REDUCTION OF THE ORDER OF A LINEAR ORDINARY DIFFERENTIAL EQUATION CONTAINING A SMALL PARAMETER

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Introduction.

Let an n^{th} order linear ordinary differential equation of the form

$$(\mathrm{E}_1) \qquad \varepsilon^{\rho} w^{(n)} + a_1(x, \varepsilon) w^{(n-1)} + \cdots + a_n(x, \varepsilon) w = 0$$

be given, where ρ is a positive integer, x is an independent variable, ε is a parameter, and the coefficients a_k are functions holomorphic in (x, ε) for

$$(\mathrm{D}_1) \qquad |x| \leqq \delta_0, \ 0 < |\varepsilon| \leqq r_0, \ |\arg \varepsilon| \leqq \theta_0,$$

δ_0, r_0 and θ_0 being positive constants. We assume that a_k admits for $|x| \leqq \delta_0$ a uniformly asymptotic expansion in powers of ε

$$a_k(x, \varepsilon) \simeq \sum_{\nu=0}^{\infty} \varepsilon^{\nu} a_{k\nu}(x)$$

as ε tends to zero in the domain $0 < |\varepsilon| \leqq r_0$, $|\arg \varepsilon| \leqq \theta_0$ with the coefficients $a_{k\nu}$ holomorphic for $|x| \leqq \delta_0$. In this paper we shall discuss reduction of the order of the equation (E_1) in the domain (D_1).

First of all, by choosing a positive rational number σ suitably, we rewrite the equation (E_1) in a form

$$(\mathrm{E}_2) \qquad \varepsilon^{n\sigma} w^{(n)} + \varepsilon^{(n-1)\sigma} b_1(x, \varepsilon) w^{(n-1)} + \cdots + b_n(x, \varepsilon) w = 0,$$

where the coefficients b_k still have the properties similar to those of a_k, and moreover we have $b_k(x, 0) \not\equiv 0$ at least for a certain k.[1] In fact, let N_k be an integer such that $\lim_{\varepsilon \to 0} \varepsilon^{-N_k} a_k(x, \varepsilon)$ exists and is not identically equal to zero, then we may choose σ so as to have

$$\min_{j=1}^{n} \{ j\sigma + N_j - \rho \} = 0.^{2)}$$

In other words, we put

$$\sigma = \max_{j=1}^{n} \frac{\rho - N_j}{j} \qquad \text{(See Fig. 1).}$$

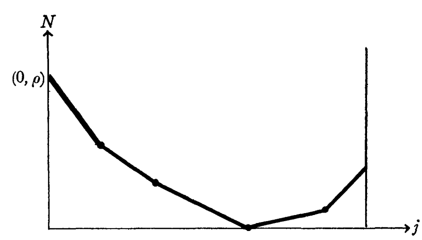

Fig. 1

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1) For simplicity's sake, we write $b_k(x, 0)$ to represent the coefficient $b_{k0}(x)$ in the asymptotic expression $b_k(x, \varepsilon) \simeq \sum_{\nu=0}^{\infty} \varepsilon^{\nu} b_{k\nu}(x)$. Therefore, $b_k(0, 0) = b_{k0}(0)$.

2) We assume that $\min\{N_1, N_2, \cdots, N_n\} = 0$.

Now we can assume without loss of generality that the algebraic equation

$$(\mathrm{C}_1) \qquad \lambda^n + b_1(0, 0)\lambda^{n-1} + \cdots + b_n(0, 0) = 0$$

has only one n-ple root, since, otherwise, the reduction of the order of the equation is always possible (See Sibuya [2]). Then the transformation

$$(\mathrm{T}) \qquad w = y \exp\left\{-\int^x \frac{b_1(\xi, \varepsilon)}{n\varepsilon^\sigma} d\xi\right\}$$

reduces the equation (E_2) to

$$(\mathrm{E}_3) \qquad \varepsilon^{n\sigma} y^{(n)} = \varepsilon^{(n-2)\sigma} p_2(x, \varepsilon) y^{(n-2)} + \cdots + p_n(x, \varepsilon) y,$$

where p_k have the properties similar to those of b_k, and we have

$$(\mathrm{A}_1) \qquad p_k(0, 0) = 0 \qquad (k = 2, 3, \cdots, n).$$

In the following chapters we shall study the equation (E_3) under the assumptions (A_1) and

$$(\mathrm{A}_2) \qquad p_k(x, 0) \not\equiv 0 \qquad \text{for some } k.$$[3]

Under these assumptions, the algebraic equation in λ:

$$(\mathrm{C}_2) \qquad \lambda^n - \{p_2(x, 0)\lambda^{n-2} + \cdots + p_n(x, 0)\} = 0$$

has only one n-ple root $\lambda = 0$ for $x = 0$, but it has at least two distinct roots for $x \neq 0$. Therefore, $x=0$ is possibly a turning point of the equation (E_3).

Generally speaking, the domain $|x| \leqq \delta_0$ can be divided into a finite number of subdomains in each of which the solution of (E_3) behaves quite differently. These subdomains can be constructed with the aid of positive rational numbers

$$0 < \rho_1 < \rho_2 < \cdots < \rho_m$$

in the following way:

$$(\mathrm{D}_2) \qquad \begin{cases} M_m |\varepsilon|^{\rho_m} \leqq |x| \leqq \delta_{m-1} |\varepsilon|^{\rho_{m-1}}, \\ M_{m-1} |\varepsilon|^{\rho_{m-1}} \leqq |x| \leqq \delta_{m-2} |\varepsilon|^{\rho_{m-2}}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ M_2 |\varepsilon|^{\rho_2} \leqq |x| \leqq \delta_1 |\varepsilon|^{\rho_1}, \\ M_1 |\varepsilon|^{\rho_1} \leqq |x| \leqq \delta_0 \end{cases}$$

and

3) In case when the assumption (A_2) does not hold, we must choose a suitable positive rational number σ' instead of σ, and bring the equation (E_3) into a desired form. As this is always possible, the assumption (A_2) does not harm any generality of our following discussions.

$$
(\mathrm{D}_3)\qquad \begin{cases} |x| \leqq M_m |\varepsilon|^{\rho_m}, \\ \delta_{m-1}|\varepsilon|^{\rho_{m-1}} \leqq |x| \leqq M_{m-1}|\varepsilon|^{\rho_{m-1}}, \\ \cdots\cdots\cdots\cdots\cdots\cdots, \\ \delta_1|\varepsilon|^{\rho_1} \leqq |x| \leqq M_1|\varepsilon|^{\rho_1}, \end{cases}
$$

where the numbers M_i are sufficiently large, while the numbers δ_i are sufficiently small. Chapter I will be devoted to the determination of those rational numbers ρ_i which can be done by introducing a convex polygon similar to the Newton's polygon in the theory of algebraic functions. In Chapter II we shall discuss reduction of the order of the equation (E_3) in each of the domains (D_2). In Chapter III we shall investigate the solutions of the equation (E_3) in each of the domains (D_3). In Chapter IV several examples will be given.

Our theory is based upon the ideas originally due to M. Iwano and simplified by Y. Sibuya.

I. Characteristic polygon.

§1. Assumptions. We shall consider an n^{th} order linear ordinary differential equation of the form

$$
(1.1)\qquad \varepsilon^{n\sigma}y^{(n)} = \varepsilon^{(n-2)\sigma}p_2(x, \varepsilon)y^{(n-2)} + \cdots + \varepsilon^{k\sigma}p_{n-k}(x, \varepsilon)y^{(k)} + \cdots + p_n(x, \varepsilon)y,
$$

where σ is a positive integer,[4] x is an independent variable, ε is a parameter, and the coefficients p_k are functions holomorphic in (x, ε) for

$$
(1.2)\qquad |x| \leqq \delta_0,\ 0 < |\varepsilon| \leqq r_0,\ |\arg \varepsilon| \leqq \theta_0,
$$

δ_0, r_0 and θ_0 being positive constants. We assume that p_k admits for

$$
(1.3)\qquad |x| \leqq \delta_0
$$

a uniformly asymptotic expansion in powers of ε

$$
(1.4)\qquad p_k(x, \varepsilon) \simeq \sum_{\nu=0}^{\infty} \varepsilon^{\nu} p_{k\nu}(x)
$$

as ε tends to zero in the domain

$$
(1.5)\qquad 0 < |\varepsilon| \leqq r_0,\ |\arg \varepsilon| \leqq \theta_0,
$$

where the coefficients $p_{k\nu}$ are holomorphic in the domain (1.3).

We shall further assume that

(i) $\quad p_{k0}(x) \not\equiv 0 \quad$ *for some* k;

(ii) $\quad p_{k0}(0) = 0 \quad (k = 2, 3, \cdots, n)$.

4) As is seen from our discussion in the Introduction, σ is generally a positive rational number. However, replacing ε, if necessary, by a suitable fractional power of ε, σ can always be regarded as an integer.

Under these assumptions, the algebraic equation in λ:

$$\lambda^n - \{p_{20}(x)\lambda^{n-2}+\cdots+p_{n0}(x)\} = 0 \tag{1. 6}$$

has only one n-ple root $\lambda = 0$ for $x = 0$, and at least two distinct roots for $x \neq 0$. Therefore, $x=0$ is possibly a turning point of the equation (1. 1).

§ 2. **Definition of the characteristic polygon.** Let

$$p_{k\nu}(x) = \sum_{h=0}^{\infty} x^h p_{k\nu h} \tag{2. 1}$$

be the expansion of $p_{k\nu}$ in powers of x, where the coefficients $p_{k\nu h}$ are constants. Suppose

$$p_{k\nu h} = 0 \qquad (h < m_{k\nu}) \tag{2. 2}$$

and

$$p_{k\nu h} \neq 0 \qquad (h = m_{k\nu}). \tag{2. 3}$$

If $p_{k\nu}(x) \equiv 0$, we put $m_{k\nu} = +\infty$.

In a plane with a rectangular coordinate system (X, Y), we plot the following points

$$\begin{cases} \mathrm{R} = (\sigma, -1), \\ \mathrm{P}_{k\nu} = \left(\dfrac{\nu}{k}, \dfrac{m_{k\nu}}{k}\right) \end{cases} \quad \begin{pmatrix} k = 2, 3, \cdots, n; \\ \nu = 0, 1, \cdots \end{pmatrix}. \tag{2. 4}$$

All of the points $\mathrm{P}_{k\nu}$ are either on the X-axis or in the upper half-plane, while the point R is in the lower half-plane. A polygon Π, convex downward, can be constructed in such a way that

(i) *its vertices are some of the points* (2. 4);

(ii) *none of the points* (2. 4) *is located below the polygon.*

Hereafter, this will be called *the characteristic polygon.* It is easily seen that the point R is a vertex of the characteristic polygon. On the other hand, the assumptions (i) and (ii) of § 1 imply that there is a vertex Q_0 on the Y-axis but Q_0 is not the origin. We shall denote the vertices between Q_0 and R by $\mathrm{Q}_1, \mathrm{Q}_2, \cdots, \mathrm{Q}_{m-1}$ successively from the left to the right. We shall also denote R by Q_m (See Fig. 2).

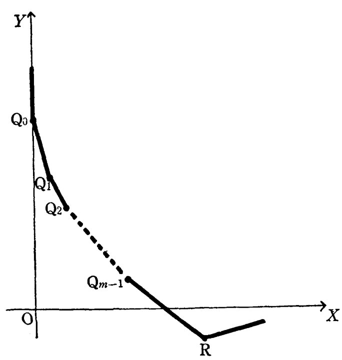

Fig. 2

§ 3. **Definition of ρ_i.** Let

$$\mathrm{Q}_i = (\alpha_i, \beta_i) \quad (i = 0, 1, \cdots, m). \tag{3. 1}$$

Since the point Q_0 is on the Y-axis, its X-coordinate must be equal to zero:

(3. 2) $$\alpha_0 = 0.$$

On the other hand, since Q_0 is not the origin, its Y-coordinate must be positive:

(3. 3) $$\beta_0 > 0.$$

Further we have

(3. 4) $$\alpha_m = \sigma\,,\ \beta_m = -1,$$

because $Q_m = R$.

Let us put

(3. 5) $$\rho_i = -\frac{\alpha_i - \alpha_{i-1}}{\beta_i - \beta_{i-1}} \qquad (i = 1, 2, \cdots, m).$$

Then ρ_i are positive rational numbers such that

(3. 6) $$\rho_0 (= 0) < \rho_1 < \rho_2 < \cdots < \rho_m.$$

The straight line passing through the points Q_i and Q_{i-1} is given by the equation

(3. 7) $$\rho_i (Y - \beta_i) + (X - \alpha_i) = 0,$$

and the straight line passing through the points Q_i and Q_{i+1} is given by

(3. 8) $$\rho_{i+1}(Y - \beta_i) + (X - \alpha_i) = 0.$$

Now, returning to the expansion (2. 1) of the function $p_{k\nu}$, let us consider a point (X, Y) with

(3. 9) $$X = \frac{\nu}{k}\,,\ Y = \frac{h}{k}$$

for some $p_{k\nu h} \neq 0$. Then, since (2. 2) implies $h \geqq m_{k\nu}$, the point (X, Y) does not lie below the characteristic polygon. Therefore, we have

(3. 10) $$\tau \equiv \rho_i (Y - \beta_i) + (X - \alpha_i) \geqq 0$$

and

(3. 11) $$\tau' \equiv \rho_{i+1}(Y - \beta_i) + (X - \alpha_i) \geqq 0.$$

In particular, we have

(3. 12) $$\tau = 0$$

if and only if the point (X, Y) is on the segment joining Q_i to Q_{i-1}, and we have

(3. 13) $$\tau' = 0$$

if and only if the point (X, Y) is on the segment joining Q_i to Q_{i+1} (See Fig. 3).

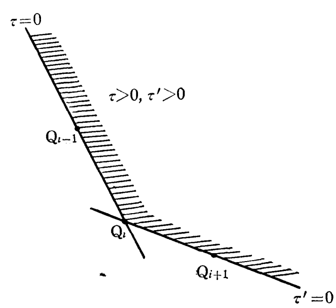

Fig. 3

§4. Properties of the characteristic polygon, I. If we put

$$(4.1)\qquad \vec{y}=\begin{pmatrix} y \\ \varepsilon^{\sigma}y' \\ \varepsilon^{2\sigma}y'' \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon^{(n-1)\sigma}y^{(n-1)} \end{pmatrix},$$

the equation (1. 1) can be written as

$$(4.2)\qquad \varepsilon^{\sigma}\frac{d\vec{y}}{dx}=A(x,\varepsilon)\vec{y},$$

where $A(x, \varepsilon)$ is an n by n matrix of the form

$$(4.3)\qquad A(x,\varepsilon)=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ p_n & p_{n-1} & p_{n-2} & \cdots & p_2 & 0 \end{pmatrix}.$$

Let us consider a linear transformation

$$(4.4)\qquad \vec{y}=\Lambda(\alpha,\beta)\vec{z},$$

where Λ is a diagonal matrix whose components on the principal diagonal are 1, $\varepsilon^{\alpha}x^{\beta}$, $(\varepsilon^{\alpha}x^{\beta})^2$, $\cdots$, $(\varepsilon^{\alpha}x^{\beta})^{n-1}$ respectively, i.e.:

$$\Lambda(\alpha,\beta)=\begin{pmatrix} 1 & & & & \\ & \varepsilon^{\alpha}x^{\beta} & & & 0 \\ & & (\varepsilon^{\alpha}x^{\beta})^2 & & \\ & & & \ddots & \\ & 0 & & & (\varepsilon^{\alpha}x^{\beta})^{n-1} \end{pmatrix}.$$

Then the equation (4. 2) will be transformed into

$$(4.5)\qquad \varepsilon^{\sigma-\alpha}x^{-\beta}\frac{d\vec{z}}{dx}=B(x,\varepsilon)\vec{z},$$

where

$$B(x,\varepsilon)=\varepsilon^{-\alpha}x^{-\beta}\left\{\Lambda(\alpha,\beta)^{-1}A(x,\varepsilon)\Lambda(\alpha,\beta)-\Lambda(\alpha,\beta)^{-1}\varepsilon^{\sigma}\frac{d\Lambda(\alpha,\beta)}{dx}\right\}.$$

Therefore we can write

$$B(x, \varepsilon) = B_1(x, \varepsilon) - \beta\, \varepsilon^{\sigma-\alpha}\, x^{-\beta-1}\, C_0, \tag{4. 6}$$

where

$$B_1(x, \varepsilon) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ (\varepsilon^{\alpha}x^{\beta})^{-n}p_n & (\varepsilon^{\alpha}x^{\beta})^{-n+1}p_{n-1} & (\varepsilon^{\alpha}x^{\beta})^{-n+2}p_{n-2} & \cdots & (\varepsilon^{\alpha}x^{\beta})^{-2}p_2 & 0 \end{pmatrix} \tag{4. 7}$$

and

$$C_0 = \begin{pmatrix} 0 & & & & \\ & 1 & & 0 & \\ & & 2 & & \\ & & & \ddots & \\ 0 & & & & n-1 \end{pmatrix}. \tag{4. 8}$$

From (1. 4) and (2. 1) it follows that

$$(\varepsilon^{\alpha}x^{\beta})^{-k}\, p_k(x, \varepsilon) \simeq \sum_{\nu, h} \varepsilon^{\nu-k\alpha}\, x^{h-k\beta}\, p_{k\nu h}. \tag{4. 9}$$

Let us put

$$\tau = \rho(Y-\beta) + (X-\alpha) \tag{4. 10}$$

and

$$\tau' = \rho'\,(Y-\beta) + (X-\alpha), \tag{4. 11}$$

where $0<\rho<\rho'$. Then $\tau=0$ defines a straight line passing through the point (α, β), and $\tau \geqq 0$ means that the point (X, Y) does not lie below the straight line. The same is true for τ' (See Fig. 4).

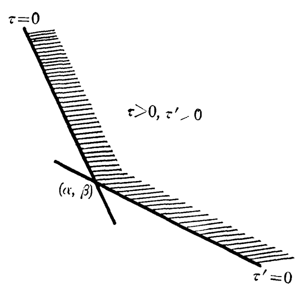

Fig. 4

Substituting

$$X = \frac{\nu}{k}, \quad Y = \frac{h}{k} \tag{4. 12}$$

into (4. 10) and (4. 11), each summand in the expression (4. 9) will be written in a form

$$\varepsilon^{\nu-k\alpha}\, x^{h-k\beta} = (x\varepsilon^{-\rho})^{k\tau'/(\rho'-\rho)}(x\varepsilon^{-\rho'})^{-k\tau/(\rho'-\rho)}, \tag{4. 13}$$

where $\tau \geqq 0$ if and only if the point $(\nu/k,\ h/k)$ does not lie below the straight line defined by $\rho(Y-\beta)+(X-\alpha)=0$, while $\tau' \geqq 0$ if and only if the point $(\nu/k,\ h/k)$ does not lie below the straight line defined by $\rho'(Y-\beta)+(X-\alpha)=0$. In case when those two lines do not intersect the characteristic polygon Π, both of τ and τ' are nonnegative.

If both of τ and τ' are nonnegative, and one of them is positive, then (4. 13) implies that $\varepsilon^{\nu-k\alpha}x^{h-k\beta}$ is small when $x\varepsilon^{-\rho}$ is small and $x\varepsilon^{-\rho'}$ is large. Therefore, in this case, $\varepsilon^{\nu-k\alpha}x^{h-k\beta}$ is small in a domain

$$M|\varepsilon|^{\rho'}\leqq|x|\leqq\delta|\varepsilon|^{\rho}, \tag{4. 14}$$

where M and $1/\delta$ are large positive constants. Furthermore, in this case, if $h/k\geqq\beta$, then $\tau'-\tau=(\rho'-\rho)(h/k-\beta)\geqq 0$ and $\varepsilon^{\nu-k\alpha}x^{h-k\beta}$ can be written as

$$\varepsilon^{\nu-k\alpha}x^{h-k\beta}=\varepsilon^{k\tau}(x\varepsilon^{-\rho})^{k(\tau'-\tau)/(\rho'-\rho)}. \tag{4. 15}$$

On the other hand, if $h/k\leqq\beta$, then $\tau-\tau'=(\rho-\rho')(h/k-\beta)\geqq 0$ and $\varepsilon^{\nu-k\alpha}x^{h-k\beta}$ can be written as

$$\varepsilon^{\nu-k\alpha}x^{h-k\beta}=\varepsilon^{k\tau'}(x\varepsilon^{-\rho'})^{-k(\tau-\tau')/(\rho'-\rho)}. \tag{4. 16}$$

The same can be proved for

$$X=\sigma, \qquad Y=-1. \tag{4. 17}$$

From (4. 9) it is easily seen that the linear transformation (4. 4) effects a change of coordinates

$$X'=X-\alpha, \qquad Y'=Y-\beta \tag{4. 18}$$

in the (X, Y)-plane. The origin of the new coordinate system is at the point (α, β) (See Fig. 5).

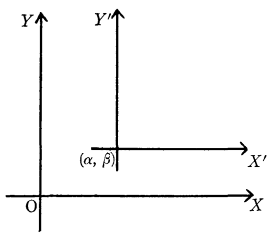

Fig. 5

§5. **Properties of the characteristic polygon, II.** Now let us consider a transformation of the form

$$\begin{cases} x=\varepsilon^{\rho}\xi, \\ \vec{y}=\Lambda(\gamma, 0)\vec{z}, \end{cases} \tag{5. 1}$$

where ρ is a positive number. If we write the transformed equation as

$$\varepsilon^{\sigma-\rho-\gamma}\frac{d\vec{z}}{d\xi}=B(\xi, \varepsilon)\vec{z}, \tag{5. 2}$$

the matrix B has the following form:

$$B(\xi, \varepsilon)=\varepsilon^{-\gamma}\Lambda(\gamma, 0)^{-1}A(\varepsilon^{\rho}\xi, \varepsilon)\Lambda(\gamma, 0)$$

$$=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \varepsilon^{-n\gamma}p_n & \varepsilon^{-(n-1)\gamma}p_{n-1} & \varepsilon^{-(n-2)\gamma}p_{n-2} & \cdots & \varepsilon^{-2\gamma}p_2 & 0 \end{pmatrix}. \tag{5. 3}$$

From (1. 4) and (2. 1) we can derive the following asymptotic expansion:

$$\varepsilon^{-k\gamma} p_k(\varepsilon^\rho \xi, \varepsilon) \simeq \sum_{\nu, h} \varepsilon^{\nu+\rho h-\gamma k} \xi^h p_{k\nu h}. \tag{5. 4}$$

Now let us put

$$\sigma' = \rho Y + (X - \gamma). \tag{5. 5}$$

Then $\sigma'=0$ defines a straight line passing through the point $(\gamma, 0)$, and $\sigma' \geqq 0$ means that the point (X, Y) does not lie below the straight line. In particular, $\sigma'=0$ if and only if the point (X, Y) is on the straight line.

If we put $X=\nu/k$, $Y=h/k$ in (5. 5), we have

$$\varepsilon^{\nu+\rho h-\gamma k} = \varepsilon^{k\sigma'}, \tag{5. 6}$$

and if we put $X=\sigma$, $Y=-1$ in (5. 5), we have

$$\varepsilon^{\sigma-\rho-\gamma} = \varepsilon^{\sigma'}. \tag{5. 7}$$

Thus we see that the transformation (5. 1) effects a change of coordinates

$$X'=\rho Y+(X-\gamma), \qquad Y'=Y \tag{5. 8}$$

in the (X, Y)-plane. The Y'-axis is the straight line defined by

$$\rho Y + (X - \gamma) = 0, \tag{5. 9}$$

and the origin of the new coordinate system is at the point $(\gamma, 0)$ (See Fig. 6).

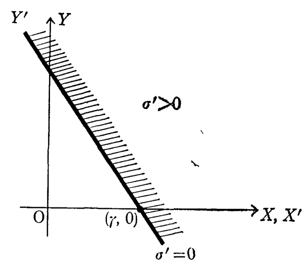

Fig. 6

II. Reduction of order.

§ 6. Transformation of the equation (1. 1). In this chapter, we shall reduce the order of the equation (1. 1) in each of the following domains:

$$\begin{cases} M_{i+1}|\varepsilon|^{\rho_{i+1}} \leqq |x| \leqq \delta_i|\varepsilon|^{\rho_i} & (i=1, 2, \cdots, m-1), \\ M_1|\varepsilon|^{\rho_1} \leqq |x| \leqq \delta_0. \end{cases} \tag{6. 1}$$

To do this, we shall use transformations of the form

$$\vec{y} = A(\alpha_i, \beta_i)\vec{z}. \tag{6. 2}$$

The equation on $\vec{z}$ can be written as

$$\varepsilon^{\sigma-\alpha_i} x^{-\beta_i} \frac{d\vec{z}}{dx} = \{B_1(x, \varepsilon)-\beta_i \varepsilon^{\sigma-\alpha_i} x^{-\beta_i-1} C_0\}\vec{z}, \tag{6. 3}$$

where B_1 is given by (4. 7) with $\alpha=\alpha_i$ and $\beta=\beta_i$, while C_0 is given by (4. 8).

Let us put

$$\tau = \rho_i(Y - \beta_i) + (X - \alpha_i) \tag{6. 4}$$

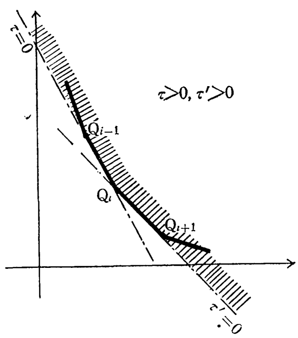

Fig. 7

and

$$\tau'=\rho_{i+1}(Y-\beta_i)+(X-\alpha_i). \tag{6.5}$$

Then we have $\tau \geqq 0$ and $\tau' \geqq 0$ for (4.12) as well as for (4.17) (See Fig. 7). Therefore in the domain

$$M_{i+1}|\varepsilon|^{\rho_{i+1}} \leqq |x| \leqq \delta_i|\varepsilon|^{\rho_i} \tag{6.6}$$

the matrix $B_1-\beta_i\varepsilon^{\sigma-\alpha_i}x^{-\beta_i-1}C_0$ can be written as

$$B_1(x, \varepsilon)-\beta_i\varepsilon^{\sigma-\alpha_i}x^{-\beta_i-1}C_0=H_0+[\cdots], \tag{6.7}$$

where $[\cdots]$ indicates the terms small in the domain (6.6), and H_0 is a constant matrix of the form

$$H_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_2 & 0 \end{pmatrix}. \tag{6.8}$$

Furthermore, at least one of the constants $c_n, c_{n-1}, \cdots, c_2$ is different from zero (because we have $\tau=\tau'=0$ in (4.13) for some k). Therefore, H_0 has at least two distinct characteristic values. Thus the order of the equation (1.1) can be reduced in the domain (6.6) by the use of the lemma which will be given in §7.

REMARK. In case when the vertex Q_{m-1} is on the X-axis, we have $\beta_{m-1}=0$ (See Fig. 8). Then by the transformation

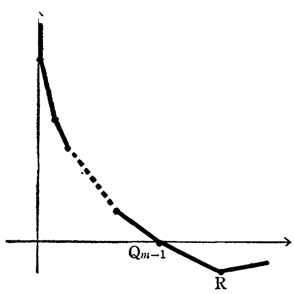

Fig. 8

$$\vec{y} = \Lambda(\alpha_{m-1}, \beta_{m-1})\vec{z} \tag{6.9}$$

the equation (4.2) is reduced to

$$\varepsilon^{\sigma-\alpha_{m-1}}x^{-\beta_{m-1}}\frac{d\vec{z}}{dx}=B_1(x, \varepsilon)\vec{z}, \tag{6.10}$$

where B_1 is given by (4.7) with $\alpha=\alpha_{m-1}$ and $\beta=\beta_{m-1}$ $(=0)$. Therefore, it is easily seen that, in this case, we can reduce the order of the equation (1.1) in the domain

$$|x| \leqq \delta_{m-1}|\varepsilon|^{\rho_{m-1}} \tag{6.11}$$

in stead of the domain

$$M_m|\varepsilon|^{\rho_m} \leqq |x| \leqq \delta_{m-1}|\varepsilon|^{\rho_{m-1}} \quad \text{(See (4.15) and (4.16)).}$$

§7. **Fundamental lemma.** Consider a system of linear ordinary differential equations

$$(x^{-\mu}\varepsilon)^q x^{q'+1}\frac{d\vec{z}}{dx}=E(x,\varepsilon)\vec{z}, \tag{7.1}$$

where μ is a positive rational number, q is a positive integer, q' is a nonnegative integer, and E is an n by n matrix whose components are holomorphic functions of (x, ε) in a domain

$$M|\varepsilon|^{1/\mu}\leqq|x|\leqq\delta,\ 0<|\varepsilon|\leqq r,\ |\arg\varepsilon|\leqq\theta, \tag{7.2}$$

M, δ, r and θ being positive constants.

Assume that the matrix E admits, for $M|\varepsilon|^{1/\mu}\leqq|x|\leqq\delta$, a uniformly asymptotic expansion in powers of $(x^{-\mu}\varepsilon)$

$$E(x,\varepsilon)\simeq\sum_{k=0}^{\infty}(x^{-\mu}\varepsilon)^k E_k(x) \tag{7.3}$$

as ε tends to zero in the domain

$$0<|\varepsilon|\leqq r,\ |\arg\varepsilon|\leqq\theta, \tag{7.4}$$

where the coefficients E_k are n by n matrices whose components are holomorphic functions of x in the domain

$$|x|\leqq\delta. \tag{7.5}$$

In other words, an inequality

$$\left|E(x,\varepsilon)-\sum_{k=0}^{N-1}(x^{-\mu}\varepsilon)^k E_k(x)\right|\leqq K_N|x^{-\mu}\varepsilon|^N \tag{7.6}$$

is satisfied uniformly in the domain (7. 2) for each N, where K_N is a positive constant independent of (x, ε). The integers q and q' are supposed to satisfy an inequality

$$q'-\mu q<0. \tag{7.7}$$

Let $\lambda_1, \lambda_2, \cdots, \lambda_s$ be distinct characteristic values of $E_0(0)$, and let n_j be the multiplicity of λ_j. Then, assuming that $E_0(0)$ has the Jordan canonical form, we can write it as

$$E_0(0)=\begin{pmatrix}\mathring{E}_1 & & & 0\\ & \mathring{E}_2 & & \\ & & \ddots & \\ 0 & & & \mathring{E}_s\end{pmatrix}, \tag{7.8}$$

where

$$\mathring{E}_j=\begin{pmatrix}\lambda_j & \delta_{j1} & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & \lambda_j & \delta_{j2} & 0 & \cdots & 0 & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_j & \delta_{jn_j-1}\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \lambda_j\end{pmatrix} \tag{7.9}$$

$$(j=1, 2, \cdots, s),$$

δ_{jk} being equal either to 1 or to zero.

Let Θ_+, Θ_-, θ_+ and θ_- be positive numbers. A domain

$$(7.10) \qquad -\Theta_- \leqq \arg x \leqq \Theta_+, \quad -\theta_- \leqq \arg \varepsilon \leqq \theta_+$$

is said to be proper with respect to the functions

$$(7.11) \qquad (\lambda_j - \lambda_k)\varepsilon^{-q}x^{q\mu-q'} \qquad (j \neq k)$$

when we can choose $\arg(\lambda_j-\lambda_k)$ in such a way that

$$|\arg\{(\lambda_j-\lambda_k)\varepsilon^{-q}x^{q\mu-q'}\}| \leqq \frac{3\pi}{2} - \gamma$$

for (7.10), where γ is a sufficiently small positive number.

Suppose that the domain (7.10) is proper with respect to the functions (7.11). The existence of such a domain is easily proved. Let $P(x, \varepsilon)$ be an n by n matrix whose components are holomorphic functions of (x, ε) for

$$(7.12) \qquad \begin{cases} M'|\varepsilon|^{1/\mu} \leqq |x| \leqq \delta', & -\Theta_- \leqq \arg x \leqq \Theta_+, \\ 0 < |\varepsilon| \leqq r', & -\theta_- \leqq \arg \varepsilon \leqq \theta_+, \end{cases}$$

where M', δ' and r' are positive constants. We shall assume that P admits for

$$(7.13) \qquad M'|\varepsilon|^{1/\mu} \leqq |x| \leqq \delta', \quad -\Theta_- \leqq \arg x \leqq \Theta_+$$

a uniformly asymptotic expansion in powers of $(x^{-\mu}\varepsilon)$

$$(7.14) \qquad P(x, \varepsilon) \simeq \sum_{k=0}^{\infty} (x^{-\mu}\varepsilon)^k P_k(x)$$

as ε tends to zero in the domain

$$(7.15) \qquad 0 < |\varepsilon| \leqq r', \quad -\theta_- \leqq \arg \varepsilon \leqq \theta_+,$$

where the coefficients P_k are n by n matrices whose components are holomorphic functions for $|x| \leqq \delta'$. In particular, $P_0(x)$ is supposed to be nonsingular for $|x| \leqq \delta'$.

Let

$$(7.16) \qquad (x^{-\mu}\varepsilon)^q x^{q'+1} \frac{d\vec{v}}{dx} = F(x, \varepsilon)\vec{v}$$

be the system of equations which is derived from (7.1) by the linear transformation

$$(7.17) \qquad \vec{z} = P(x, \varepsilon)\vec{v}.$$

LEMMA. *If we choose the positive numbers M', δ' and r' in a suitable way, there exists a matrix P satisfying the conditions given above such that the matrix F of the system* (7.16) *has the following form:*

(7. 18)
$$F(x, \varepsilon)=\begin{pmatrix} F_1(x, \varepsilon) & & & 0 \\ & F_2(x, \varepsilon) & & \\ & & \ddots & \\ 0 & & & F_s(x, \varepsilon) \end{pmatrix},$$

where F_j is an n_j by n_j matrix which has an asymptotic expansion

(7. 19)
$$F_j(x, \varepsilon) \simeq \sum_{k=0}^{\infty} (x^{-\mu}\varepsilon)^k F_{jk}(x)$$

with

(7. 20)
$$F_{j0}(0) = \mathring{E}_j.$$

§8. **Application of the fundamental lemma.** Now we shall consider the equation

(6. 3)
$$\varepsilon^{\sigma-\alpha_i} x^{-\beta_i} \frac{d\vec{z}}{dx} = \{B_1(x, \varepsilon) - \beta_i \varepsilon^{\sigma-\alpha_i} x^{-\beta_i-1} C_0\}\vec{z},$$

where B_1 is given by (4. 7) with $\alpha=\alpha_i$ and $\beta=\beta_i$, and C_0 is given by (4. 8).

According to the facts stated in §4, we can express the matrix of the right-hand member of (6. 3) as

(8. 1)
$$B_1(x, \varepsilon) - \beta_i \varepsilon^{\sigma-\alpha_i} x^{-\beta_i-1} C_0 \simeq \sum_{k=0}^{\infty} (x^{-1}\varepsilon^{\rho_{i+1}})^{k/\nu_0} \mathfrak{B}_k(x\varepsilon^{-\rho_i})$$

in the domain

(8. 2)
$$M_{i+1}|\varepsilon|^{\rho_{i+1}} \leqq |x| \leqq \delta_i |\varepsilon|^{\rho_i}, \quad 0<|\varepsilon|\leqq r_0, \quad |\arg \varepsilon| \leqq \theta_0,$$

where ν_0 is a positive integer, and the coefficients $\mathfrak{B}_k(X)$ are holomorphic functions of X^{1/ν_0} for

(8. 3)
$$|X| \leqq \delta_i$$

if we choose ν_0 in a suitable way. Actually, for $i \neq 0$, all of the $\mathfrak{B}_k$ are polynomials of X^{1/ν_0}. In particular, we have

(8. 4)
$$\mathfrak{B}_0(0) = H_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ c_n & c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_2 & 0 \end{pmatrix},$$

where some of the constants c_k are different from zero.

Making a change of the independent variable

(8. 5)
$$x = \varepsilon^{\rho_i} \xi,$$

the equation (6. 3) is reduced to

$$\varepsilon^{\tau_i}\xi^{-\beta_i}\frac{d\vec{z}}{d\xi}=E_i(\xi,\ \varepsilon)\vec{z}, \tag{8. 6}$$

where

$$\tau_i=-\rho_i(1+\beta_i)+(\sigma-\alpha_i) \tag{8. 7}$$

and

$$E_i(\xi,\ \varepsilon)\simeq\sum_{k=0}^{\infty}(\xi^{-1}\varepsilon^{\rho_{i+1}-\rho_i})^{k/\nu_0}\mathfrak{B}_k(\xi) \tag{8. 8}$$

in the domain

$$M_{i+1}|\varepsilon|^{\rho_{i+1}-\rho_i}\leqq|\xi|\leqq\delta_i,\ 0<|\varepsilon|\leqq r_0,\ |\arg\varepsilon|\leqq\theta_0. \tag{8. 9}$$

Since the point R is not on the straight line passing through the points Q_{i-1} and Q_i for $i<m$, we have

$$\tau_i>0 \qquad (i=0,1,\cdots,m-1). \tag{8. 10}$$

Here we put $\rho_0=0$ so that $\tau_0=\sigma>0$.

On the other hand

$$\varepsilon^{\tau_i}\xi^{-\beta_i}=(\xi^{-\mu_i}\varepsilon)^{\tau_i}\xi^{\mu_i\tau_i'+1}, \tag{8. 11}$$

where

$$\begin{cases}\mu_i=1/(\rho_{i+1}-\rho_i)>0,\\ \tau_i'=-\rho_{i+1}(1+\beta_i)+(\sigma-\alpha_i).\end{cases} \tag{8. 12}$$

Since the point R is not on the straight line passing through the points Q_i and Q_{i+1} for $i<m-1$, we have

$$\tau_i'>0 \qquad (i=0,1,\cdots,m-2). \tag{8. 13}$$

However, we have

$$\tau_{m-1}'=0, \tag{8. 14}$$

because $R=Q_m$.

Thus (8. 6) is given the form

$$(\xi^{-\mu_i}\varepsilon)^{\tau_i}\xi^{\tau_i'\mu_i+1}\frac{d\vec{z}}{d\xi}=E_i(\xi,\ \varepsilon)\vec{z}, \tag{8. 15}$$

where we have the asymptotic expansion

$$E_i(\xi,\ \varepsilon)\simeq\sum_{k=0}^{\infty}(\xi^{-\mu_i}\varepsilon)^{k(\rho_{i+1}-\rho_i)/\nu_0}\mathfrak{B}_k(\xi). \tag{8. 16}$$

Finally, put

$$\xi=\eta^p, \qquad \varepsilon=\hat{\varepsilon}^p, \tag{8. 17}$$

where p is a positive integer. Then (8. 15) is given the form

$$(\eta^{-\mu_i}\hat{\varepsilon})^{p\tau_i}\eta^{p\tau_i'\mu_i+1}\frac{d\vec{z}}{d\eta}=pE_i(\eta^p,\hat{\varepsilon}^p)\vec{z}, \tag{8. 18}$$

where

$$E_i(\eta^p,\hat{\varepsilon}^p)\simeq\sum_{k=0}^{\infty}(\eta^{-\mu_i}\hat{\varepsilon})^{kp(\rho_{i+1}-\rho_i)/\nu_0}\mathfrak{B}_k(\eta^p) \tag{8. 19}$$

for

$$M_{i+1}{}^{1/p}|\hat{\varepsilon}|^{1/\mu_i}\leqq|\eta|\leqq\delta_i{}^{1/p},\ 0<|\hat{\varepsilon}|\leqq r_0{}^{1/p},\ |\arg\hat{\varepsilon}|\leqq\theta_0/p. \tag{8. 20}$$

Here we remark that

$$p\tau_i'\mu_i-p\tau_i\mu_i=-p(1+\beta_i)<0. \tag{8. 21}$$

Therefore, if we choose p in a suitable way, the equation (8. 18) satisfies all of the assumptions imposed on the equation (7. 1) in the lemma of § 7.

Since $\mathfrak{B}_0(0)$ has at least two distinct characteristic values, the order of the equation (1. 1) is reduced by the use of the lemma.

§ 9. **Sketch of the proof.** We shall give here only a sketch of the proof of our fundamental lemma. The details of the proof will be given in a forthcoming paper by M. Iwano for more general cases. The proof is quite similar to that of a theorem of Sibuya [3] concerned with the perturbation of irregular singular points of linear ordinary differential equations.

A lemma due to Sibuya [3] says that there exists a nonsingular matrix $P_0(x)$ whose components are holomorphic functions of x for $|x|\leqq\delta'$, δ' being a sufficiently small positive number, such that

$$P_0(x)^{-1}E_0(x)P_0(x)=\begin{pmatrix} E^{(1)}(x) & & & 0 \\ & E^{(2)}(x) & & \\ & & \ddots & \\ 0 & & & E^{(s)}(x)\end{pmatrix}, \tag{9. 1}$$

where $E^{(j)}(x)$ is an n_j by n_j matrix and

$$E^{(j)}(0)=\mathring{E}_j \qquad (j=1, 2, \cdots, s). \tag{9. 2}$$

Let us denote the matrix (9. 1) by $\hat{E}_0(x)$. Making a change of variables

$$\vec{z}=P_0(x)\vec{u}, \tag{9. 3}$$

we have

$$(x^{-\mu}\varepsilon)^q x^{q'+1}\frac{d\vec{u}}{dx}=\hat{E}(x,\varepsilon)\vec{u}, \tag{9. 4}$$

where

$$\hat{E}(x, \varepsilon)=P_0(x)^{-1}\left\{E(x, \varepsilon)P_0(x)-(x^{-\mu}\varepsilon)^q x^{q'+1}\frac{dP_0(x)}{dx}\right\}.$$

Therefore $\hat{E}$ admits an asymptotic expansion in powers of $(x^{-\mu}\varepsilon)$

$$\hat{E}(x, \varepsilon)\simeq \hat{E}_0(x)+\sum_{k=1}^{\infty}(x^{-\mu}\varepsilon)^k\hat{E}_k(x), \tag{9.5}$$

$\hat{E}_k(x)$ being n by n matrices whose components are holomorphic for $|x|\leqq\delta'$.

Let us consider now a linear transformation

$$\vec{u}=Q(x, \varepsilon)\vec{v} \tag{9.6}$$

and let (7.16) be the equation on $\vec{v}$. Then

$$(x^{-\mu}\varepsilon)^q x^{q'+1}\frac{dQ}{dx}=\hat{E}(x, \varepsilon)Q-QF(x, \varepsilon). \tag{9.7}$$

We shall determine Q in such a way that the matrix

$$P(x, \varepsilon) = P_0(x)\, Q(x, \varepsilon) \tag{9.8}$$

and the matrix $F(x, \varepsilon)$ of (7.16) satisfy all of the conditions stated in our fundamental lemma.

To do this, we put

$$\begin{cases} \hat{E}(x, \varepsilon)=\hat{E}_0(x)+H(x, \varepsilon), \\ F(x, \varepsilon)=\hat{E}_0(x)+K(x, \varepsilon), \\ Q(x, \varepsilon)=\mathbf{1}_n+T(x, \varepsilon), \end{cases} \tag{9.9}$$

where $\mathbf{1}_n$ is the n by n unit-matrix. Then the equation (9.7) can be written as

$$(x^{-\mu}\varepsilon)^q x^{q'+1}\frac{dT}{dx}=\hat{E}_0T-T\hat{E}_0+HT-TK+H-K. \tag{9.10}$$

Put

$$T=\begin{pmatrix} T_{11} & \cdots & T_{1s} \\ \cdots & \cdots & \cdots \\ T_{s1} & \cdots & T_{ss} \end{pmatrix},\quad H=\begin{pmatrix} H_{11} & \cdots & H_{1s} \\ \cdots & \cdots & \cdots \\ H_{s1} & \cdots & H_{ss} \end{pmatrix},\quad K=\begin{pmatrix} K_{11} & \cdots & K_{1s} \\ \cdots & \cdots & \cdots \\ K_{s1} & \cdots & K_{ss} \end{pmatrix}, \tag{9.11}$$

where T_{jk}, H_{jk}, and K_{jk} are n_j by n_k matrices. Then (9.10) can be given the form

$$\begin{aligned}(x^{-\mu}\varepsilon)^q x^{q'+1}\frac{dT_{jk}}{dx}&=E^{(j)}T_{jk}-T_{jk}E^{(k)} \\ &\quad+\sum_{h=1}^{s}(H_{jh}T_{hk}-T_{jh}K_{hk})+H_{jk}-K_{jk}.\end{aligned} \tag{9.12}$$

Now let us put

(9.13) $$\begin{cases} K_{jk}(x, \varepsilon) \equiv 0 & (j \neq k), \\ T_{jk}(x, \varepsilon) \equiv 0 & (j = k). \end{cases}$$

Firstly from the equations (9.12) for $j=k$ we can derive

$$0=\sum_{h \neq j} H_{jh} T_{hj}+H_{jj}-K_{jj} \qquad (j=1, 2, \cdots, s)$$

or

(9.14) $$K_{jj} = \sum_{h \neq j} H_{jh} T_{hj} + H_{jj} \qquad (j=1, 2, \cdots, s).$$

Secondly from (9.12) with $j \neq k$ and (9.14) we can derive

(9.15) $$\begin{aligned}(x^{-\mu}\varepsilon)^q x^{q'+1} \frac{dT_{jk}}{dx} &= E^{(j)} T_{jk}-T_{jk}E^{(k)} \\ &+ \sum_{h \neq k} H_{jh} T_{hk}-T_{jk}\{\sum_{h \neq k} H_{kh} T_{hk}+H_{kk}\}+H_{jk} \quad (j \neq k).\end{aligned}$$

We shall determine T_{jk} $(j \neq k)$ by (9.15) in such a way that they admit uniformly asymptotic expansions in powers of $(x^{-\mu}\varepsilon)$

(9.16) $$T_{jk}(x, \varepsilon) \simeq \sum_{l=1}^{\infty} (x^{-\mu}\varepsilon)^l T_{jkl}(x)$$

in the domain (7.12), where the coefficients T_{jkl} are n_j by n_k matrices whose components are holomorphic for $|x| \leqq \delta'$. K_{jj} will then be determined by (9.14). This will complete the proof of our fundamental lemma.

§ 10. Problem of nonlinear differential equations. Now we know that, in order to prove our fundamental lemma, it is sufficient to find a suitable solution of the equations (9.15). The following facts should be remarked:

(i) $H_{jk}(x, \varepsilon) = O(|x^{-\mu}\varepsilon|)$;

(ii) $E^{(j)}(0) = \overset{\circ}{E}_j \qquad (j = 1, 2, \cdots, s)$;

(iii) $\lambda_j - \lambda_k \neq 0$;

(iv) *the equations* (9.15) *are nonlinear.*

Therefore the problem of solving the equations (9.15) is reduced to the following problem of nonlinear ordinary differential equations:

We consider a system of nonlinear ordinary differential equations

(10.1) $$(x^{-\mu}\varepsilon)^q x^{q'+1} \frac{d\eta_j}{dx} = \kappa_j \eta_j+\nu_{j+1} \eta_{j+1}+f_j(x, \varepsilon, \eta_1, \cdots, \eta_m) \qquad (j=1, 2, \cdots, m),$$

where we suppose that

1) *μ is a positive rational number;*

2) *q is a positive integer and q' is a nonnegative integer;*

3) $q' - \mu q < 0$;

4) κ_j *is a constant different from zero, and* ν_j *is equal to* 1 *or* 0;

5) f_j *are holomorphic functions of* (x, ε, η) *for*

$$M|\varepsilon|^{1/\mu} \leqq |x| \leqq \delta,\ 0<|\varepsilon| \leqq r_0,\ |\arg \varepsilon| \leqq \theta_0,\ |\eta_k| \leqq r_1 \quad (k=1, 2, \cdots, m)$$

and admitting uniformly convergent expansions in powers of η:

$$f_j(x, \varepsilon, \eta) = \mathring{f}_j(x, \varepsilon) + \sum_{k=1}^{m} f_{jk}(x, \varepsilon)\eta_k + \sum_{k_1+\cdots+k_m \geqq 2} f_{jk_1\cdots k_m}(x, \varepsilon)\eta_1^{k_1} \cdots \eta_m^{k_m};$$

6) *the coefficients* $\mathring{f}_j, f_{jk}$ *and* $f_{jk_1\cdots k_m}$ *are holomorphic for*

$$M|\varepsilon|^{1/\mu} \leqq |x| \leqq \delta,\ 0<|\varepsilon| \leqq r_0,\ |\arg \varepsilon| \leqq \theta_0 \tag{10. 2}$$

and expressible in uniformly asymptotic series in powers of $(x^{-\mu}\varepsilon)$ *as* ε *tends to* 0 *in the domain*

$$0<|\varepsilon| \leqq r_0, \qquad |\arg \varepsilon| \leqq \theta_0;$$

7) *the following inequalities*

$$|f_{jk}(x, \varepsilon)| \leqq L(|x| + |x^{-\mu}\varepsilon|),$$
$$|\mathring{f}_j(x, \varepsilon)| \leqq K|x^{-\mu}\varepsilon|$$

hold for (10. 2), *where* L *and* K *are positive constants independent of* (x, ε).

It should be remarked that the equations (9. 15) satisfy all of these assumptions. For example, among the four facts (i), (ii), (iii) and (iv) which were mentioned above, the third corresponds to the assumption 4), while the first and the second correspond to the assumption 7).

Let

$$-\Theta_- \leqq \arg x \leqq \Theta_+, \quad -\theta_- \leqq \arg \varepsilon \leqq \theta_+$$

be a domain proper with respect to the functions

$$(\mu q - q')\int^x \frac{\kappa_j}{(x^{-\mu}\varepsilon)^q x^{q'+1}} dx = \kappa_j \varepsilon^{-q} x^{\mu q - q'} \qquad (j=1, 2, \cdots, m).$$

Then we can prove the following:

The equations (10. 1) *admit a solution* $\eta_j = p_j(x, \varepsilon)$ *holomorphic for*

$$\begin{cases} M'|\varepsilon|^{1/\mu} \leqq |x| \leqq \delta', & -\Theta_- \leqq \arg x \leqq \Theta_+, \\ 0 < |\varepsilon| \leqq r_0', & -\theta_- \leqq \arg \varepsilon \leqq \theta_+ \end{cases}$$

and expressible in uniformly asymptotic series in powers of $(x^{-\mu}\varepsilon)$:

$$p_j(x, \varepsilon) \simeq \sum_{k=1}^{\infty} (x^{-\mu}\varepsilon)^k p_{jk}(x)$$

as ε *tends to* 0 *in the domain*

$$0<|\varepsilon|\leqq r_0', \qquad -\theta_-\leqq \arg\varepsilon\leqq\theta_+,$$

where p_{jk} *are holomorphic for* $|x|\leqq\delta'$, *and* M', δ' *and* r_0' *are suitably chosen positive constants.*

This will be proved by M. Iwano in his succeeding paper. Relying on this result, we can complete the proof of our fundamental lemma.

III. Behavior of solutions in intermediate domains.

§ 11. Transformation of the equation (1. 1). In this chapter, we shall study the behavior of solutions of the equation (1. 1) in a domain between any two of the domains (6. 1). Such a domain is given by

$$\delta_i|\varepsilon|^{\rho_i}\leqq|x|\leqq M_i|\varepsilon|^{\rho_i} \qquad (i=1, 2, \cdots, m-1)$$

or

$$|x|\leqq M_m|\varepsilon|^{\rho_m}.$$

Let $(\gamma_i, 0)$ be the point at which the straight line passing through the points Q_{i-1} and Q_i intersects the X-axis (See Fig. 9).

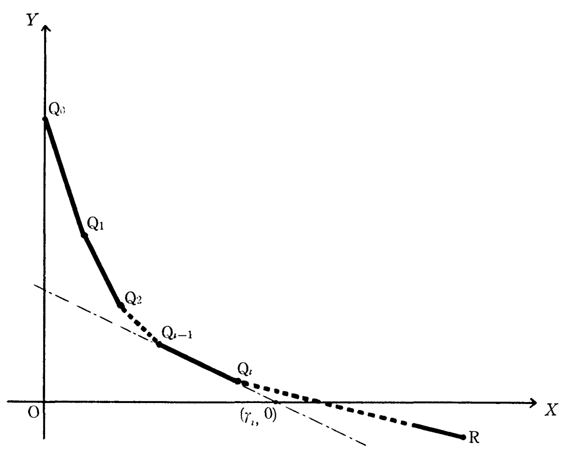

Fig. 9

Then we consider a transformation

(11. 1) $$x=\varepsilon^{\rho_i}\xi, \qquad \vec{y}=\Lambda(\gamma_i, 0)\vec{z}.$$

If we write the equation on $\vec{z}$ as

(11. 2) $$\varepsilon^{\sigma-\rho_i-\gamma_i}\frac{d\vec{z}}{d\xi}=B(\xi, \varepsilon)\vec{z},$$

the matrix B is given by (5. 3) with $\rho=\rho_i$ and $\gamma=\gamma_i$. In order to investigate the behavior of solutions of the equation (4. 2) in the domain

(11. 3) $$\delta_i|\varepsilon|^{\rho_i} \leqq |x| \leqq M_i|\varepsilon|^{\rho_i},$$

it is sufficient to study the behavior of solutions of the equation (11. 2) in the domain

(11. 4) $$\delta_i \leqq |\xi| \leqq M_i.$$

Let us put

(11. 5) $$\sigma_i = \sigma - \rho_i - \gamma_i.$$

Then we have

(11. 6) $$\sigma_i > 0 \qquad (i = 1, 2, \cdots, m-1)$$

and

(11. 7) $$\sigma_m = 0,$$

because the point $\mathrm{R}=\mathrm{Q}_m$ is not on the straight line passing through the points Q_{i-1} and Q_i for $i<m$, while it lies on this line for $i=m$ (See Figs. 10 and 11).

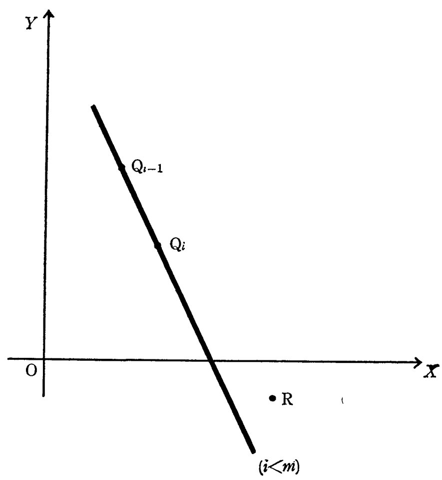

Fig. 10

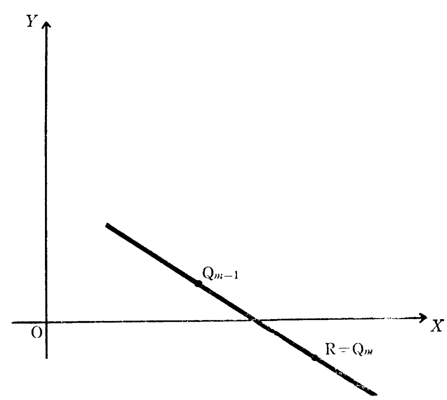

Fig. 11

Therefore, for $i=m$, we shall consider the corresponding equation (11. 2) in the domain

(11. 8) $$|\xi| \leqq M_m.$$

The domain (11. 8) corresponds to the domain

(11. 9) $$|x| \leqq M_m|\varepsilon|^{\rho_m}.$$

REMARK. In certain cases, it might be more reasonable to study the behavior of the equation (11. 2) in the larger domain

$$|\xi| \leqq M_\iota. \tag{11. 10}$$

At the present moment, we shall not go into these details.

§12. Study of behavior of solutions. Since none of the points R and $P_{k\nu}$ lies below the straight line passing through the points Q_{i-1} and Q_i (See Fig. 12), the matrix $B(\xi, \varepsilon)$ does not contain any negative powers of ε.

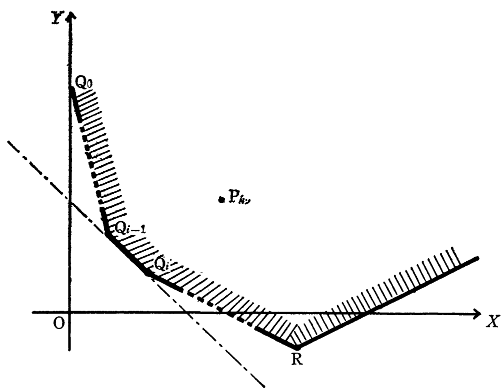

Fig. 12

Now the equation (11. 2) with $i=m$ is of the form

$$\frac{d\vec{z}}{d\xi}=B(\xi, \varepsilon)\vec{z}. \tag{12. 1}$$

It is easy to study the behavior of solutions of (12. 1) in the domain (11. 8). Therefore, we shall consider the equation (11. 2) only for $i<m$.

The equation (11. 2) can be written as

$$\varepsilon^{\sigma_i}\frac{d\vec{z}}{d\xi}=B(\xi, \varepsilon)\vec{z}, \tag{12. 2}$$

where σ_ι is given by (11. 5). It should be remarked that

$$0<\sigma_\iota<\sigma \qquad (i=1, 2, \cdots, m-1). \tag{12. 3}$$

We can write the matrix B as

$$B(\xi, \varepsilon) = K_0(\xi) + [\cdots], \tag{12. 4}$$

where $[\cdots]$ indicates the terms which are small in the domain

$$|\xi| \leqq M_\iota, \quad 0<|\varepsilon| \leqq r_0, \quad |\arg \varepsilon| \leqq \theta_0, \tag{12. 5}$$

and

$$K_0(\xi) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_n(\xi) & c_{n-1}(\xi) & c_{n-2}(\xi) & \cdots & c_2(\xi) & 0 \end{pmatrix}. \tag{12. 6}$$

The functions $c_k(\xi)$ are polynomials of ξ, and at least one of them is not identically equal to zero. In particular, if $c_j \not\equiv 0$, then at least one of the points $P_{j\nu}$ lies on the segment joining Q_{i-1} to Q_i, and c_j is of the following form:

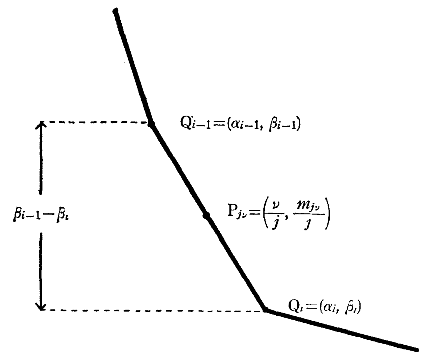

Fig. 13

(12. 7) $c_j(\xi)=\xi^{\omega_j}\{c_{j0}+c_{j1}\xi+\cdots+c_{j\kappa_j}\xi^{\kappa_j}\}$,

where c_{jk} are constants and

(12. 8) $c_{j0}\neq 0, \quad c_{j\kappa_j}\neq 0,$

(12. 9) $0\leqq \kappa_j \leqq j(\beta_{i-1}-\beta_i), \ \omega_j \geqq j\beta_i$

(See Fig. 13).

Let us consider the algebraic equation

$$\lambda^n - \{c_2(\xi)\lambda^{n-2}+c_3(\xi)\lambda^{n-3} + \cdots + c_n(\xi)\} = 0. \tag{12. 10}$$

This equation may have multiple roots somewhere in the domain (11. 4), but the number of those points are at most finite. They are possibly turning points of the equation (12. 2).

The domain (11. 4) can be covered by a finite number of open sets, each containing at most a single point where all of the roots of (12. 10) are equal to zero, such that, if there is not such a point in one of them, at least two roots of (12. 10) do not coincide with each other all over this open set. In the latter case, the order of the equation (1. 1) is reduced in the open set by the use of a theorem due to Sibuya [2].

Now assume that all roots of (12. 10) are equal to zero at a point $\xi=\xi_0$ in one of the open sets. Then, making a change of the independent variable

$$\xi = x + \xi_0, \tag{12. 11}$$

we go back to the same situation as what we were in at the start of our discussions. We construct again a characteristic polygon as in § 2. This polygon has the following properties:

(i) *σ is less than that of the original equation* (1. 1);

(ii) *the point Q_0 does not lie above the corresponding point of the original equation* (1. 1).

Therefore, after repeating a finite number of transformations similar to those given above, we shall be led to the situation with $\sigma=0$.

IV. Examples.

§ 13. **Second order equations.** Consider a second order linear ordinary differential equation of the form

$$\varepsilon^2 \frac{d^2y}{dx^2} = \{x^q + \varepsilon\phi(x, \varepsilon)\} y, \tag{13. 1}$$

where q is a positive integer, and ϕ is a holomorphic function of (x, ε) in a domain

$$|x| \leqq \delta_0, \quad 0 < |\varepsilon| \leqq r_0, \quad |\arg \varepsilon| \leqq \theta_0. \tag{13. 2}$$

$x=0$ is possibly a turning point. Assume that ϕ has the following form:

$$\phi(x, \varepsilon) = \phi_0(x) + O(|\varepsilon|), \tag{13. 3}$$

where ϕ_0 is a holomorphic function of x for

$$|x| \leqq \delta_0. \tag{13. 4}$$

Let m_0 be the order of zero of ϕ_0 at $x=0$. If $\phi_0(0) \neq 0$ we put $m_0=0$, and if $\phi_0(x) \equiv 0$ we put $m_0 = +\infty$.

Let us construct a characteristic polygon as in § 2. Put

$$\begin{cases} \mathrm{R} = (1, -1), \\ \mathrm{Q}_0 \equiv \mathrm{P}_{20} = \left(0, \dfrac{q}{2}\right), \quad \text{(See (2. 4)).} \\ \mathrm{P}_0 \equiv \mathrm{P}_{21} = \left(\dfrac{1}{2}, \dfrac{m_0}{2}\right) \end{cases} \tag{13. 5}$$

Then, it is easily seen that, if the point P_0 lies below the straight line passing through the points Q_0 and R, the point P_0 is a vertex of the polygon. If it does

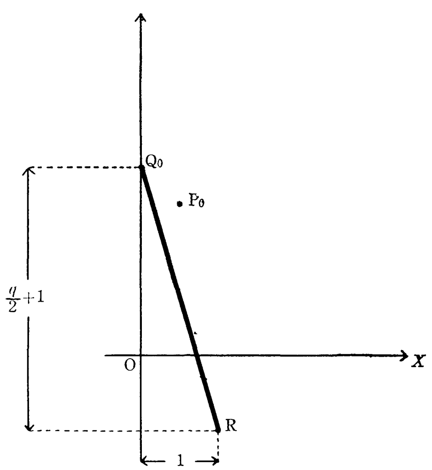

Fig. 14

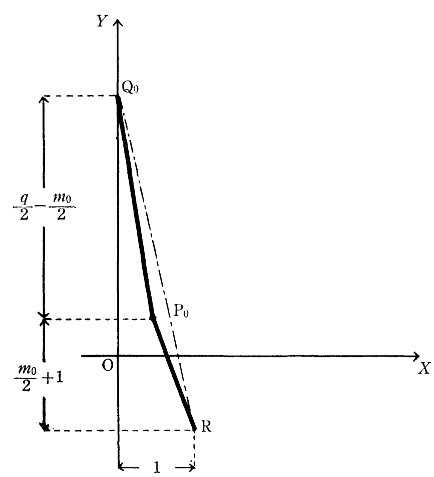

Fig. 15

not lie below the straight line, then there is no vertex between Q_0 and R (See Figs. 14 and 15).

In case when there is no vertex between Q_0 and R, the order of the equation (13. 1) can be reduced in the domain

$$M|\varepsilon|^{\rho} \leqq |x| \leqq \delta_0, \tag{13. 6}$$

where M is a positive constant, and

$$\rho = \frac{2}{q+2}. \tag{13. 7}$$

On the other hand, to investigate the solution in the domain $|x| \leqq M|\varepsilon|^{\rho}$, we put

$$x = \varepsilon^{\rho}\xi. \tag{13. 8}$$

Then the equation (13. 1) can be written as

$$\frac{d^2y}{d\xi^2} = \{\xi^q + \varepsilon^{1-\rho q}\phi(\varepsilon^{\rho}\xi, \varepsilon)\}y, \tag{13. 9}$$

and the function

$$\varepsilon^{1-\rho q}\phi(\varepsilon^{\rho}\xi, \varepsilon) \tag{13. 10}$$

is bounded in the domain

$$|\xi| \leqq M,\ 0<|\varepsilon| \leqq r_0,\ |\arg\varepsilon| \leqq \theta_0. \tag{13. 11}$$

Here it is to be remarked that $\rho q = 2-2\rho = 2q/(q+2)$. Thus the expressions of solutions of (13. 1) can be obtained in each of the domains

$$M|\varepsilon|^{\rho} \leqq |x| \leqq \delta_0 \quad \text{and} \quad |x| \leqq M|\varepsilon|^{\rho} \tag{13. 12}$$

(See Fig. 16).

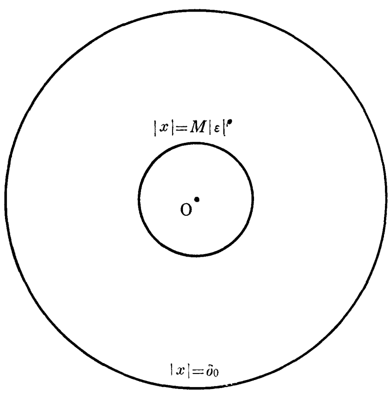

Fig. 16

The expressions of solutions of (13. 1) in the domain (13. 6) correspond to the classical asymptotic expansions of solutions of (13. 1).

In case when the point P_0 is a vertex between Q_0 and R, we must consider the following domains:

$$\begin{cases} |x| \leqq M_2|\varepsilon|^{\rho_2}, \\ M_2|\varepsilon|^{\rho_2} \leqq |x| \leqq \delta_1|\varepsilon|^{\rho_1}, \\ \delta_1|\varepsilon|^{\rho_1} \leqq |x| \leqq M_1|\varepsilon|^{\rho_1}, \\ M_1|\varepsilon|^{\rho_1} \leqq |x| \leqq \delta_0, \end{cases} \tag{13. 13}$$

where

$$\rho_1 = \frac{1}{q-m_0}, \quad \rho_2 = \frac{1}{m_0+2} \quad \text{(See Figs. 15 and 17).} \tag{13. 14}$$

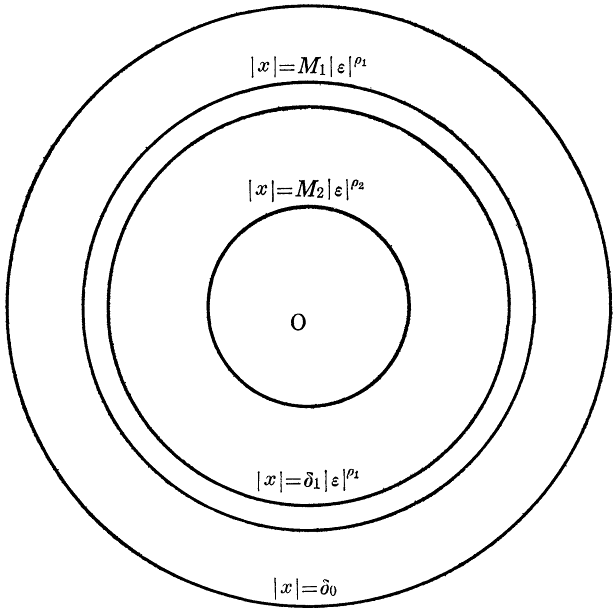

Fig. 17

In each of the domains

(13. 15) $$M_2|\varepsilon|^{\rho_2} \leqq |x| \leqq \delta_1|\varepsilon|^{\rho_1} \text{ and } M_1|\varepsilon|^{\rho_1} \leqq |x| \leqq \delta_0,$$

the order of the equation (13. 1) is reduced. In the domain

(13. 16) $$|x| \leqq M_2|\varepsilon|^{\rho_2},$$

if we put

(13. 17) $$x = \varepsilon^{\rho_2}\xi,$$

then the equation (13. 1) can be written as

(13. 18) $$\frac{d^2y}{d\xi^2} = \{c\xi^{m_0} + [\cdots]\}y,$$

where c is a constant different from zero, and $[\cdots]$ indicates the terms which are small in the domain

(13. 19) $$|\xi| \leqq M_2, \quad 0<|\varepsilon| \leqq r_0, \quad |\arg\varepsilon| \leqq \theta_0.$$

On the other hand, in the domain

(13. 20) $$\delta_1|\varepsilon|^{\rho_1} \leqq |x| \leqq M_1|\varepsilon|^{\rho_1},$$

if we put

(13. 21) $$x = \varepsilon^{\rho_1}\xi,$$

the equation (13. 1) can be written as

(13. 22) $$\varepsilon^{\sigma'}\frac{d^2y}{d\xi^2}=\{\xi^q+c\xi^{m_0}+[\cdots]\}y,$$

where

(13. 23) $$\sigma'=1-\frac{\rho_1}{\rho_2}>0,$$

and $[\cdots]$ indicates the terms which are small in the domain

(13. 24) $$|\xi|\leqq M_1,\quad 0<|\varepsilon|\leqq r_0,\quad |\arg\varepsilon|\leqq\theta_0.$$

The points $\xi=0$ and $\xi=(-c)^{\rho_1}$ are possibly turning points of the equation (13. 22).

In case when there is no vertex between Q_0 and R, the equation (13. 1) can be simplified by a linear transformation with coefficients holomorphic in a sector whose vertex is the origin and whose radius is independent of ε. Both domains (13. 12) can be patched together by the use of this sector. Such results were obtained by several authors. See, for example, Sibuya [4].

In case when P_0 is a vertex between Q_0 and R, such results have not been obtained yet, although the equation (13. 1) can be formally simplified by a linear transformation whose coefficients are formal power series of x and ε; [4].

In order that the point P_0 is not a vertex, it is necessary and sufficient to have the inequality

(13. 25) $$m_0\geqq\frac{q-2}{2}$$

satisfied. This inequality means that the point P_0 does not lie below the straight line passing through the points Q_0 and R. This is always satisfied if $q=1$ or 2 (See Fig. 18).

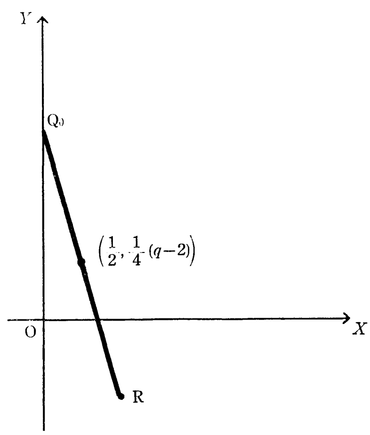

Fig. 18

REMARK. If P_0 is a vertex on the X-axis, then we have $m_0=0$. Therefore, $\xi=0$ is not a turning point of (13. 22). Thus we can reduce the order of the equation (13. 22) in the domain

(13. 26) $$|\xi|\leqq\delta_1.$$

This corresponds to the domain

(13. 27) $$|x|\leqq\delta_1|\varepsilon|^{\rho_1}.$$

Hence we can reduce the order of the equation (13. 1) in each of the domain

(13. 28) $$|x|\leqq\delta_1|\varepsilon|^{\rho_1}\quad\text{and}\quad M_1|\varepsilon|^{\rho_1}\leqq|x|\leqq\delta_0,$$

instead of the domains (13. 15).

§ 14. n^{th} order equations. Consider now an n^{th} order equation of the form

$$\varepsilon^{n-m}y^{(n)}=(x+\varepsilon\phi(x,\varepsilon))y^{(m)}+\sum_{j=1}^{m}g_j(x,\varepsilon)y^{(m-j)}, \tag{14. 1}$$

where ϕ, $g_1, \cdots, g_m$ are holomorphic functions of (x, ε) in a domain

$$|x|\leqq\delta_0, \qquad 0<|\varepsilon|\leqq r_0, \qquad |\arg\varepsilon|\leqq\theta_0. \tag{14. 2}$$

Assume that

$$n-2\geqq m\geqq 0. \tag{14. 3}$$

Let us construct a characteristic polygon as in § 2. Put

$$\mathrm{R}=(1,\,-1),\ \mathrm{Q}_0\equiv\mathrm{P}_{p0}=\left(0,\,\frac{1}{p}\right) \qquad \text{(See (2. 4))}, \tag{14. 4}$$

where

$$p=n-m. \tag{14. 5}$$

Then it is easily seen that there is no vertex between Q_0 and R. However, the point

$$\mathrm{P}_0=\left(\frac{1}{p+1},\ 0\right) \tag{14. 6}$$

is on the straight line passing through the points Q_0 and R (See Fig. 19), and it coincides with the point P_{p+11} if and only if $g_1(0, 0)\neq 0$.

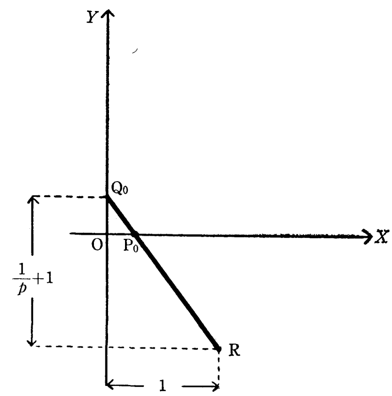

Fig. 19

Therefore, the order of the equation (14. 1) is reduced in the domain

$$M|\varepsilon|^{\rho}\leqq|x|\leqq\delta_0, \tag{14. 7}$$

where

$$\rho=\frac{p}{p+1}. \tag{14. 8}$$

On the other hand, if we put

$$x=\varepsilon^{\rho}\xi, \tag{14. 9}$$

the equation (14. 1) can be written as

$$\frac{d^ny}{d\xi^n}=(\xi+E_0)\frac{d^my}{d\xi^m}+(c+E_1)\frac{d^{m-1}y}{d\xi^{m-1}}+\sum_{j=2}^{m}E_j\frac{d^{m-j}y}{d\xi^{m-j}}, \tag{14. 10}$$

where $E_0, \cdots, E_m$ are quantities small in the domain

$$|\xi|\leqq M, \qquad 0<|\varepsilon|\leqq r_0, \qquad |\arg\varepsilon|\leqq\theta_0, \tag{14. 11}$$

and c is a constant.

Roughly speaking, in the domain (14. 7), the equation (14. 1) is reduced to a system of $(p+1)$ equations. One of the equations is of the m^{th} order, while the others are of the first order. Thus we can construct expressions of solutions of (14. 1) in the domain (14. 7). These expressions correspond to the classical asymptotic expansions of solutions of (14. 1). On the other hand, by the use of (14. 10), we can construct expressions of solutions in the domain

$$|x| \leqq M|\varepsilon|^{\rho}. \tag{14. 12}$$

In order to find the relations between the expressions of solutions in (14. 7) and those in (14. 12), we may try to simplify the equation (14. 1) in a sector whose vertex is the origin and whose radius is independent of ε. Actually, in the domain (14. 2), the equation (14. 1) can be formally simplified by a linear transformation whose coefficients are formal power series of ε with coefficients holomorphic in x. Simplification of the equation (14. 1) by a transformation with analytic coefficients in such a sector as mentioned above was also proved for the cases when $m=0$ and $m=1$. See, for example, Okubo [1] and Sibuya [4, 5].

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