction. If $f_2(z)\equiv 0$, then f reduces to a single-valued function of z, whence follows $P(f)\leqq 2$. This is also untenable. Thus we have the desired result: $P(R)=2$.

§4. We shall here give three examples, which belong to the non-existence case of analytic mappings.

(i) Let R be the proper existence domain of an n-valued algebroid function on $|z|<\infty$. Then $P(R)\leqq 2n$ by Selberg's generalization of Picard's theorem. Let S be a Riemann surface defined by an equation $e^w=g(z)$, where $g(z)$ is an entire function with at least $2n+1$ zeros. Then $P(S)\geqq 2n+1$, since the projection map $(z, w)\to z$ omits at least $2n+1$ points, which are the zeros of $g(z)$. Thus there is no analytic mapping from R into S. If $g(z)$ is a polynomial of degree at most n, then S may be considered as a proper existence domain of an algebroid function whose defining equation is $g(z)=e^w$. Thus there may exist an analytic mapping from R into S, when R is suitably chosen.

(ii) Let R be an open Riemann surface with one ideal boundary $\mathfrak{J}$. We assume that there is an infinite number of disjoint annuli $\{\mathfrak{A}_j\}$ satisfying the conditions: The modulus of $\mathfrak{A}_j$ satisfies an inequality mod $\mathfrak{A}_j\geqq\delta>1$ for every j, $\mathfrak{A}_j$ separates $\mathfrak{A}_{j-1}$ from $\mathfrak{J}$ and $\mathfrak{A}_j$ tend to $\mathfrak{J}$ if j tends to ∞. Then R belongs to the class O_G and the end of R is of Heins' harmonic dimension one. Then by the Schottky theorem we have $P(R)\leqq 2$. Since generally $P(R)\geqq 2$, we have $P(R)=2$. Let S be the proper existence domain of an algebroid function

$$\sqrt{\frac{e^z-1}{e^z+1}}.$$

Then $P(S)\leqq 4$ by Rémoundos-Selberg's theorem and evidently $P(S)\geqq 4$, since the above algebroid function omits four values $1, -1, i, -i$. Therefore $P(S)=4$. Thus there is no analytic mapping from R into S.

(iii) Let R and S be the proper existence domains of two algebroid functions

$$\sqrt{\frac{e^z-1}{e^z+1}} \quad\text{and}\quad \sqrt{\frac{e^z+a^2}{e^z+1}},\ a^2\neq 0,\ 1,\ -1,$$

respectively. In this case $P(R)=P(S)=4$. Thus we cannot apply Theorem 1.

Let T be an analytic mapping from R into S. Let $g(w)$ be a two-valued entire algebroid function with four Picard's exceptional values, say

$$g(w)=\frac{1}{a^2-1}\left(2e^w+a^2+1+2\sqrt{(e^w+a^2)(e^w+1)}\right).$$

For this function we have evidently

$$g(w)\neq 0,\ \infty,\ \frac{a+1}{a-1},\ \frac{a-1}{a+1}.$$

Then $f=g\circ T$ should exclude the above four values and hence f should be a two-valued entire algebroid function of z. Thus we have a representation

$$f(z)=f_1(z)+f_2(z)\sqrt{e^{2z}-1}$$

with two entire functions f_1 and f_2 both of which are of order at most one. Then we have

$$F(z,f)\equiv f^2-2f_1(z)f+f_1(z)^2-f_2(z)^2(e^{2z}-1)=0.$$

Thus by the method of proof of Rémoundos' theorem we have

$$\begin{pmatrix} F(z,0) \\ F\left(z,\dfrac{a+1}{a-1}\right) \\ F\left(z,\dfrac{a-1}{a+1}\right)\end{pmatrix}=\begin{pmatrix} c \\ e^{\alpha_1 z+\beta_1} \\ e^{\alpha_2 z+\beta_2}\end{pmatrix} \text{ or } \begin{pmatrix} e^{\alpha_1 z+\beta_1} \\ c \\ e^{\alpha_2 z+\beta_2}\end{pmatrix} \text{ or } \begin{pmatrix} e^{\alpha_1 z+\beta_1} \\ e^{\alpha_2 z+\beta_2} \\ c\end{pmatrix},$$

where c is a constant and α_1, α_2 are non-zero constants and β_1, β_2 are two constants.

Now we shall discuss the first case. Then we have

$$\left(\frac{a+1}{a-1}\right)^2-2\left(\frac{a+1}{a-1}\right)f_1+c=e^{\alpha_1 z+\beta_1},$$

$$\left(\frac{a-1}{a+1}\right)^2-2\left(\frac{a-1}{a+1}\right)f_1+c=e^{\alpha_2 z+\beta_2}.$$

Eliminating f_1, we have

$$(a-1)^2e^{\alpha_1 z+\beta_1}-(a+1)^2e^{\alpha_2 z+\beta_2}=4a(1-c).$$

If $c\neq 1$, then this contradicts Borel's formulation of Picard's theorem, since $a\neq 0$. Thus $c=1$ and hence $\alpha_1=\alpha_2$ and $(a-1)^2e^{\beta_1}=(a+1)^2e^{\beta_2}$. Then we have

$$f_1^2-1=f_2^2(e^{2z}-1),$$

$$f_1=\frac{1}{2}\left(\frac{a^2+1}{a^2-1}-\frac{a-1}{a+1}e^{\alpha_1 z+\beta_1}\right).$$

Let $z=0$, then $f_1(0)=\pm 1$ and hence $e^{\beta_1}=(3-a^2)^2/(a-1)^2$ or $-2/(a-1)^2$. Thus we have

$$f_1=\frac{1}{2}\left(\frac{a^2+1}{a^2-1}+\frac{a^2-3}{a^2-1}e^{\alpha_1 z}\right) \tag{1}$$

or

$$f_1=\frac{1}{2}\left(\frac{a^2+1}{a^2-1}+\frac{2}{a^2-1}e^{\alpha_1 z}\right). \tag{2}$$

If $a^2=3$, then this contradicts the reasoning in Rémoundos' proof. Therefore we may assume that $a^2\neq 3$. We shall first discuss the case (1). If we put $z=n\pi i$, then we have by $f_1(n\pi i)=\pm 1$

$$e^{\alpha_1 \pi i}=1 \quad \text{or} \quad \frac{-3a^2+1}{a^2-3}.$$

Thus either α_2 is equal to an even integer $2m$ $(\neq 0)$ or satisfies $e^{2\alpha_1\pi i}=(3a^2-1)^2 \div (a^2-3)^2$, which is either 1 or $(-3a^2+1)/(a^2-3)$. In the latter case we have either $a^2=-1$ or 1 or 1/3. When $a^2=1/3$, $e^{\alpha_1\pi i}=0$, which is also a contradiction. Thus we have that α_2 is $2m$ $(\neq 0)$. Then we have

$$f_2{}^2(e^{2z}-1)=f_1{}^2-1$$

$$=\frac{(a^2-3)^2}{4(a^2-1)^2}(e^{2mz}-1)\left(e^{2mz}-\frac{1-3a^2}{a^2-3}\right).$$

Since $a^2\neq 1/3$, this equation is untenable. Because the function

$$\frac{(e^{2mz}-1)\left(e^{2mz}-\dfrac{1-3a^2}{a^2-3}\right)}{e^{2z}-1}$$

is not a square of an entire function for every m. Though the case (2) is somewhat complicated, a quite similar discussion leads to the same contradiction as in the case (1). Thus we can conclude that the first case does not occur. The remaining two cases are also untenable by the similar discussion. Thus we have the following fact: There is no analytic mapping from R into S. Quite similarly we have that there is no analytic mapping from S into R.

The above example suggests the following general theorem, whose proof is immediate.

THEOREM 2. *If there is a meromorphic function $g\in\mathfrak{M}(S)$ in such a manner that for an arbitrary member f of $\mathfrak{M}(R)$ there is at least one Picard's exceptional value of g, which is not a Picard value of f, then there is no analytic mapping from R into S.*

§5. Our original intention in the present paper is to emphasize that the problem of analytic mappings especially their existence problem is beyond the scope of Nevanlinna theory. It seems to the present author that the existence problem would be the problem of moduli of open surfaces, for which our knowledges in the present status are extremely meagre. Our theorems, which seem to be applicable, show that the quantity $P(R)$ and the distribution of Picard's values are important as the first step in order to test the existence of analytic mappings. Our two theorems have less effectivity in two cases of $P(S)=2$ and $P(S)=3$. This is a weak point of our criteria. Further they never contribute to the existence part, which is the weakest point of our criteria. It seems to the present author that Sario's result ($\mathbf{A}_2$) or its generalization would give many striking applications. Although its original form in Sario's paper has an extremely implicite form, anyhow it contains very significant meanings.

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