ON THE GROWTH OF ANALYTIC FUNCTIONS

BY MITSURU OZAWA

1. In our previous papers [3], [4] we made use of the rigidity of projection map in order to establish some results on the value distribution of analytic functions on some Riemann surfaces. In the present paper we shall construct two open Riemann surfaces on which there is no analytic function of order lower than any given number. In the first example our Riemann surface belongs to the class O_G. In the second example we shall construct an open Riemann surface of hyperbolic type having the similar property. Our fundamental tools are (1) the rigidity property of projection map for functions of lower growth and (2) the unsymmetric welding of two surfaces.

Acknowledgement. The author wishes to express his heartiest gratitude to Professors J. Tamura and K. Oikawa for stimulating discussions he has had with them.

2. The first example. Let E be a plane with an infinite number of slits S_j clustering only at the point at infinity. We assume that S_1 is the unit segment $[0, 1]$. Let E_1 and E_2 be two copies of E. We shall connect E_1 and E_2 along each slits S_j $(j>1)$ in the standard manner and along S_1 in the following manner. Let $\sigma(t)$ be a monotone increasing function in $[0, 1]$ such that

$$\sigma(t)=t+t^2(t-a)^2(t-b)^2(t-1)^2, \qquad 0<a<b<1.$$

The upper shore S_1^+ of S_1 on E_1 and the lower shore S_1^- of S_1 on E_2 are welded in such a manner that $t\in S_1^+$ corresponds to $\sigma(t)\in S_1^-$. The lower shore of S_1 on E_1 and the upper shore of S_1 on E_2 are welded at the points with the same coordinate. This process is called a σ-process. There are many other σ-processes. The resulting surface is denoted by W, which is a Riemann surface belonging to the class O_G and two points 0 and 1 correspond to two inner points of new surface W. See Courant [1], p. 69 "Sewing theorem".

Let $n(r, E)$ be the number of end points of slits S_j $(j>1)$ lying in $|z|<r$. Let $T(r, f)$ be the Nevanlinna-Selberg characteristic of f on W over the ring domain $r_0<|z|<r$, $r_0>2$ or more general one defined by Sario.

THEOREM. *There is no non-constant single-valued meromorphic function f on W satisfying the growth condition*

Received November 14, 1963.

$$\varlimsup_{r\to\infty} \frac{\log T(r, f)}{\log r} < \varlimsup_{r\to\infty} \frac{\log n(r, E)}{\log r}.$$

Proof. Let $f(p)$ be a non-constant single-valued meromorphic function on W. Let $P(p)$ be the projection map: $p \to z$, which is defined on a part of W lying on the punctured disc $r_0 < |z| < \infty$. Let $F(z)$ be $f(P^{-1}(z))$, then $F(z)$ is either a single-valued meromorphic or two-valued algebroid function in $r_0 < |z| < \infty$. If $F(z)$ is two-valued there, then we can apply the Selberg ramification theorem [6]

$$N(r, W) < 2T(r, F) + O(1),$$

where $N(r, W)$ is equal to a quantity defined by

$$\frac{1}{2}\int_{r_0}^{r} \frac{n(r, E)}{r} dr.$$

Thus we have

$$\varlimsup_{r\to\infty} \frac{\log n(r, E)}{\log r} \leqq \varlimsup_{r\to\infty} \frac{\log N(2r, W)}{\log r} \leqq \varlimsup_{r\to\infty} \frac{\log T(2r, F)}{\log r},$$

which contradicts our assumption, since $T(r, f) = T(r, F) + O(1)$. Thus $F(z)$ must be reduced to a single-valued meromorphic function in $r_0 < |z| < \infty$, that is, $F(z) = f(p_j)$ for $P(p_j) = z$, $j = 1, 2$. This single-valuedness relation can be continued along any curve in W', which is the surface W with a cut along S_1.

For simplicity's sake we shall denote the points with the coordinate t on S_1^+ S_1^- by $p(t)$ and by $q(t)$, respectively. Then we have for $a < t < b$

$$\begin{aligned} A &= f(p(t)) = f(q(t)) = f(p(\sigma^{-1}(t))) = f(q(\sigma^{-1}(t))) \\ &= \cdots = f(p(\sigma^{-n}(t))) = \cdots, \end{aligned}$$

where $\sigma^{-n}(t) = \sigma^{-1}(\sigma^{-n+1}(t))$. However $\sigma^{-n}(t) < \sigma^{-n+1}(t)$ for any n and for any $t (a < t < b)$ and further $\lim_{n\to\infty} \sigma^{-n}(t) = a$ for any $t (a < t < b)$. Thus the point $z = a$ is a cluster points of $f(p)$. By the method of welding the point $z = a$ can be considered as an inner point of W. Thus $f(p)$ must be reduced to a constant. This is a contradiction.

Let P be the number of Picard's exceptional values of single-valued meromorphic functions on the surface W. By the Selberg theory on algebroid functions [6] we have the following facts in our Riemann surface W:

In Picard's great theorem

$$P \leqq \begin{cases} 2 & \text{if } \rho_f > \rho_n, \\ 4 & \text{if } \rho_f = \rho_n, \\ 2 & \text{if } \rho_f < \rho_n \end{cases}$$

with the exception of meromorphic functions of finite spherical area on $r_0 < |z| < \infty$, where ρ_f and ρ_n are the orders of $T(r, f)$ and $n(r, E)$, respectively.

In Picard's small theorem

$$P \leqq \begin{cases} 2 & \text{if } \rho_f > \rho_n, \\ 4 & \text{if } \rho_f = \rho_n \end{cases}$$

with the exception of constant functions. There is no non-trivial function of order ρ_f satisfying $\rho_f < \rho_n$.

Further we can impose a condition (if necessary) to W guaranteeing that $P \leqq 2$ or $P \leqq 3$.

The above fact shows that the existence of meromorphic functions of lower growth is not the boundary property. For the analytic bounded functions this phenomenon was pointed out by Myrberg [2]. Further the above proof shows that there is no non-constant single-valued meromorphic function $f(p)$ on W which is reduced to a single-valued function of z in $r_0 < |z| < \infty$.

3. The second example. Let E be the unit disc with an infinite number of slits which cluster only at the circumference $|z| = 1$. Let E_1 and E_2 be two copies of E. E_1 and E_2 are welded along all the slits S_j with the exception of only one pair of S_1 by the standard process. We make a σ-process between E_1 and E_2 along S_1. The resulting surface is denoted by W, which is a Riemann surface belonging to the class P_G (hyperbolic type.) In a quite similar manner we can infer that there is no non-constant single-valued meromorphic function on W satisfying

$$\overline{\lim_{r\to 1}} \frac{\log T(r, f)}{\log \dfrac{1}{1-r}} < \overline{\lim_{r\to 1}} \frac{\log n(r, E)}{\log \dfrac{1}{1-r}}.$$

References

[1] Courant, R., Dirichlet's principle, conformal mapping, and minimal surfaces. New York, 1950.

[2] Myrberg, P. J., Über die analytische Fortsetzung von beschränkten Funktionen. Ann. Acad. Sci. Fenn. A I. **58** (1949), 7 pp.

[3] Ozawa, M., Picard's theorem on some Riemann surfaces. Kōdai Math. Sem. Rep. **15** (1963), 245–256.

[4] Ozawa, M., Rigidity of projection map and the growth of analytic functions. Kōdai Math. Sem. Rep. **16** (1964), 40–43.

[5] Sario, L., Meromorphic functions and conformal metrics on Riemann surfaces. Pacific Journ. Math. **12** (1962), 1079–1097.

[6] Selberg, H. L., Algebroide Funktionen und Umkehrfunktionen Abelscher Integrale. Avh. Norske Vid.-Akad. Oslo (1934), Nr. 8.

[7] Selberg, H. L., Ein Satz über beschränkte endlichvieldeutige analytische Funktionen. Comm. Math. Helv. **9** (1937), 104–108.

Department of Mathematics,
Tokyo Institute of Technology.