REMARK ON GALOIS THEORY OF SIMPLE RING

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1. Bortfeld [1] proved the following: Let D be a division ring which has (left) finite dimensionality over a division subring L of D and we suppose that the center C of D is an infinite field. If L is the invariant ring of an automorphism ω of D, then, for each intermediate division subring T between L and A, there exists an automorphism ρ such that T is an invariant ring of ρ.

Recently, Nagahara and Tominaga [2] extended this therorem to simple rings, in case the commutator of an invariant ring is a division ring.

In this note, we shall prove that Nagahara and Tominaga's result is still valid in the case where the commutator is a simple ring.

2. By a simple ring, we shall mean a two sided simple ring with a unit element satisfying minimum condition for left ideals, and we suppose that its center is an infinite field. Let A be a simple ring which has finite dimensionality over a subring S of A. If S is the invariant ring of a group $\mathfrak{G}$ of automorphisms of A and $V_A(S)$ is simple ring then

$$(A:S) \geqq (\mathfrak{G}:\mathfrak{J})(V_A(S):C),$$

where $\mathfrak{J}$ is a subgroup which consists of all the inner automorphisms of $\mathfrak{G}$ and C is the center of A.

LEMMA. *Let B be a finite-dimensional simple algabra over center K and F be a subfield contained in K. If F is the invariant field of an automorphism ω of B, and $(K:F)<\infty$ then B is commutative.*

Proof. Since $(K:F)=m<\infty$, K is a finite Galois extension of F. Let $\bar{K}$ be a field isomorphic with K. Then

$$B \underset{F}{\times} \bar{K} \simeq e_1\bar{B} \oplus e_2\bar{B} \oplus \cdots \oplus e_m\bar{B}$$

where $e_1, \cdots, e_m$ are mutually orthogonal idempotent elements and $\bar{B} \simeq B$. If $u_1, \cdots, u_m$ be basis of $\bar{K}$ over F, then $B \times_F K \ni a = \sum_{i=1}^m a_i u_i$, where $a_i \in B$. And we can extend ω to the automorphism $\bar{\omega}$ of $B \times_F K$ in the following way:

$$a^{\bar{\omega}} = \sum_{i=1}^m a_i{}^{\omega} u_i.$$

Then, the invariant ring of $\bar{\omega}$ is $\bar{K}$ in $B \times_F K$. Clearly, $\bar{\omega}(e_i) = e_j$; that is, $\bar{\omega}$ induces the permutation of $(e_1, \cdots, e_m)$. This permutation is cyclic. Indeed, if it has a cyclic component $(e_{i_1}, e_{i_2}, \cdots, e_{i_r})$, where $r<m$, then.

$$\bar{\omega}(xe_{i_1} + \cdots + xe_{i_r}) = xe_{i_1} + \cdots + xe_{i_r},$$

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for $x \in \bar{K}$. Hence we may suppose that $\bar{\omega}(e_i) = e_{i+1}$, where $1 \leqq i \leqq m-1$ and $\bar{\omega}(e_m) = e_1$. If, for $y \in \bar{B}$, $\bar{\omega}(e_i y) = e_{i+1} y'$ then, by $y \to y'$, an automorphism of $\bar{B}$ is given and this automorphism leaves $\bar{K}$ fixed elementwise, so it is an inner automorphism σ_{λ_i}, where $\lambda_i \in \bar{B}$ and $1 \leqq i \leqq m$. Let us take an element z of B such that $\sigma_{\lambda_m} \cdots \sigma_{\lambda_2} \sigma_{\lambda_1}(z) = z$ and let

$$t = e_1 z + e_2 \sigma_{\lambda_1}(z) + e_3 \sigma_{\lambda_2} \sigma_{\lambda_1}(z) + \cdots + e_m \sigma_{\lambda_{m-1}} \sigma_{\lambda_{m-2}} \cdots \sigma_{\lambda_1}(z).$$

Then, clearly, $\bar{\omega}(t) = t$, so $z \in \bar{K}$. But, if B is non-commutative, the element z such as above is taken outside of $\bar{K}$, so $\bar{B}$ is commutative. Thus B is commutative.

THEOREM. *Let A be a simple ring which has finite dimensionality over a simple subring S, and let $V_A(S)$ be a simple ring. If S is the invariant ring of an automorphism ω of A then, for each intermediate ring T between A and S, there exists an automorphism ρ such that T is an invariant of ρ.*

Proof. Clearly, $V_A(S)$ is ω-normal, and, in $V_A(S)$, ω-invariant ring is $S \frown V_A(S)$. Since $S \frown V_A(S)$ is contained in the center of $V_A(S)$, $V_A(S)$ is a field. Hence, from the result of Nagahara and Tominaga, the above fact holds good.

REFERENCES

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