ON CONFORMAL SLIT MAPPING

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Dedicated to Professor K. Kunugi on his sixtieth birthday

1. Let B be a bounded slit domain in the sense of Löwner [6], which is obtained from the unit circle $|w|<1$ by cutting it along a Jordan arc L starting at a point on $|w|=1$ and not passing through the origin. Let $w=f(z)$ be the function mapping $|z|<1$ univalently onto B which is normalized by $f(0)=0$ and $f'(0)>0$. A well-known fundamental theorem was established by Löwner [6]; cf. also [2], [7]:

THEOREM 1. *For each bounded slit mapping*

$$f(z)=e^{-t_0}(z+\cdots), \qquad t_0>0 \qquad (|z|<1)$$

there exists a function $\kappa(t)$ *continuous and* $|\kappa(t)|=1$ *for* $0\leqq t\leqq t_0$ *such that* $f(z)=f(z, t_0)$ *is determined as the integral of the differential equation*

$$\frac{\partial f(z, t)}{\partial t}=-f(z, t)\frac{1+\kappa(t)f(z, t)}{1-\kappa(t)f(z, t)}$$

with the initial condition $f(z, 0)=z$.

Each function

$$w_t=f(z, t)=e^{-t}(z+\cdots) \qquad (0<t\leqq t_0)$$

carries out also a univalent mapping of $|z|<1$ onto a bounded slit domain B_t with a slit L_t. The continuous function $\kappa(t)=e^{i\theta(t)}$ involved in the Löwner equation shows an interesting behavior if the original slit $L=L_{t_0}$ is supposed to be analytic. Every slit L_t $(0<t<t_0)$ is then also analytic and meets $|w_t|=1$ orthogonally at its endpoint $\kappa(t)=e^{-i\theta(t)}$. Let the curvature of L_t at $\bar{\kappa}(t)$ be denoted by $\rho(t)$. The following theorem is known; cf. [3], [9]:

THEOREM 2. *Suppose that the slit* L *is an analytic arc. Then the function* $\theta(t)\equiv\arg\kappa(t)$ *involved in the Löwner equation is differentiable and satisfies*

$$\frac{d\theta(t)}{dt}=3\rho(t) \qquad (0<t<t_0).$$

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2. A function univalent in $|z|<1$, normalized by $f(0)=0$ and $f'(0)>0$ and with the image domain symmetric with respect to the real axis is characterized by the condition that all the coefficients of its Taylor expansion about the origin are real. Let B be a bounded slit domain which is obtained from $|w|<1$ by cutting it along two disjoint Jordan arcs L and $\bar{L}$ symmetric with respect to the real axis. For the class of these functions the Löwner equation may be modified as stated in the following theorem; cf. [1], [8]:

THEOREM 3. *For each bounded slit mapping* $f(z)=e^{-t_0}(z+\cdots)$, $t_0>0$, $(|z|<1)$ *with the real coefficients alone, there exists a real-valued function* $\theta(t)$ *continuous for* $0\leqq t\leqq t_0$ *such that* $f(z)=f(z,t_0)$ *is determined as the integral of the differential equation*

$$\frac{\partial f(z,t)}{\partial t}=-f(z,t)\frac{1-f(z,t)^2}{1-2f(z,t)\cos\theta(t)+f(z,t)^2}$$

with the initial condition $f(z,0)=z$.

Now, it will be aimed to derive a property of $\theta(t)$ which corresponds to the statement in theorem 2. Every function

$$w_t=f(z,t)=e^{-t}(z+\cdots)\qquad(0<t<t_0)$$

determined by the equation in theorem 3 maps $|z|<1$ onto a bounded slit domain B_t of the same nature as $B=B_{t_0}$. The endpoints of two slits L_t and $\bar{L}_t$ of B_t lying on $|w_t|=1$ are $e^{\mp i\theta(t)}$. If $L=L_{t_0}$ is supposed to be analytic, then every slit L_t is also analytic and meets $|w_t|=1$ orthogonally. Let the curvature of L_t at $w_t=e^{-i\theta(t)}$ be denoted by $\rho(t)$; that of $\bar{L}_t$ at $e^{i\theta(t)}$ is then $-\rho(t)$.

THEOREM 4. *Suppose that the slits of the domain B in theorem* 3 *are analytic. Then the function* $\theta(t)$ *involved there is differentiable and satisfies*

$$\frac{d\theta(t)}{dt}=\frac{1}{2}(3\rho(t)-\cot\theta(t))$$

i. e.,

$$\frac{d}{dt}(e^{-t/2}\cos\theta(t))=-\frac{3}{2}e^{\pm t/2}\rho(t)\sin\theta(t).$$

Proof. Though the proof can be performed by suitably modifying the procedure used in the earlier paper [3], the whole step will be stated here fully for the sake of completeness. The function Φ defined by the relations

$$w_\tau=\Phi(w_t;\ \tau,t),\quad w_\tau=f(z,\tau),\quad w_t=f(z,t)\qquad(\tau>t)$$

maps $|w_t|<1$ onto a bounded slit domain consisting of $|w_\tau|<1$ slit along the beginning pieces of L_τ and $\bar{L}_\tau$. The interior endpoints on the piece of L_τ lies at the point

$$\omega(t, \tau)=\Phi(e^{-i\theta(t)};\ \tau, t)$$

which tends to $\omega(t, t)=e^{-i\theta(t)}$ as $\tau\to t+0$. The curvature of L_τ at $e^{-i\theta(\tau)}$ is by definition given by

$$\rho(\tau)=\lim_{t\to\tau-0}\frac{2\,\Im\omega(t, \tau)e^{i\theta(\tau)}}{|1-\omega(t, \tau)e^{i\theta(\tau)}|^2};$$

the sign of $\rho(\tau)$ is understood positive or negative when the point $\omega(t, \tau)$ on the beginning piece of L_τ satisfies $\arg\omega(t, \tau)>-\theta(\tau)$ or $\arg\omega(t, \tau)<-\theta(\tau)$, respectively. It is shown that this limit relation defining $\rho(\tau)$ is valid uniformly on any closed interval contained in $0<\tau<t_0$; in particular, $\rho(\tau)$ is continuous there. It is further verified that the limit relation

$$\lim_{t\to\tau-0}\frac{1-|\omega(t, \tau)|}{|1-\omega(t, \tau)e^{i\theta(\tau)}|}=1$$

holds also uniformly there. Cf. the corresponding arguments stated in [3]. Hence it follows that the limit relation

$$\begin{aligned}&\lim_{t\to\tau-0}\frac{\Im\omega(t, \tau)e^{i\theta(\tau)}}{|1-\omega(t, \tau)e^{i\theta(\tau)}|^2}\Big/\frac{\arg(\omega(t, \tau)e^{i\theta(\tau)})}{(1-|\omega(t, \tau)|)^2}\\ &=\lim_{t\to\tau-0}|\omega(t, \tau)|\frac{\sin\arg(\omega(t, \tau)e^{i\theta(\tau)})}{\arg(\omega(t, \tau)e^{i\theta(\tau)})}\left(\frac{1-|\omega(t, \tau)|}{|1-\omega(t, \tau)e^{i\theta(\tau)}|}\right)^2=1\end{aligned}$$

is valid uniformly. Consequently, the curvature may be expressed also in the form

$$\rho(\tau)=\lim_{t\to\tau-0}\frac{2\arg(\omega(t, \tau)e^{i\theta(\tau)})}{(1-|\omega(t, \tau)|)^2}.$$

Now, by separating the differential equation

$$\frac{\partial\log w}{\partial t}=-\frac{1-w^2}{1-2w\cos\theta+w^2}\qquad (w=f(z, t),\ \ \theta=\theta(t))$$

into real and imaginary parts, it becomes

$$\begin{aligned}\frac{\partial|w|}{\partial t}&=-|w|\Re\frac{1-w^2}{1-2w\cos\theta+w^2}=-\frac{|w|(1-|w|^2)}{2}\left(\frac{1}{|1-we^{i\theta}|^2}+\frac{1}{|1-we^{-i\theta}|^2}\right),\\ \frac{\partial\arg w}{\partial t}&=-\Im\frac{1-w^2}{1-2w\cos\theta+w^2}=-\left(\frac{\Im we^{i\theta}}{|1-we^{i\theta}|^2}+\frac{\Im we^{-i\theta}}{|1-we^{-i\theta}|^2}\right).\end{aligned}$$

The first equation implies for $\tau>t$

$$\begin{aligned}&-(1-|\omega(t, \tau)|)^2=2\int_{|\omega(t,t)|}^{|\omega(t,\tau)|}(1-|\omega|)\,d|\omega|\\ &=-\int_t^\tau|\omega(t, \tau)|(1+|\omega(t, \tau)|)\left(\left(\frac{1-|\omega(t, \tau)|}{|1-\omega(t, \tau)e^{i\theta(\tau)}|}\right)^2+\left(\frac{1-|\omega(t, \tau)|}{|1-\omega(t, \tau)e^{-i\theta(\tau)}|}\right)^2\right)d\tau.\end{aligned}$$

Since the integrand of the last integral tends to 2 uniformly as $\tau-t\to+0$, one finds

$$(1-|\omega(t,\tau)|)^2=2(\tau-t)(1+\varepsilon_1)$$

where ε_1 as well as the ε's used below denote the quantities tending to zero with $\tau-t$. On the other hand, the second equation implies

$$2(\arg\omega(t,\tau)+\theta(t))=2\int_{\arg\omega(t,t)}^{\arg\omega(t,\tau)}d\arg\omega$$
$$=-\int_t^\tau 2\left(\frac{\Im\omega(t,\tau)e^{i\theta(\tau)}}{|1-\omega(t,\tau)e^{i\theta(\tau)}|^2}+\frac{\Im\omega(t,\tau)e^{-i\theta(\tau)}}{|1-\omega(t,\tau)e^{-i\theta(\tau)}|^2}\right)d\tau.$$

Since the integrand of the last integral tends to $\rho(\tau)-\cot\theta(\tau)$ uniformly as $\tau-t\to+0$, this becomes

$$2\arg(\omega(t,\tau)e^{i\theta(t)})=-\int_t^\tau(\rho(\tau)-\cot\theta(\tau))\,d\tau+\varepsilon_2(\tau-t).$$

Consequently, by taking account of the expression for $\rho(\tau)$ derived above, it follows that the relation

$$2(\tau-t)(1+\varepsilon_1)(\rho(\tau)+\varepsilon_3)=2\arg(\omega(t,\tau)e^{i\theta(\tau)})$$
$$=2(\theta(\tau)-\theta(t))-\int_t^\tau(\rho(\tau)-\cot\theta(\tau))\,d\tau+\varepsilon_2(\tau-t)$$

holds good which is written in the form

$$\frac{\theta(\tau)-\theta(t)}{\tau-t}=(1+\varepsilon_1)(\rho(\tau)+\varepsilon_3)+\frac{1}{\tau-t}\int_t^\tau\frac{1}{2}(\rho(t)-\cot\theta(\tau))\,d\tau-\varepsilon_2.$$

Thus, in view of the continuity of ρ and θ, the desired relation follows.

3. Analogues of theorem 1 as well as theorem 2 have been established for doubly-connected case in previous papers [4], [5].

Let B be a ring domain with the modulus $-\log Q$ $(0<Q<1)$ which is obtained from an annulus R: $Q_0<|w|<1$ by cutting it along a Jordan arc L lying in R save only for one endpoint $\bar{\gamma}$ on $|w|=1$. Let $w=f(z)$ be the function mapping $Q<|z|<1$ univalently onto B which is normalized by $f(Q)=Q_0$. The following two theorems are known.

THEOREM 5. *For each bounded slit mapping $f(z)$ of the nature explained just above, there exists a function $\gamma(q)$ continuous and $|\gamma(q)|=1$ for $Q_0\leqq q\leqq Q$ such that $f(z)=f(z,Q_0)$ is determined as the integral of the differential equation*

$$\frac{\partial\log f(z,q)}{\partial\log q}=-2i(\zeta(i\log\gamma(q)f(z,q))-\zeta_3(i\log\gamma(q)))-\frac{2\eta_1}{\pi}\log f(z,q)$$
$$\equiv 1+2\sum_{n=1}^{\infty}\frac{1}{1-q^{2n}}(f(z,q)^n-q^n)\left(\gamma(q)^n+\frac{1}{\gamma(q)^nf(z,q)^n}\right)$$

with the initial condition $f(z, Q)=z$, *the zeta-functions being those of Weierstrassian theory of elliptic functions with primitive periods* $2\omega_1=2\pi$ *and* $2\omega_3=-2i\log q$.

The right-hand member of the differential equation in theorem 5 can be written in an alternative form

$$\frac{1+\gamma w}{1-\gamma w}+1-\frac{1+\gamma q}{1-\gamma q}+2\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}(w^n-q^n)\left(\gamma^n+\frac{1}{q^n\gamma^n w^n}\right) \qquad (w=f(z, q),\ \gamma=\gamma(q)).$$

THEOREM 6. *Suppose that the slit* L *of the original domain* B *is analytic. Let* $\rho(q)$ *denote the curvature of the slit* L_q *at* $w_q=\bar{\gamma}(q)$ *of the domain* B_q *which is obtained as the image of* $Q<|w|<1$ *by the mapping* $w_q=f(z, q)$. *Then the function* $\theta(q)\equiv\arg\gamma(q)$ *is differentiable and satisfies*

$$\frac{d\theta(q)}{d\log q}=-3\rho(q)+2\left(\zeta_3(\theta(q))-\frac{\eta_1}{\pi}\theta(q)\right)$$

$$\equiv-3\rho(q)+4\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\sin n\theta(q).$$

4. Finally, let B be a bounded slit ring domain with the modulus $-\log Q$ $(0<Q<1)$ which is obtained from an annulus $Q_0<|w|<1$ by cutting it along two disjoint Jordan arcs L and $\bar{L}$ symmetric with respect to the real axis. The function $w=f(z)$ mapping $Q<|z|<1$ univalently onto B and normalized by $f(Q)=Q_0$ is subject to the condition that all the coefficients of its Laurent expansion in powers of z are real.

Theorems corresponding to theorem 5 and theorem 6 can be derived for this case. The proofs of those theorems being performable similarly to these ones, only the final results will be stated here.

THEOREM 7. *For each bounded slit mapping* $f(z)$ *of the nature explained just above, there exists a real-valued function* $\theta(q)$ *continuous for* $Q_0\leqq q\leqq Q$ *such that* $f(z)=f(z, Q_0)$ *is determined as the integral of the differential equation*

$$\frac{\partial\log f(z, q)}{\partial\log q}=-i(\zeta(i\log f(z, q)-\theta(q))+\zeta(i\log f(z, q)+\theta(q)))-\frac{2\eta_1}{\pi}\log f(z, q)$$

$$\equiv 1+2\sum_{n=1}^{\infty}\frac{1}{1-q^{2n}}(f(z, q)^n-q^n)\left(1+\frac{q^n}{f(z, q)^n}\right)\cos n\theta(q)$$

with the initial condition $f(z, Q)=z$.

It may be convenient to replace the right-hand member of the differential equation in theorem 7 by an alternative form

$$\frac{1-w^2}{1-2w\cos\theta+w^2}+1-\frac{1-q^2}{1-2q\cos\theta+q^2}+2\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}(w^n-q^n)\left(1+\frac{1}{q^n w^n}\right)\cos n\theta$$

$$(w=f(z, q),\ \theta=\theta(q)).$$

THEOREM 8. *Suppose that the slit L is an analytic arc. Let $\rho(q)$ denote the curvature of the slit L_q at $w_q=e^{-i\theta(q)}$ of the domain B_q which is the image of $Q<|z|<1$ by the mapping $w_q=f(z,q)$. Then the function $\theta(q)$ is differentiable and satisfies*

$$\begin{aligned}\frac{d\theta(q)}{d\log q}&=-\frac{1}{2}(3\rho(q)-\cot\theta(q))+2\left(\zeta_3(\theta(q))-\frac{\eta_1}{\pi}\theta(q)\right)\\&\equiv-\frac{1}{2}(3\rho(q)-\cot\theta(q))+4\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\sin n\theta(q).\end{aligned}$$

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