ON STARLIKE AND CONVEX MAPPINGS OF A CIRCLE

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The classical theory of analytic functions regular and univalent in a circle has been attacked by several authors from various points of view. Tools of attack are also various. Two subclasses are among others especially investigated in detail; namely, the starlike and convex classes. Let the class of functions $\{f(z)\}$ regular and starlike (with respect to the origin) or convex in the unit circle and satisfying the normalization $f(0)=0$ and $f'(0)=1$ be denoted by $\mathfrak{St}$ or $\mathfrak{K}$, respectively. It is well known that these classes $\mathfrak{St}=\{f_s(z)\}$ and $\mathfrak{K}=\{f_c(z)\}$ correspond each other by means of the Alexander's relation [1]

$$(1) \qquad f_s(z)=zf_c'(z) \quad \text{or} \quad f_c(z)=\int_0^z \frac{f_s(z)}{z}\,dz.$$

Two authors, Marx [3] and Strohhäcker [5], developed almost simultaneously but independently systematic treatments of these subclasses. Strohhäcker's principal theorem states that any function $f(z)\in\mathfrak{K}$ satisfies the relations

$$(2) \qquad \Re\frac{f(z)}{z}>\frac{1}{2} \quad \text{and} \quad \Re\frac{zf'(z)}{f(z)}>\frac{1}{2} \qquad (|z|<1),$$

which are mutually equivalent, the bound $1/2$ being best possible. He then proved that, for $f(z)\in\mathfrak{K}$, the points $f(z)/z$ and $zf'(z)/f(z)$ for any assigned z with $|z|\leqq r<1$ are contained in the closed circular disc with the diameter $((1+r)^{-1},(1-r)^{-1})$, and that the point $f'(z)$ lies for $|z|<r$ in the image of $|z|<r$ by the mapping $w=(1-z)^{-2}$. The last statement implies

$$(3) \qquad |f'(z)^{-1/2}-1|<1 \quad \text{or} \quad \Re f'(z)^{1/2}>\frac{1}{2} \qquad (|z|<1).$$

On the other hand, Marx proceeded on a way opposite to Strohhäcker's. In fact, he first proved that any $f(z)\in\mathfrak{K}$ satisfies (3) and then showed that it satisfies the first inequality of (2). Now, it is verified that the first inequality of (2) is a consequence of (3) without any additional condition. Hence, at least from this point of view, Marx's theory seems more natural und straightforward than Strohhäcker's. However, in Marx's own proof of deriving this fact, an elementary but very troublesome computation was contained, so that his way has appeared unreasonably complicated. It seems, therefore, desirable to give a brief proof for this procedure.

Though the results which will be derived in the present paper are not new but they are contained really in the papers of Marx and Strohhäcker, the

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method used below will be perhaps methodically unified and simplified to a considerable extent. Now, an essential part of the troublesome procedure in Marx's original proof consists in verifying an elementary inequality. We shall give here a very brief proof for this inequality which is stated as follows:

LEMMA. *For any z_1 and z_2 in $|z|<1$, we have*

$$\Re \frac{1}{z_2 - z_1} \lg \frac{1-z_1}{1-z_2} > \frac{1}{2}.$$

Proof. The linear function $\Phi(z) = (1+z)/(1-z)$ maps $|z|<1$ onto the right half-plane. Hence, for any z_1 and z_2 in $|z|<1$, we get

$$0 < \int_0^1 \Re\Phi(z_1 + (z_2 - z_1)t)\,dt$$
$$= \Re \frac{1}{z_2 - z_1} \int_{z_1}^{z_2} \frac{1+z}{1-z}\,dz = \Re \frac{2}{z_2 - z_1} \lg \frac{1-z_1}{1-z_2} - 1.$$

Marx proved his main theorem on $\mathfrak{K}$ by making use of a corresponding theorem on $\mathfrak{St}$. But it can be derived more directly and independently of the latter. Here we shall give a proof of his main theorem in two following theorems.

THEOREM 1. *For the class $\mathfrak{K} = \{f(z)\}$, the variability region of $f'(z)$ for $|z| \leqq r$ (<1) coincides with its image by $(1-z)^{-2}$. A boundary point of the latter is attained only by $f(z) = z/(1-\varepsilon z)$ with $|\varepsilon| = 1$. In particular, any $f(z) \in \mathfrak{K}$ satisfies* (3).

Proof. It is known [2] that any $f(z) \in \mathfrak{K}$ admits the integral representation of Herglotz type, i. e.,

$$f'(z) = \exp 2 \int_{-\pi}^{\pi} \lg \frac{e^{i\varphi}}{e^{i\varphi} - z}\,d\rho(\varphi)$$

where $\rho(\varphi)$ is a real-valued increasing function defined for $-\pi \leqq \varphi \leqq \pi$ with the total variation equal to unity. On the other hand, the function

$$h(z;\varphi) = \lg \frac{e^{i\varphi}}{e^{i\varphi} - z} = e^{-i\varphi}z + \cdots$$

maps $|z|<r$ univalently onto a convex domain which is independent of φ. In fact, we have

$$\Re z \frac{h''(z;\varphi)}{h'(z;\varphi)} = \frac{1}{2} \Re \frac{e^{i\varphi} + z}{e^{i\varphi} - z} - \frac{1}{2} > -\frac{1}{2} > -1 \qquad (|z|<1).$$

Consequently, it is evident that the variability region of $f'(z)$ is the image of $|z| \leqq r$ by $\exp(2\lg(1-z)^{-1}) = (1-z)^{-2}$. The latter part of the theorem is an immediate consequence of the former.

THEOREM 2. *For any $f(z)$ regular in $|z|<1$, satisfying $f(0)=0$ and $f'(0)=1$,* (3) *implies the first inequality of* (2).

Proof. Since $2f'(z)^{1/2}-1=1+\cdots$ has the real part positive in $|z|<1$, its Herglotz representation leads to

$$f'(z)=\left(\int_{-\pi}^{\pi}\frac{e^{\imath\varphi}}{e^{\imath\varphi}-z}d\mu(\varphi)\right)^2=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e^{\imath\varphi}}{e^{\imath\varphi}-z}\,\frac{e^{\imath\psi}}{e^{\imath\psi}-z}\,d\mu(\varphi)d\mu(\psi);$$

$$d\mu(\varphi)\geqq 0\ \ (-\pi\leqq\varphi\leqq\pi),\qquad \int_{-\pi}^{\pi}d\mu(\varphi)=1.$$

Integrating with respect to z, we have

$$f(z)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{1}{e^{-\imath\varphi}-e^{-\imath\psi}}\lg\frac{1-e^{-\imath\psi}z}{1-e^{-\imath\varphi}z}\,d\mu(\varphi)d\mu(\psi),$$

whence follows, by virtue of the lemma,

$$\Re\frac{f(z)}{z}=\Re\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{1}{e^{-\imath\varphi}z-e^{-\imath\psi}z}\lg\frac{1-e^{-\imath\psi}z}{1-e^{-\imath\varphi}z}\,d\mu(\varphi)d\mu(\psi)>\frac{1}{2}.$$

By the way, we remark here that two inequalities (2) in Strohhäcker's main theorem are equivalent. In fact, for any ζ with $|\zeta|<1$, the function defined by

$$g(z)=\frac{1}{f'(\zeta)(1-|\zeta|^2)}\left(f(\zeta)-f\left(\frac{\zeta-z}{1-\bar{\zeta}z}\right)\right)=z+\cdots\qquad(|z|<1)$$

belongs to $\mathfrak{K}$ if $f(z)$ so does. It satisfies

$$g(\zeta)=\frac{f(\zeta)}{f'(\zeta)(1-|\zeta|^2)}\qquad\text{and}\qquad\frac{g'(\zeta)}{g(\zeta)}=\frac{1}{f(\zeta)(1-|\zeta|^2)}.$$

Now, the first inequality of (2) applied to $g(z)$ yields $\Re(g(\zeta)/\zeta)>1/2$, i.e., $|\zeta/g(\zeta)-1|<1$. It follows by Schwarz's lemma that $|\zeta/g(\zeta)-1|\leqq|\zeta|$ and hence

$$\Re\frac{\zeta f'(\zeta)}{f(\zeta)}=\frac{1}{1-|\zeta|^2}\,\Re\frac{\zeta}{g(\zeta)}\geqq\frac{1}{1+|\zeta|}>\frac{1}{2}.$$

Next, the second inequality of (2) applied to $g(z)$ gives similarly $|g(\zeta)/(\zeta g'(\zeta))-1|\leqq|\zeta|$ and hence

$$\Re\frac{f(\zeta)}{\zeta}=\frac{1}{1-|\zeta|^2}\,\Re\frac{g(\zeta)}{\zeta g'(\zeta)}\geqq\frac{1}{1+|\zeta|}>\frac{1}{2}.$$

As mentioned above, in order to prove theorem 2, Marx used the corresponding theorem on $\mathfrak{St}$, which he derived by means of a theorem of Rogosinski [4]; cf. theorem 4 stated below. But we can derive Marx's theorem on $\mathfrak{St}$ independently of Rogosinski's theorem and quite readily from the corresponding theorem on $\mathfrak{K}$, which has been established above. It is also possible to give a direct and brief proof. The theorem in question is stated as follows:

THEOREM 3. *For the class* $\mathfrak{St}=\{f(z)\}$, *the variability region of* $f(z)/z$ *for* $|z|\leqq r$ (<1) *coincides with its image by* $(1-z)^{-2}$. *A boundary point is attained only by* $(1-\varepsilon z)^{-2}$ *with* $|\varepsilon|=1$.

Proof. The theorem is an immediate consequence of theorem 1 combined with Alexander's relation (1). ——Alternatively, it can be proved directly in a similar manner as for $\mathfrak{K}$. In fact, in view of the representation (cf. [2])

$$f(z) = z \exp 2\int_{-\pi}^{\pi} \lg \frac{e^{i\varphi}}{e^{i\varphi} - z}\, d\rho(\varphi);$$

$$d\rho(\varphi) \geqq 0 \ (-\pi \leqq \varphi \leqq \pi), \qquad \int_{-\pi}^{\pi} d\rho(\varphi) = 1,$$

valid for $f(z) \in \mathfrak{St}$, the range in question is the range of $|z| \leqq r$ by $\exp 2 \lg(1 - z)^{-1} = (1-z)^{-2}$.

Finally, we give here as a supplement a brief proof of a similar nature of a theorem of Rogosinski referred to above which is stated as follows:

THEOREM 4. *Let* $g(z)$ *be regular in* $|z| < 1$ *and satisfy there* $\Re zg'(z) > -1$ *and* $g(0) = 0$. *Then the range of* $g(z)$ *for* $|z| \leqq r$ (<1) *is contained in its image by* $\lg(1-z)^{-2}$. *A boundary point of the latter is attained only by* $g(z) = \lg(1-\varepsilon z)^{-2}$ *with* $|\varepsilon| = 1$.

Proof. By using Herglotz's representation, we can write

$$1 + zg'(z) = \int_{-\pi}^{\pi} \frac{e^{i\varphi} + z}{e^{i\varphi} - z}\, d\chi(\varphi);$$

$$d\chi(\varphi) \geqq 0 \ (-\pi \leqq \varphi \leqq \pi), \qquad \int_{-\pi}^{\pi} d\chi(\varphi) = 1.$$

Hence we get

$$g(z) = \int_0^z g'(z)\, dz = \int_{-\pi}^{\pi} \lg \frac{1}{(1 - ze^{-i\varphi})^2}\, d\chi(\varphi).$$

It is only necessary to observe that $w = \lg(1 + re^{i\varphi})^{-2}$ describes a strictly convex Jordan curve as φ varies from $-\pi$ to π.

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