ON SUMMABILITIES OF DOUBLE FOURIER SERIES

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1. Let $f(x, y)$ be a Lebesgue-integrable function of period 2π with respect to each x and y. Let the double Fourier series of $f(x, y)$ be

$$\mathfrak{S}(f) \equiv \sum_{m,n=0}^{\infty} A_{m,n}(x, y), \tag{1}$$

where

$$A_{0,0}(x, y) = \frac{1}{4} a_{0,0}$$

$$A_{m,0}(x, y) = \frac{1}{2}(a_{m,0} \cos mx + b_{m,0} \sin mx),$$

$$A_{0,n}(x, y) = \frac{1}{2}(a_{0,n} \cos ny + b_{0,n} \sin ny),$$

$$A_{m,n}(x, y) = a_{m,n} \cos mx \cos ny + b_{m,n} \cos mx \sin ny + c_{m,n} \sin mx \cos ny + d_{m,n} \sin mx \sin ny,$$

m and n being positive. Further, let the conjugate double Fourier series of $f(x, y)$ be

$$\bar{\mathfrak{S}}(f) \equiv \sum_{m,n=1}^{\infty} \bar{A}_{m,n}(x, y), \tag{2}$$

where

$$\bar{A}_{m,n}(x, y) = a_{m,n} \sin mx \sin ny - b_{m,n} \sin mx \cos ny - c_{m,n} \cos mx \sin ny + d_{m,n} \cos mx \cos ny.$$

The first arithmetic means $\sigma_{m,n}(x, y)$ of the series (1) are given by the formula

$$\begin{aligned}\sigma_{m,n}(x, y) &= \sum_{p=0}^{m} \sum_{q=0}^{n} \left(1 - \frac{p}{m+1}\right)\left(1 - \frac{q}{n+1}\right) A_{p,q}(x, y) \\ &= \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x+t, y+s) K_m(t) K_n(s)\, dt ds,\end{aligned} \tag{3}$$

where $K_m(t)$ is the Fejér kernel

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$$K_m(t)=\frac{2}{m+1}\left\{\frac{\sin\frac{1}{2}(m+1)t}{2\sin\frac{1}{2}t}\right\}^2 \tag{4}$$

satisfying the inequalities

$$|K_m(t)|<2m, \tag{5}$$

$$|K_m(t)|<C_1 m^{-1}t^{-2}\ \left(\frac{1}{m}\leqq|t|\leqq\pi;\ C_1 \text{ an absolute const.}\right). \tag{6}$$

Further, the first arithmetic means $\bar{\sigma}_{m,\,n}(x,y)$ of the series (2) are given by the formula

$$\begin{aligned}\bar{\sigma}_{m,\,n}(x,y)&=\sum_{p=1}^{m}\sum_{q=1}^{n}\left(1-\frac{p}{m+1}\right)\left(1-\frac{q}{n+1}\right)\bar{A}_{p,\,q}(x,y)\\ &=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(x+t,y+s)\bar{K}_m(t)\bar{K}_n(s)\,dtds,\end{aligned} \tag{7}$$

where $\bar{K}_m(t)$ is the conjugate Fejér kernel

$$\bar{K}_m(t)=\frac{1}{2}\cot\frac{1}{2}t-\frac{1}{m+1}\ \frac{\sin(m+1)t}{(2\sin\frac{1}{2}t)^2}. \tag{8}$$

Let

$$\bar{K}_m(t)=\frac{1}{2}\cot\frac{1}{2}t-H_m(t).$$

Then we have

$$|\bar{K}_m(t)|\leqq m \qquad \text{for all } t, \tag{9}$$

$$|H_m(t)|\leqq C_2 t^{-1} \qquad \text{for}\ \ |t|\leqq\frac{1}{m} \tag{10}$$

and

$$|H_m(t)|\leqq C_2 m^{-1}t^{-2} \qquad \text{for}\ \ \frac{1}{m}<|t|\leqq\pi, \tag{11}$$

C_2 being an absolute constant. The integral

$$\int_{-\pi}^{(m)\pi}\int_{-\pi}^{(n)\pi}g(x,y)\,dxdy \tag{12}$$

will mean the one extended over the set

$$\left\{(x,y);\ \frac{1}{m}\leqq|x|\leqq\pi,\ \ \frac{1}{n}\leqq|y|\leqq\pi\right\}.$$

We shall consider continuous functions $f(x,y)$ of period 2π with respect to each x and y, satisfying a Lipschitz condition, and we say that $f(x,y)$ belongs to Lip(α,β) if

$$|f(x+t, y+s)-f(x, y)|=O(|t|^{\alpha}+|s|^{\beta}) \tag{13}$$

uniformly in the point (x, y) as t and s tend to zero independently of each other, where $0<\alpha\leqq 1$ and $0<\beta\leqq 1$. We shall say that $f(x, y)$ belongs to $\mathrm{lip}(\alpha, \beta)$ if

$$|f(x+t, y+s)-f(x, y)|=o(|t|^{\alpha}+|s|^{\beta}) \tag{13$'$}$$

uniformly in the point (x, y) as t and s tend to zero independently of each other, where $0<\alpha\leqq 1$ and $0<\beta\leqq 1$.

As regards the first arithmetic means of Fourier series in the one-dimensional case, the following result is known. (see Zygmund [1], p. 91.)

THEOREM. *Let $\sigma_n(x)$ be the first arithmetic means of $\mathfrak{S}(f)$. If $f\in \mathrm{Lip}\,\alpha$, $0<\alpha<1$, then*

$$\sigma_n(x)-f(x)=O(n^{-\alpha}) \qquad \text{uniformly in } x.$$

If $\alpha=1$, then

$$\sigma_n(x)-f(x)=O(n^{-1}\log n).$$

We shall generalize this theorem in the two-dimensional case.

THEOREM 1.1. a) *If a continuous function $f(x, y)$ of period 2π with respect to each x and y belongs to $\mathrm{Lip}(\alpha, \beta)$, where $0<\alpha<1$ and $0<\beta<1$, then*

$$|\sigma_{m,n}(x, y)-f(x, y)|=O(m^{-\alpha}+n^{-\beta}) \tag{14}$$

uniformly in (x, y) as m and n tend to infinity independently of each other.

If $\alpha=\beta=1$, then

$$|\sigma_{m,n}(x, y)-f(x, y)|=O(m^{-1}\log m+n^{-1}\log n) \tag{15}$$

uniformly in (x, y) as m and n tend to infinity independently of each other.

b) *If a continuous function $f(x, y)$ of period 2π with respect to each x and y belongs to $\mathrm{lip}(\alpha, \beta)$, where $0<\alpha<1$ and $0<\beta<1$, then we can replace the symbol "O" of the formula* (14) *by the symbol "o".*

If $\alpha=\beta=1$, then we can replace the symbol "O" of the formula (15) *by the symbol "o".*

Proof. a) First, we shall prove (14). From (3), we have

$$\pi^2\{\sigma_{m,n}(x, y)-f(x, y)\}=\int_0^{\pi}\int_0^{\pi}\lambda_{x,y}(t, s)K_m(t)K_n(s)\,dtds, \tag{16}$$

where

$$\lambda_{x,y}(t, s)=f(x+t, y+s)+f(x-t, y+s)+f(x+t, y-s)+f(x-t, y-s)-4f(x, y).$$

Since $f(x, y)$ belongs to $\mathrm{Lip}(\alpha, \beta)$, where $0<\alpha<1$ and $0<\beta<1$,

$$\lambda_{x,y}(t, s)=o(t^{\alpha}+s^{\beta})$$

uniformly in (x, y) as t and s tend to $+0$ independently of each other. Therefore, from this, (5) and (6),

$$\pi^2|\sigma_{m,n}(x,y)-f(x,y)|$$
$$\leqq\left(\int_0^{1/m}\int_0^{1/n}+\int_0^{1/m}\int_{1/n}^{\pi}+\int_{1/m}^{\pi}\int_0^{1/n}+\int_{1/m}^{\pi}\int_{1/n}^{\pi}\right)|\lambda_{x,y}(t,s)|K_m(t)K_n(s)\,dtds$$
$$<\int_0^{1/m}\int_0^{1/n}O(t^\alpha+s^\beta)4mn\ dtds+\int_0^{1/m}\int_{1/n}^{\pi}O(t^\alpha+s^\beta)2C_1m\,n^{-1}\,s^{-2}\,dtds$$
$$+\int_{1/m}^{\pi}\int_0^{1/n}O(t^\alpha+s^\beta)2C_1m^{-1}t^{-2}n\ dtds+\int_{1/m}^{\pi}\int_{1/n}^{\pi}O(t^\alpha+s^\beta)C_1^2m^{-1}n^{-1}t^{-2}s^{-2}dtds$$
$$=O(m^{-\alpha}+n^{-\beta})$$

uniformly in (x,y) as m and n tend to infinity independently of each other.

If $\alpha=\beta=1$, we shall obtain (15) by the same method as in (14).

b) This will be proved by the same method as in a). q. e. d.

Further, we shall obtain the following theorem similar to Theorem 1.1 with respect to $\bar{\sigma}_{m,n}$.

THEOREM 1.2. a) *If a Lebesgue-integrable function $f(x, y)$ of period 2π with respect to each x and y satisfies*

(17) $$|f(x+t,y+s)-f(x-t,y+s)-f(x+t,y-s)+f(x-t,y-s)|=O(|t|^\alpha|s|^\beta)$$

uniformly in (x, y) as t and s tend to zero independently of each other, where $0<\alpha<1$, and $0<\beta<1$, then

(18) $$\left|\bar{\sigma}_{m,n}(x,y)-\frac{1}{\pi^2}\int_{-\pi}^{(m)\pi}\int_{-\pi}^{(n)\pi}f(x+t,y+s)\left(\frac{1}{2}\cot\frac{1}{2}t\right)\left(\frac{1}{2}\cot\frac{1}{2}s\right)dtds\right|$$
$$=O(m^{-\alpha}+n^{-\beta})$$

uniformly in (x, y) as m and n tend to infinity independently of each other.

If $\alpha=\beta=1$, then

(19) $$\left|\bar{\sigma}_{m,n}(x,y)-\frac{1}{\pi^2}\int_{-\pi}^{(m)\pi}\int_{-\pi}^{(n)\pi}f(x+t,y+s)\left(\frac{1}{2}\cot\frac{1}{2}t\right)\left(\frac{1}{2}\cot\frac{1}{2}s\right)dtds\right|$$
$$=O(m^{-1}\log m+n^{-1}\log n)$$

uniformly in (x, y) as m and n tend to infinity independently of each other.

b) *If we replace the symbol "O" of the condition (17) by the symbol "o", then we can replace the symbol "O" of the formula (18) by the symbol "o".*

If $\alpha=\beta=1$, then we can replace the symbol "O" of (19) by the symbol "o".

Proof. a) First we shall prove (18). Let

$$\bar{\lambda}_{x,y}(t,s)=f(x+t,y+s)-f(x-t,y+s)-f(x+t,y-s)+f(x-t,y-s).$$

From this and (7), we have

$$\pi^2\left\{\bar{\sigma}_{m,n}(x,y)-\frac{1}{\pi^2}\int_{-\pi}^{(m)\pi}\int_{-\pi}^{(n)\pi}f(x+t,y+s)\left(\frac{1}{2}\cot\frac{1}{2}t\right)\left(\frac{1}{2}\cot\frac{1}{2}s\right)dtds\right\}.$$

$$(20)\quad =\left(\int_0^{1/m}\int_0^{1/n}+\int_0^{1/m}\int_{1/n}^{\pi}+\int_{1/m}^{\pi}\int_0^{1/n}\right)\bar{\lambda}_{x,\,y}(t,s)\bar{K}_m(t)\bar{K}_n(s)\,dtds$$

$$+\int_{1/m}^{\pi}\int_{1/n}^{\pi}\bar{\lambda}_{x,\,y}(t,s)\left\{-H_n(s)\frac{1}{2}\cot\frac{1}{2}t-H_m(t)\frac{1}{2}\cot\frac{1}{2}s+H_m(t)H_n(s)\right\}dtds$$

$$=(I_1+I_2+I_3)+I_4,\text{ say.}$$

From (9), (17) and the definition of $\bar{\lambda}_{x,\,y}(t, s)$ we have that

$$(21)\qquad |I_1|\leqq\int_0^{1/m}\int_0^{1/n}O(t^{\alpha}\,s^{\beta})mn\,dtds\leqq O(m^{-\alpha}\,n^{-\beta})$$

uniformly in (x, y) as m and n tend to infinity independently of each other. From (9), (10), (11) and (17), we obtain that

$$(22)\qquad |I_2|\leqq\int_0^{1/m}\int_{1/n}^{\pi}O(t^{\alpha}s^{\beta})m\{O(s^{-1})+C_2n^{-1}s^{-2}\}dtds=O(m^{-\alpha}+m^{-\alpha}n^{-\beta})$$

uniformly in (x, y) as m and n tend to infinity independently of each other. Similarly, we have that

$$(23)\qquad |I_3|\leqq O(m^{-\alpha}n^{-\beta}+n^{-\beta})$$

uniformly in (x, y) as m and n tend to infinity independently of each other. Further, from (10), (11), and (17), we obtain that

$$(24)\qquad \begin{aligned}|I_4|&\leqq\int_{1/m}^{\pi}\int_{1/n}^{\pi}O(t^{\alpha}s^{\beta})\{O(t^{-1})C_2n^{-1}s^{-2}+C_2m^{-1}t^{-2}O(s^{-1})+C_2^2m^{-1}t^{-2}n^{-1}s^{-2}\}dtds\\&=O(n^{-\beta}+m^{-\alpha}+m^{-\alpha}n^{-\beta})\end{aligned}$$

uniformly in (x, y) as m and n tend to infinity independently of each other. Therefore, from (20), (21), (22), (23) and (24), we obtain (18).

If $\alpha=\beta=1$, we shall obtain (19) by the same method as in (18).

b) This will be proved by the same method as in a). q. e. d.

2. We shall prove two theorems for the Abel summability of double Fourier series.

Let $f(x, y)$ be a Lebesgue-integrable function of period 2π with respect to each x and y. Let the double Fourier series of $f(x, y)$ be of the form (1). Further, let the Abel means of the series (1) be

$$(25)\qquad \begin{aligned}f(r, x; R, y)&\equiv\sum_{m,n=0}^{\infty}A_{m,\,n}(x, y)r^mR^n,\qquad 0\leqq r<1\text{ and }0\leqq R<1,\\&=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(x+t, y+s)P(r, t)P(R, s)\,dtds,\end{aligned}$$

where $P(r, t)$ is Poisson's kernel $(1-r^2)/2(1-2r\cos t+r^2)$. For $P(r, t)$, we have two inequalities

$$(26)\qquad P(r, t)<\frac{1}{1-r}\qquad(0\leqq t\leqq\pi)$$

and

$$P(r, t) < \frac{1-r}{4r(\sin \frac{1}{2}t)^2} \qquad (0 < t \leqq \pi). \tag{27}$$

If $P_t'(r, t)$ denotes the derivative of Poisson's kernel with respect to t, we have

$$P_t'(r, t) = -\frac{(1-r^2)r \sin t}{(1-2r \cos t + r^2)^2} = -\frac{(1-r^2)r \sin t}{\{(1-r)^2 + 4r(\sin \frac{1}{2}t)^2\}^2}. \tag{28}$$

Thus we have

$$|P_t'(r, t)| \leqq \frac{2t}{(1-r)^3} \qquad (0 \leqq t \leqq \pi) \tag{29}$$

and

$$|P_t'(r, t)| < \frac{(1-r)t}{16r(\sin \frac{1}{2}t)^4} \qquad (0 < t \leqq \pi). \tag{30}$$

We shall prove that Theorem 1.1 holds if we replace the first arithmetic means $\sigma_{m,n}(x, y)$ by the Abel means $f(r, x; R, y)$. (In the one-dimensional case, see Salem and Zygmund [2], p. 30, Lemma 1.)

THEOREM 2.1. a) *If a continuous function* $f(x, y)$ *of period* 2π *with respect to each* x *and* y *belongs to* Lip(α, β), *where* $0 < \alpha < 1$ *and* $0 < \beta < 1$, *then*

$$|f(r, x; R, y) - f(x, y)| = O\{(1-r)^\alpha + (1-R)^\beta\} \tag{31}$$

uniformly in (x, y) *as* r *and* R *tend to* $1-0$ *independently of each other.*

If $\alpha = \beta = 1$, *then*

$$|f(r, x; R, y) - f(x, y)| = O\left\{(1-r)\log\frac{1}{1-r} + (1-R)\log\frac{1}{1-R}\right\} \tag{32}$$

uniformly in (x, y) *as* r *and* R *tend to* $1-0$ *independently of each other.*

b) *If a continuous function* $f(x, y)$ *of period* 2π *with respect to each* x *and* y *belongs to* lip(α, β), *where* $0 < \alpha < 1$ *and* $0 < \beta < 1$, *then we can replace the symbol "O" of the formula* (31) *by the symbol "o"*

If $\alpha = \beta = 1$, *then we can replace the symbol "O" of the formula* (32) *by the symbol "o".*

Proof. a) First, we shall prove (31). From (25), we have

$$\pi^2\{f(r, x; R, y) - f(x, y)\} = \int_0^\pi \int_0^\pi \lambda_{x,y}(t, s) P(r, t) P(R, s)\, dt ds,$$

where

$$\lambda_{x,y}(t, s) = f(x+t, y+s) + f(x-t, y+s) + f(x+t, y-s) + f(x-t, y-s) - 4f(x, y).$$

Since $f(x, y)$ belongs to Lip(α, β), where $0 < \alpha < 1$ and $0 < \beta < 1$,

$$\lambda_{x,y}(t, s) = O(t^\alpha + s^\beta)$$

uniformly in (x, y) as t and s tend to $+0$ independently of each other. From this, (26) and (27), we have

$$\pi^2|f(r,x;R,y)-f(x,y)|$$
$$\leqq\left(\int_0^{1-r}\int_0^{1-R}+\int_0^{1-r}\int_{1-R}^{\pi}+\int_{1-r}^{\pi}\int_0^{1-R}+\int_{\alpha-r}^{\pi}\int_{1-R}^{\pi}\right)|\lambda_{x,y}(t,s)|P(r,t)P(R,s)\,dtds$$
$$\leqq\frac{1}{(1-r)(1-R)}\int_0^{1-r}\int_0^{1-R}O(t^\alpha+s^\beta)dtds+\frac{1-R}{1-r}\int_0^{1-r}\int_{1-R}^{\pi}O(t^\alpha+s^\beta)O(s^{-2})\,dtds$$
$$+\frac{1-r}{1-R}\int_{1-r}^{\pi}\int_0^{1-R}O(t^\alpha+s^\beta)O(t^{-2})\,dtds$$
$$+(1-r)(1-R)\int_{1-r}^{\pi}\int_{1-R}^{\pi}O(t^\alpha+s^\beta)O(t^{-2})O(s^{-2})\,dtds$$
$$=\frac{1}{(1-r)(1-R)}[O\{(1-r)^{1+\alpha}(1-R)\}+O\{(1-r)(1-R)^{1+\beta}\}]$$
$$+\frac{1-R}{1-r}\left[O\left\{(1-r)^{1+\alpha}\frac{1}{1-R}\right\}+O\left\{(1-r)(1-R)^{\beta-1}\right\}\right]$$
$$+\frac{1-r}{1-R}\left[O\left\{(1-r)^{\alpha-1}(1-R)\right\}+O\left\{\frac{1}{1-r}(1-R)^{1+\beta}\right\}\right]$$
$$+(1-r)(1-R)\left[O\left\{(1-r)^{\alpha-1}\frac{1}{1-R}\right\}+O\left\{\frac{1}{1-r}(1-R)^{\beta-1}\right\}\right]$$
$$=O\{(1-r)^\alpha+(1-R)^\beta\}$$

uniformly in (x, y) as r and R tend to $1-0$ independently of each other.

If $\alpha=\beta=1$, we shall obtain (32) by the same method as in (31).

b) This will be proved by the same method as in a). q. e. d.

In the one-dimensional case, Salem and Zygmund [2] proved the following theorem.

THEOREM. *Let*

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

be the Fourier series of a continuous function $f(x)$ of period 2π, and belonging to lip α, *where* $0<\alpha<1$.

Then the difference

$$-\frac{1}{\pi}\Gamma(\alpha+1)\cos\frac{\pi\alpha}{2}\int_{1-r}^{\infty}\frac{f(x+t)-f(x-t)}{t^{1+\alpha}}dt-\sum_{n=1}^{\infty}(a_n\sin nx-b_n\cos nx)n^\alpha r^n$$

tends to zero uniformly, as $r\to1-0$.

If f belongs to Lip α, *the above difference is bounded, uniformly in x.*

In the two-dimensional case, we shall prove the analogue of this theorem. It is as follows:

THEOREM 2.2. *Let a continuous function $f(x, y)$ of period 2π with respect to each x and y belong to* Lip(ρ, ρ), *where* $0<\rho<1$. *Let*

$$A \leqq \frac{1-r}{1-R} \leqq B \qquad (0<A<B: \text{ constant}) \tag{33}$$

as r and R tend to $1-0$. Then the difference

$$\begin{aligned} &\frac{1}{\pi^2}\Gamma(\varphi+1)\Gamma(\psi+1)\cos\frac{\pi\varphi}{2}\cos\frac{\pi\psi}{2} \\ &\cdot\int_{1-r}^{\infty}\int_{1-R}^{\infty}\frac{f(x+t, y+s)-f(x-t, y+s)-f(x+t, y-s)+f(x-t, y-s)}{t^{1+\varphi}s^{1+\psi}}dtds \\ &-\sum_{m,n=1}^{\infty} m^{\varphi}n^{\psi}(a_{m,n}\sin mx\sin ny-b_{m,n}\sin mx\cos ny \\ &\qquad -c_{m,n}\cos mx\sin ny+d_{m,n}\cos mx\cos ny)r^mR^n \end{aligned} \tag{34}$$

is bounded, uniformly in (x, y) as r and R tend to $1-0$ in such a way that the condition (33) *is satisfied, where*

$$\rho=\varphi+\psi, \quad \varphi>0 \text{ and } \psi>0. \tag{35}$$

If $f(x, y)$ belongs to lip(ρ, ρ), *then the above difference tends to zero uniformly in (x, y).*

In order to prove this theorem, we need the following lemma.

LEMMA 1. *Let a continuous function $g(x, y)$ of period 2π with respect to each x and y belong to* Lip(α, β), *where* $0<\alpha<1$ *and* $0<\beta<1$. *Let $g(r, x; R, y)$ be the Abel means of the double Fourier series of $g(x, y)$. Then*

$$\left|\frac{\partial}{\partial x}g(r, x: R, y)\right|=O\{(1-r)^{\alpha-1}+(1-r)^{-1}(1-R)^{\beta}\}, \tag{36}$$

$$\left|\frac{\partial}{\partial y}g(r, x: R, y)\right|=O\{(1-r)^{\alpha}(1-R)^{-1}+(1-R)^{\beta-1}\} \tag{37}$$

and

$$\left|\frac{\partial^2}{\partial x\partial y}g(r, x: R, y)\right|=O\{(1-r)^{\alpha-1}(1-R)^{-1}+(1-r)^{-1}(1-R)^{\beta-1}\} \tag{38}$$

uniformly in (x, y) as r and R tend to $1-0$ independently of each other.

If $g(x, y)$ belongs to lip(α, β), *where* $0<\alpha<1$ *and* $0<\beta<1$, *then we can replace the symbols "O" of the formulas* (36), (37) *and* (38) *by the symbols "o".*

Proof. We shall prove the first half of this lemma. Fist, we prove the formula (36). We have

$$\frac{\partial}{\partial x} g(r, x;\ R, y) = -\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(t, s)P_t'(r, t-x)P(R, s-y)\, dtds$$

$$= -\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \{g(x+t, y+s)-g(x-t, y+s)+g(x+t, y-s)$$

$$-g(x-t, y-s)\}\, P_t'(r, t)P(R, s)dtds$$

$$= -\frac{1}{\pi^2}\Big[\int_0^{\pi}\int_0^{\pi} \{g(x+t, y+s)-g(x, y\}P_t'(r, t)P(R, s)\, dtds$$

$$-\int_0^{\pi}\int_0^{\pi} \{g(x-t, y+s)-g(x, y)\}\, P_t'(r, t)P(R, s)\, dtds$$

$$+\int_0^{\pi}\int_0^{\pi} \{g(x+t, y-s)-g(x, y)\}\, P_t'(r, t)P(R, s)\, dtds$$

$$-\int_0^{\pi}\int_0^{\pi} \{g(x-t, y-s)-g(x, y)\}\, P_t'(r, t)P(R, s)\, dtds\Big].$$

It is enough for us to show that

$$\left|\int_0^{\pi}\int_0^{\pi} \{g(x+t, y+s)-g(x, y\}\, P_t'(r, t)P(R, s)\, dtds\right| = O\{(1-r)^{\alpha-1}+(1-r)^{-1}(1-R)^{\beta}\} \tag{39}$$

uniformly in (x, y) as r and R tend to $1-0$ independently of each other, because the other terms are similar to this. Since $g(x, y)$ belongs to Lip(α, β), where $0<\alpha<1$ and $0<\beta<1$, we obtain from (26), (27), (29) and (30) that

$$\left|\int_0^{\pi}\int_0^{\pi} \{g(x+t, y+s)-g(x, y)\}\, P_t'(r, t)P(R, s)\, dtds\right|$$

$$\leqq \int_0^{\pi}\int_0^{\pi} |g(x+t, y+s)-g(x, y)|\ |P_t'(r, t)|P(R, s)\, dtds$$

$$\leqq (1-r)^{-3}(1-R)^{-1}\int_0^{1-r}\int_0^{1-R} O(t^{\alpha}+s^{\beta})2t\, dtds$$

$$+(1-r)^{-3}(1-R)\int_0^{1-r}\int_{1-R}^{\pi} O(t^{\alpha}+s^{\beta})2t\, O(s^{-2})\, dtds$$

$$+(1-r)(1-R)^{-1}\int_{1-r}^{\pi}\int_0^{1-R} O(t^{\alpha}+s^{\beta})O(t^{-3})\, dtds$$

$$+(1-r)(1-R)\int_{1-r}^{\pi}\int_{1-R}^{\pi} O(t^{\alpha}+s^{\beta})O(t^{-3})O(s^{-2})dtds$$

$$=(1-r)^{-3}(1-R)^{-1}O\{(1-r)^{2+\alpha}(1-R)+(1-r)^{2}(1-R)^{\beta+1}\}$$

$$+(1-r)^{-3}(1-R)O\{(1-r)^{2+\alpha}(1-R)^{-1}+(1-r)^{2}(1-R)^{-1+\beta}\}$$

$$+(1-r)(1-R)^{-1}O\{(1-r)^{-2+\alpha}(1-R)+(1-r)^{-2}(1-R)^{1+\beta}\}$$

$$+(1-r)(1-R)O\{(1-r)^{-2+\alpha}(1-R)^{-1}+(1-r)^{-2}(1-R)^{-1+\beta}\}$$

$$=O\{(1-r)^{\alpha-1}+(1-r)^{-1}(1-R)^{\beta}\}$$

uniformly in (x, y) as r and R tend to $1-0$ independently of each other.

The formula (37) will be proved by the same method as in (36).

In order to prove (38), it is enough for us to notice that

$$\frac{\partial^2}{\partial x \partial y} g(r, x; R, y)=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(t, s)P_t'(r, t-x)P_s'(R, s-y)\, dtds$$

$$=\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\{g(x+t, y+s)-g(x-t, y+s)-g(x+t, y-s)$$

$$+g(x-t, y-s)\}P_t'(r, t)P_s'(R, s)\, dtds.$$

Thus we have proved the first half of this lemma.

The second half of this lemma will be proved by the same method as in the first half. q. e. d.

Proof of Theorem 2.2. We shall prove the first half of this theorem. Let

(40) $$g_{x, y}(t, s)=f(x+t, y+s)-f(x-t, y+s)-f(x+t, y-s)+f(x-t, y-s).$$

Thus, the double Fourier series of $g_{x, y}(t, s)$ is

$$\mathfrak{S}\{g_{x, y}(t, s)\} \equiv \sum_{m,n=1}^{\infty} 4 \sin mt \sin ns(a_{m, n} \sin mx \sin ny-b_{m, n} \sin mx \cos ny$$

$$-c_{m, n} \cos mx \sin ny+d_{m, n} \cos mx \cos ny).$$

Let the Abel means of the double Fourier series of $g_{x, y}(t, s)$ be

(41) $$g_{x, y}(r, t; R, s)=\sum_{m,n=1}^{\infty} 4 \sin mt \sin ns(a_{m, n} \sin mx \sin ny-b_{m, n} \sin mx \cos ny$$

$$-c_{m, n} \cos mx \cos ny+d_{m, n} \cos mx \cos ny)r^m R^n.$$

For simplicity, we shall omit the suffices x and y of $g_{x, y}(t, s)$ and $g_{x, y}(r, t; R, s)$. For given (x, y), r and R, the series

$$\frac{g(r, t; R, s)}{t^{1+\varphi}s^{1+\psi}}=4\sum_{m,n=1}^{\infty}\frac{\sin mt}{t^{1+\varphi}}\frac{\sin ns}{s^{1+\psi}}(a_{m, n} \sin nx \sin ny-b_{m, n} \sin mx \cos ny$$

$$-c_{m, n} \cos mx \sin ny+d_{m, n} \cos mx \cos ny)r^m R^n$$

is uniformly convergent in (t, s) for $t>\varepsilon_1>0$ and $s>\varepsilon_2>0$. Hence we can integrate term by term in the rectangle $(\varepsilon_1, T; \varepsilon_2, S)$. Observing that

$$\left|\int_0^{\varepsilon_1}\frac{\sin mt}{t^{1+\varphi}}dt\right|<C_\varphi m\varepsilon_1^{1-\varphi}, \quad \left|\int_T^{\infty}\frac{\sin mt}{t^{1+\varphi}}dt\right|<\frac{C_\varphi}{T^\varphi},$$

$$\left|\int_0^{\varepsilon_2}\frac{\sin ns}{s^{1+\psi}}ds\right|<C_4 n\varepsilon_2^{1-\psi}, \quad \left|\int_S^{\infty}\frac{\sin ns}{s^{1+\psi}}ds\right|<\frac{C_\psi}{S^\psi},$$

where C_φ depends on φ only and C_ψ depends on ψ only, we deduce immediately that

$$\int_0^\infty\int_0^\infty \frac{g(r,t;R,s)}{t^{1+\varphi}s^{1+\psi}}dtds$$

$$(42)\qquad =4\sum_{m,n=1}^\infty (a_{m,n}\sin mx\sin ny-b_{m,n}\sin mx\cos ny$$
$$-c_{m,n}\cos mx\sin ny+d_{m,n}\cos mx\cos ny)r^mR^n\int_0^\infty\frac{\sin mt}{t^{1+\varphi}}dt\int_0^\infty\frac{\sin ns}{s^{1+\psi}}ds.$$

By (42) and the identity

$$\int_0^\infty \frac{\sin mt}{t^{1+\varphi}}dt=\frac{\pi m^\varphi}{2\cos(\pi\varphi/2)\Gamma(\varphi+1)},$$

we obtain

$$\frac{1}{2}\cos\frac{\pi\varphi}{2}\cos\frac{\pi\psi}{2}\Gamma(\varphi+1)\Gamma(\psi+1)\int_0^\infty\int_0^\infty\frac{g(r,t;R,s)}{t^{1+\varphi}s^{1+\psi}}dtds$$

$$(43)\qquad =4\sum_{m,n=1}^\infty (a_{m,n}\sin mx\sin ny-b_{m,n}\sin mx\cos ny$$
$$-c_{m,n}\cos mx\sin ny+d_{m,n}\cos mx\cos ny)m^\varphi n^\psi r^mR^n.$$

Since $f(x,y)$ belongs to $\mathrm{Lip}(\rho,\rho)$, g satisfies that

$$(44)\qquad \begin{aligned} &|g(u+t,v+s)-g(u,v)|\\ &\leqq |f(x+u+t,y+v+s)-f(x+u,y+v)|+|f(x+u+t,y-v-s)-f(x+u,y-v)|\\ &\quad+|f(x-u-t,y+v+s)-f(x-u,y+v)|+|f(x-u-t,y-v-s)-f(x-u,y-v)|\\ &=O(|t|^\rho+|s|^\rho)\end{aligned}$$

uniformly in (x,y) and (u,v) as t and s tend to zero independently of each other. This fact shows that $g(t,s)$ belongs to $\mathrm{Lip}(\rho,\rho)$ in each point (x,y). Write

$$\begin{aligned} D\equiv&\int_0^\infty\int_0^\infty\frac{g(r,t;R,s)}{t^{1+\varphi}s^{1+\psi}}dtds-\int_{1-r}^\infty\int_{1-R}^\infty\frac{g(t,s)}{t^{1+\varphi}s^{1+\psi}}dtds\\ =&\int_0^{1-r}\int_{1-R}^\infty\frac{g(r,t;R,s)}{t^{1+\varphi}s^{1+\psi}}dtds+\int_{1-r}^\infty\int_0^{1-R}\frac{g(r,t;R,s)}{t^{1+\varphi}s^{1+\psi}}dtds\\ &+\int_0^{1-r}\int_0^{1-R}\frac{g(r,t;R,s)}{t^{1+\varphi}s^{1+\psi}}dtds+\int_{1-r}^\infty\int_{1-R}^\infty\frac{g(r,t;R,s)-g(t,s)}{t^{1+\varphi}s^{1+\psi}}dtds.\end{aligned}$$

On the other hand, we note that, since $g(r,0;R,s)=g(r,t;R,0)=g(r,0;R,0)=0$, we have

$$g(r,t;R,s)=g(r,t;R,s)-g(r,0;R,s)$$

$$=t\left\{\frac{\partial}{\partial t}g(r,t;R,s)\right\}_{t=t_1} \quad \text{for } 0<t_1<t,$$

$$g(r,t;R,s)=g(r,t;R,s)-g(r,t;R,0)$$

$$=s\left\{\frac{\partial}{\partial s}g(r,t;R,s)\right\}_{s=s_1} \quad \text{for } 0<s_1<s$$

and

$$g(r,t;R,s)=g(r,t;R,s)-g(r,0;R,s)-g(r,t;R,0)+g(r,0;R,0)$$

$$=ts\left\{\frac{\partial^2}{\partial t\partial s}g(r,t;R,s)\right\}_{t=t_2,\,s=s_2} \quad \text{for } 0<t_2<t \text{ and } 0<s_2<s.$$

From these three formulas, we obtain

$$D\equiv\int_0^{1-r}\int_{1-R}^{\infty}\frac{t\left\{\frac{\partial}{\partial t}g(r,t;R,s)\right\}_{t=t_1}}{t^{1+\varphi}s^{1+\psi}}dtds+\int_{1-r}^{\infty}\int_0^{1-R}\frac{s\left\{\frac{\partial}{\partial s}g(r,t;R,s)\right\}_{s=s_1}}{t^{1+\varphi}s^{1+\psi}}dtds$$

(45) $$+\int_0^{1-r}\int_0^{1-R}\frac{ts\left\{\frac{\partial^2}{\partial t\partial s}g(r,t;R,s)\right\}_{t=t_2,\,s=s_2}}{t^{1+\varphi}s^{1+\varphi}}dtds$$

$$+\int_{1-r}^{\infty}\int_{1-R}^{\infty}\frac{g(r,t;R,s)-g(t,s)}{t^{1+\varphi}s^{1+\psi}}dtds.$$

If we put $\alpha=\beta=\rho$ in Theorem 2.1 and Lemma 1, we have from (44), (45), (31), (36), (37), (38) and (35) that

$$|D|\leqq\int_0^{1-r}\int_{1-R}^{\infty}\frac{tO\{(1-r)^{\rho-1}+(1-r)^{-1}(1-R)^{\rho}\}}{t^{1+\varphi}s^{1+\psi}}dtds$$

$$+\int_{1-r}^{\infty}\int_0^{1-R}\frac{sO\{1-r)^{\rho-1}(1-R)^{-1}+(1-R)^{\rho-1}\}}{t^{1+\varphi}s^{1+\psi}}dtds$$

$$+\int_0^{1-r}\int_0^{1-R}\frac{tsO\{(1-r)^{\rho-1}(1-R)^{-1}+(1-r)^{-1}(1-R)^{\rho-1}\}}{t^{1+\varphi}s^{1+\psi}}dtds$$

$$+\int_{1-r}^{\infty}\int_{1-R}^{\infty}\frac{O\{(1-r)^{\rho}+(1-R)^{\rho}\}}{t^{1+\varphi}s^{1+\psi}}dtds$$

$$=O\{(1-r)^{\rho-1}+(1-r)^{-1}(1-R)^{\rho}\}O\{(1-r)^{1-\varphi}(1-R)^{-\psi}\}$$

$$+O\{(1-r)^{\rho}(1-R)^{-1}+(1-R)^{\rho-1}\}O\{(1-r)^{-\varphi}(1-R)^{1-\psi}\}$$

$$+O\{(1-r)^{\rho-1}(1-R)^{-1}+(1-r)^{-1}(1-R)^{\rho-1}\}O\{(1-r)^{1-\varphi}(1-R)^{1-\psi}\}$$

$$+O\{(1-r)^{\rho}+(1-R)^{\rho}\}O\{1-r)^{-\varphi}(1-R)^{-\psi}\}$$

$$=O\{(1-r)^{\rho-\varphi}(1-R)^{-\psi}+(1-r)^{-\varphi}(1-R^{\rho-\psi}\}=O(1)$$

uniformly in (x,y) as r and R tendto $1-0$ in such a way that the condition (33) is satisfied. This completes the proof of the first half of the theorem.

The second half of this theorem will be proved by the same method as in the first half. q. e. d.

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