SOME THEOREMS IN AN EXTENDED RENEWAL THEORY, III

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1. Let X_k, $k=1, 2, \cdots$, be a sequence of independent, non-negative, identically distributed random variables and set

$$H(t) = E\{N(t)\} = \sum_{n=1}^{\infty} Pr(S_n < t), \tag{1.1}$$

where $S_n = \sum_{k=1}^{n} X_k$, $n=1, 2, \cdots$, and $N(t)$ is the number of sums $S_1, S_2, \cdots$ which are less than t. Then it holds under some restrictions that

$$\lim_{t\to\infty} [H(t+h) - H(t)] = \lim_{t\to\infty} \sum_{n=1}^{\infty} Pr(t < S_n \leqq t+h) = \frac{h}{m}, \tag{1.2}$$

where $m = E(X_k) > 0$ and $h > 0$ is a constant. This is known as renewal theorem and discussed by many authors. Extending (1.2) Smith [6], [7], [8] has shown for instance that if (i) $\psi(t)$ is bounded for $t \geqq 0$, (ii) $\psi \in L(0, \infty)$, (iii) $\lim_{t\to\infty} \psi(t) = 0$ and (iv) for some n the n-th iterated convolution of $F(x)$ with itself has an absolutely continuous part, where $F(x)$ is the distribution function of X_k, then it holds that

$$\lim_{t\to\infty} \int_0^t \psi(t-u)\, dH(u) = \frac{1}{m} \int_0^{\infty} \psi(t)\, dt. \tag{1.3}$$

If X_k, $k=1, 2, \cdots$, do not necessarily have the same distribution, (1.2) does not necessarily hold. But assuming that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} E(X_k) = m \tag{1.4}$$

exists, we can see under some additional restrictions that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n=1}^{\infty} Pr(t < S_n \leqq t+h)\, dt = \frac{h}{m}. \tag{1.5}$$

This was proved by Kawata [4] and extended by the author [1], [2], [3]. Recently Kawata [5] has obtained for X_k, $k=1, 2, \cdots$, satisfying (1.4) as the result coresponding to (1.3) that if (i) a_k, $k=1, 2, \cdots$, is a sequence of non-negative real numbers satisfying the restrictions that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k = a \tag{1.6}$$

exists and is positive, and (ii) $\psi(x)$ is a non-negative function of bounded variation over every finite interval and is bounded and integrable over $(0, \infty)$, then it holds that

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(1.7) $$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\int_0^t \psi(t-u)\,dJ(u)=\frac{a}{m}\int_0^\infty \psi(t)\,dt$$

under the restrictions on X_k, $k=1,2,\cdots$, used in [3], where

$$J(t)=\sum_{n=1}^{\infty} a_n Pr(S_n\leqq t).$$

The purpose of the present paper is to obtain a more extended result of (1.7). In the following discussion we need only the integrability over $(0,\infty)$ as the restiction on ψ and can show that

(1.8) $$\lim_{T\to\infty}\frac{1}{T^{\alpha+1}}\int_0^T dt\int_0^t \psi(t-u)\,dK(u)=\frac{a}{(\alpha+1)m^{\alpha+1}}\int_0^\infty \psi(t)\,dt,$$

where

$$K(t)=\sum_{n=1}^{\infty} a_n n^\alpha Pr(S_n\leqq t)$$

and α is any non-negative integer. Actually we have considered the case where X_k may take the negative values but for the sake of simplicity we shall restrict ourselves here with non-negative X_k.

2. To show our statement, we need the known following lemma which is found as Theorem 1 in [3].

LEMMA. *Let X_k, $k=1,2,\cdots$, be non-negative independent random variables having finite mean values m_k, $k=1,2,\cdots$, and*

(2.1) $$\lim_{A\to\infty}\int_A^\infty x\,dF_n(x)=0$$

holds uniformly regarding n, $F_n(x)$ being the distribution function of X_n. If

(2.2) $$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} m_k=m,\qquad m>0,$$

exists and the sequence a_k, $k=1,2,\cdots$, of non-negative real numbers satisfies

(2.3) $$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} a_k=a,\qquad a>0,$$

then

$$K(t)=\sum_{n=1}^{\infty} a_n n^\alpha Pr(S_n\leqq t)$$

is convergent for every t where $S_n=\sum_{k=1}^n X_k$ and $\alpha=0,1,2,\cdots$, and it holds that

(2.4) $$\lim_{t\to\infty}\frac{1}{t^{\alpha+1}}K(t)=\frac{a}{(\alpha+1)m^{\alpha+1}}\qquad for\quad \alpha=0,1,2,\cdots.$$

we shall state the following

THEOREM 1. *If, in addition to the assumptions of Lemma, we assume* *that* ψ *is a Baire function integrable over* $(0, \infty)$, *then we have*

$$\lim_{T\to\infty} \frac{1}{T^{\alpha+1}} \int_0^T dt \int_0^t \psi(t-u)\, dK(u) = \frac{a}{(\alpha+1)m^{\alpha+1}} \int_0^\infty \psi(t)\, dt \quad \textit{for } \alpha = 0, 1, 2, \cdots. \tag{2.5}$$

Proof. Since $\psi(t-u)$ is absolutely integrable over the two-dimensional set $\{(u, t);\ 0 \leqq u \leqq t,\ 0 \leqq t \leqq T\}$ with respect to the product measure $dK(u) \times dt$, we have that

$$\begin{aligned} &\int_0^T dt \int_0^t \psi(t-u)\, dK(u) \\ &= \int_0^T dK(u) \int_u^T \psi(t-u)\, dt = \int_0^T dK(u) \int_0^{T-u} \psi(v)\, dv \\ &= K(T) \cdot \int_0^\infty \psi(v)\, dv - \int_0^T dK(u) \int_{T-u}^\infty \psi(v)\, dv. \end{aligned} \tag{2.6}$$

For any small number $\varepsilon > 0$, we can choose a positive number ξ such that

$$\int_\tau^\infty |\psi(v)|\, dv < \varepsilon \qquad \text{for} \quad \tau \geqq \xi.$$

Then, if $T > \xi$, we have

$$\begin{aligned} &\int_0^T dK(u) \int_{T-u}^\infty \psi(v) dv \\ &= \int_0^{T-\xi} dK(u) \int_{T-u}^\infty \psi(v)\, dv + \int_{T-\xi}^T dK(u) \int_{T-u}^\infty \psi(v)\, dv \\ &= I + J, \qquad \text{say,} \end{aligned} \tag{2.7}$$

and

$$|I| \leqq \varepsilon \int_0^{T-\xi} dK(u) = \varepsilon K(T-\xi),$$

because $K(u)$ is a monotone non-decreasing function of u. So we have

$$\overline{\lim_{T\to\infty}} \frac{1}{T^{\alpha+1}} |I| \leqq \varepsilon \cdot \frac{a}{(\alpha+1)m^{\alpha+1}}. \tag{2.8}$$

On the other hand, we have

$$|J| \leqq \int_0^\infty |\psi(v)|\, dv \cdot [K(T) - K(T-\xi)]$$

and so

$$\overline{\lim_{T\to\infty}} \frac{1}{T^{\alpha+1}} |J| = 0 \tag{2.9}$$

by (2.4) in the above Lemma. Suming up (2.4), (2.6), (2.7), (2.8) and (2.9), we get (2.5) which was to be proved.

Now we shall note to be able to obtain a more extended result in some sense. We set the following assumptions:

(i) X_k $(k=1, 2, \cdots)$; Y_k $(k=1, 2, \cdots)$; $\cdots$; Z_k $(k=1, 2, \cdots)$ are non-negative mutually independent random variables,

(ii) X_k $(k=1, 2, \cdots)$; Y_k $(k=1, 2, \cdots)$; $\cdots$; Z_k $(k=1, 2, \cdots)$ have finite means a_k $(k=1, 2, \cdots)$; b_k $(k=1, 2, \cdots)$; $\cdots$; c_k $(k=1, 2, \cdots)$, respectively and there exists a positive constant L such that $a_k \geqq L$, $b_k \geqq L$, $\cdots$, and $c_k \geqq L$ for $k=1, 2, \cdots$,

(iii) there exists a positive constant K such that $\mathrm{Var}(X_k) \leqq K$, $\mathrm{Var}(Y_k) \leqq K$, $\cdots$, $\mathrm{Var}(Z_k) \leqq K$ for $k=1, 2, \cdots$,

(iv) the limits

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k = a, \qquad \lim_{k\to\infty} \frac{1}{n}\sum_{k=1}^{n} b_k = b, \qquad \cdots, \qquad \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} c_k = c$$

exist,

(v) $\phi(x, y, \cdots, z)$ is monotone non-decreasing,

(vi) there exists a positive constant γ such that

$$\phi(x, y, \cdots, z) \geqq \gamma \min(x, y, \cdots, z) \quad \text{for sufficient large } x, y, \cdots, z$$

and

(vii)
$$\lim_{n\to\infty} \frac{1}{n}\phi(xn, yn, \cdots, zn)$$

exists for all $x, y, \cdots, z$ and is equal to a continuous function $\Phi(x, y, \cdots, z)$.

Then, setting

$$V_n = \phi\left(\sum_{k=1}^{n} X_k, \sum_{k=1}^{n} Y_k, \cdots, \sum_{k=1}^{n} Z_k\right)$$

we have

$$\lim_{T\to\infty} \frac{1}{T^{\alpha+1}} \sum_{n=1}^{\infty} n^{\alpha} Pr(V_n \leqq T) = \frac{1}{(\alpha+1)\Phi(a, b, \cdots, c)^{\alpha+1}} \quad \text{for } \alpha=0, 1, 2, \cdots. \tag{2.10}$$

This was proved in Theorem 3 and Theorem 5 of [2]. By making use of this fact, we get again the following

THEOREM 2. *If, in addition to the assumptions* (i)~(vii), *we assume that* ψ *is a Baire function integrable over* $(0, \infty)$, *then we have*

$$\lim_{T\to\infty} \frac{1}{T^{\alpha+1}} \int_0^T dt \int_0^t \psi(t-u)\, dQ(t) = \frac{1}{(\alpha+1)\Phi(a, b, \cdots, c)^{\alpha+1}} \int_0^\infty \psi(t)\, dt \tag{2.11}$$

where

$$Q(t) = \sum_{n=1}^{\infty} n^{\alpha} Pr(V_n \leqq t)$$

and α *is any non-negative integer.*

COROLLARY. *Under the assumption of Theorem* 2, *we have*

$$\lim_{T\to\infty} \frac{1}{T^{\alpha+1}} \int_0^T dt \int_0^t \psi(t-u)\, dQ_1(u) = \frac{1}{(\alpha+1)a^{\alpha+1}} \int_0^\infty \psi(t)\, dt \tag{2.12}$$

where

$$Q_1(t)=\sum_{n=1}^{\infty} n^{\alpha} Pr(S_n \leqq t), \qquad S_n=\sum_{k=1}^{n} X_k$$

and α is any non-negative integer.

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