ON THE EXISTENCE AND UNIQUENESS THEOREMS OF DIFFERENCE-DIFFERENTIAL EQUATIONS

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Introduction.

It is convenient to make use of differential equations in order to represent physical phenomena by certain functional relations. The differential equations, however, represent such phenomena by a relation between their present state and its instant change. Then, it is not apparent that the influence of past time does explicitly affect the present.

On the other hand, economic phenomena, generally speaking, have to be dependent on the past time, for example, on the influence before one year. In other words, we should better construct the equations by making use of the difference in order to forecast the result of the year before. They are, of course, to be observed in some physical phenomena.

For the sake of simplicity, we consider the following equation

$$F(t,\, x(t),\, x'(t),\, \cdots,\, x^{(m)}(t),\, x(t-1),\, x'(t-1),\, \cdots,\, x^{(n)}(t-1)) = 0,$$

where F, t, x represent scalars.

For $m > n$, it is supposed to be sovable for $x^{(m)}$, i.e.,

$$\frac{d^m x}{dt^m} = f(t,\, x(t),\, x'(t)\, \cdots,\, x^{(m-1)}(t),\, x(t-1),\, x'(t-1),\, \cdots,\, x^{(n)}(t-1)).$$

Putting $x(t) = y_1(t)$, $x'(t) = y_2(t)$, $\cdots$, $x^{(m-1)} = y_m(t)$, the above equation is reduced to

$$\begin{aligned}
\frac{dy_1(t)}{dt} &= y_2(t), \\
\frac{dy_2(t)}{dt} &= y_3(t), \\
&\vdots \\
\frac{dy_{m-1}(t)}{dt} &= y_m(t), \\
\frac{dy_m(t)}{dt} &= f(t,\, y_1(t),\, y_2(t),\, \cdots,\, y_m(t),\, y_1(t-1),\, y_2(t-1),\, \cdots,\, y_{n+1}(t-1)).
\end{aligned}$$

If we consider

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

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as an m-dimensional vector, we obtain a general equation

$$(1) \qquad \frac{dy(t)}{dt}=f(t,\ y(t),\ y(t-1)),$$

where f represents an m-dimensional vector.

For $m<n$, it follows by the same reason as above that

$$(2) \qquad \frac{dy(t)}{dt}=f(t,\ y(t),\ y(t-1)),$$

where y and f represent n-dimensional vectors.

For $m=n$, we obtain

$$(3) \qquad \frac{dy(t)}{dt}=f\left(t,\ y(t),\ y(t-1),\ \frac{dy(t-1)}{dt}\right)$$

or

$$(4) \qquad \frac{dy(t)}{dt}=f\left(t,\ y(t),\ y(t+1),\ \frac{dy(t+1)}{dt}\right)$$

where y and f represent n-dimensional vectors. We call (3) or (4) neutral system.

The equations (1) and (2) are essentially different from (3) and (4). In this paper, however, we shall mainly deal with the equations (1) and with certain types of (3).

Now, if we consider the solutions of (1) for $0\leqq t\leqq t_1$, it happens that we obtain a function $y(t-1)$ unable to define as a solution for $0\leqq t<1$. Hence, we have to impose on $y(t-1)$ some condition, for example, $y(t-1)\equiv\phi(t)$ for $0\leqq t<1$, where $\phi(t)$ is a given function. Then, it is sufficient to consider the ordinary differential equation

$$\frac{dy(t)}{dt}=f(t,\ y(t),\ \phi(t))$$

for $0\leqq t<1$ with the initial condition $y(0)=y_0$. Here, it is to be noted that it is essential to obtain the solutions of (1) for $0\leqq t\leqq t_1$, where t_1 is greater than 1.

In Chapter 1, we shall discuss the existence theorem by making use of the topological method and the method of prolongation of intervals with the examples whose solutions are not always determined uniquely. In Chapter 2, we summarize the results concerning the uniqueness of solutions. It is fortunate for us to be able to apply the similar methods as in the theory of ordinary differential equations. For instance, the successive approximation method and the comparison method, which is originally due to Perron, are applicable. Hence, we shall just state the results without any proofs.

Throughout this paper, the independent variable may not be confined to be real, but it may be complex, and the dependent variable may be scalar or vector which is complex.

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Chapter 1. Existence Theorem.

§1. Topological Method.

In this section, we shall deal with the existence of solutions of the difference-differential equation

$$\frac{dx(t)}{dt}=f(t,\ x(t),\ x(t-1)) \tag{1.1}$$

for $0\leqq t\leqq t_0$, with the conditions

$$x(t-1)=\phi(t) \qquad (0\leqq t<1) \tag{1.2}$$

and

$$x(0)=x_0. \tag{1.3}$$

First of all, we shall prove the following fundamental

THEOREM 1. *We suppose that in the equation* (1.1)

(i) *$f(t, x, y)$ is continuous for*

$$0\leqq t\leqq t_0,\ |x-x_0|\leqq K,\ |y-x_0|\leqq K; \tag{1.4}$$

(ii) *$|f(t, x, y)|\leqq M$ for the region* (1.4);

(iii) *$\phi(t)$ is a given function continuous for $0\leqq t<1$, $\lim_{t\to 1-0}\phi(t)$ exists, for which we denote by $\phi(1-0)$, and $|\phi(t)-x_0|\leqq K$ for $0\leqq t<1$.*

Then, there exists a continuous solution $x(t)$ which satisfies the difference-differential equation (1.1) *for $0\leqq t\leqq t_1$ with the conditions* (1.2) *and* (1.3), *where*

$$t_1=\min(t_0,\ K/M). \tag{1.5}$$

In the theory of differential equations, there are a lot of methods to establish the existence of solutions. Above all, one of the most elegant method is the one having recourse to the topology. The fundamental theorem used in the theory is a fix-point theorem due to Tychonoff. Tychonoff's theorem is stated as follows (Cf. [1]):

Let R be a linear Hausdorff space which is locally convex. Let K be a set, convex on R, for which the Borel-Lebesgue's covering theorem remains valid. Let T be a continuous mapping of K and Tx, the image of $x\in K$ by T, be contained in K. Then, there exists at least a point x in K such that the equality $x=Tx$ holds good.

By the way, if t_1 which is defined by (1.5) is less than 1, the problem is reduced to that of the theory of differential equations, that is, it is sufficient to consider the existence of differential equation

$$\frac{dx(t)}{dt}=f(t,\ x(t),\ \phi(t))$$

for $0\leqq t\leqq t_1$ with the initial condition $x(0)=x_0$, so that we suppose, in the sequel, t_1 is not less than 1. In this case, we note that t_0 and K/M should not be less than 1.

In order to obtain a solution of (1.1) with (1.2) and (1.3), it is necessary and sufficient to prove that the integral equations which are equivalent to (1.1) with (1.2) and (1.3),

$$x(t)=x_0+\int_0^t f(t,\ x(t),\ \phi(t))\,dt$$

for $0\leqq t<1$, and

$$x(t)=x_1+\int_0^t f(t,\ x(t),\ x(t-1))\,dt$$

for $1\leqq t\leqq t_0$, where

$$x_1=x_0+\int_0^1 f(t,\ x(t),\ \phi(t))\,dt,$$

have at least a solution. In the following, our purpose is to obtain a continuous solution of (1.1), so that the term "continuous" may often be omitted.

Let F be a family of all functions $x(t)$ which are continuous on $0\leqq t\leqq t_1$, where t_1 is defined by (1.6), and satisfy the inequality

$$|x(t)-x_0|\leqq K, \tag{1.6}$$

where $x(0)=x_0$ and we fix the value x_0 for any function $x(t)$ belonging to F. Then, we define a mapping T for any $x(t)$ in F as follows:

$$Tx(t)=x_0+\int_0^t f(t,\ x(t),\ \phi(t))\,dt \tag{1.7}$$

for $0\leqq t<1$ and

$$Tx(t)=x_0+\int_0^1 f(t,\ x(t),\ \phi(t))\,dt+\int_1^t f(t,\ x(t),\ x(t-1))\,dt \tag{1.8}$$

for $1\leqq t\leqq t_1$. Then, in order to establish Theorem 1, it is necessary and sufficient to prove that there exists at least a function $x(t)$ in F such that the equality $x(t)=Tx(t)$ remains valid for $0\leqq t\leqq t_1$.

In order to make use of Tychonff's theorem cited before, it is sufficient to prove five lemmas, which will be stated in the sequel.

LEMMA 1. *For any $x(t)$ in F, $Tx(t)$ defined above is also contained in F.*

Proof It follows by the definitions (1.7) and (1.8) that

$$|Tx(t)-x_0|\leqq\begin{cases} Mt & \text{for}\quad 0\leqq t<1,\\ M+M(t-1)=Mt & \text{for}\quad 1\leqq t\leqq t_1.\end{cases}$$

Since t_1 is defined by (1.5), we have

$$|Tx(t)-x_0| \leqq K \tag{1.9}$$

for $0 \leqq t \leqq t_1$, which satisfies (1.6). This shows that referring to the continuity of $Tx(t)$ proved in Lemma 5 $Tx(t)$ is also contained in F.

The following two lemmas are obvious.

LEMMA 2. *If two functions $x_1(t)$ and $x_2(t)$ are contained in F, $\lambda x_1(t) + (1-\lambda)x_2(t)$ is also contained in F for any λ such that $0<\lambda<1$.*

The lemma shows that F is convex for $0 \leqq t \leqq t_1$.

LEMMA 3. *Let $x_n(t)$ $(n=0, 1, 2, \cdots)$ be a sequence chosen in F. If the sequence converges uniformly to $x(t)$, the limiting function $x(t)$ is an element of F.*

LEMMA 4. *Let $x_n(t)$ $(n=0, 1, 2, \cdots)$ be the elements of F converging uniformly to $x(t)$. Then, $Tx_n(t)$ $(n=0, 1, 2, \cdots)$ converges uniformly to $Tx(t)$.*

This lemma shows that the mapping T is continuous.

Proof. On account of the continuity of $f(t, x, y)$, $x(t)$, and $\phi(t)$, it follows that $f(t, x(t), \phi(t))$ and its integral are continuous for $0 \leqq t < 1$, and $f(t, x(t), x(t-1))$ for $1 \leqq t \leqq t_1$ is continuous with its integral. Thus, it follows by a well-known theorem in the calculus that Lemma 4 holds good.

LEMMA 5. *The family $G=\{Tx(t)\}_{x\in F}$ is normal, that is, in any sequence of G there exists a sequence which converges uniformly.*

Proof. The necessary and sufficient conditions in order that G is a normal family are that each element of G is bounded at any point in $0 \leqq t \leqq t_1$ and it is equi-continuous.

If follows from (1.9) that

$$|Tx(t)| \leqq |x_0| + K$$

for $0 \leqq t \leqq t_1$, which implies that any $Tx(t)$ is bounded for the fixed x_0 and $0 \leqq t \leqq t_1$.

In order to prove that any $Tx(t)$ is equi-continuous, it is sufficient to prove that the inequality

$$|Tx(t)-Tx(t')| \leqq M|t-t'|$$

is satisfied for any t and t' in the interval $0 \leqq t \leqq t_1$. Thus, we must consider three cases.

(i) The case where t and t' are contained in $0 \leqq t < 1$. It follows by means of the definition (1.7) that

$$|Tx(t)-Tx(t')| \leqq \left|\int_{t'}^{t} |f(t, x(t), \phi(t))|\,dt\right| \leqq M|t-t'|.$$

(ii) The case where t and t' are contained in the interval $1 \leqq t \leqq t_1$. It follows by the definition (1.8) that

$$|Tx(t)-Tx(t')| \leqq \left| \int_{t'}^{t} |f(t, x(t), x(t-1))| \, dt \right| \leqq M|t-t'|.$$

(iii) The case where t is contained in $0 \leqq t < 1$ and t' in $1 \leqq t \leqq t_1$. It follows by (1.7) and (1.8) that

$$Tx(t)-Tx(t') = \int_0^t f(t, x(t), \phi(t))\, dt - \int_0^1 f(t, x(t), \phi(t))\, dt - \int_1^{t'} f(t, x(t), x(t-1))\, dt.$$

Hence, we obtain

$$|Tx(t)-Tx(t')| \leqq \int_t^1 |f(t, x(t), \phi(t))|\, dt + \int_1^{t'} |f(t, x(t), x(t-1))|\, dt$$
$$\leqq M(1-t) + M(t'-1) = M(t'-t),$$

which proves the equi-continuity of $Tx(t)$.

Applying five lemmas proved above, we can easily establish the fundamental existence theorem.

REMARK 1. (i) The approximation method by using polygonal lines is applicable to the proof of Theorem 1.

(ii) It is to be noted that Theorem 1 does not always guarantee the uniqueness of solutions even if $f(t, x, y)$ is supposed to be continuous. This result is parallel to that of the theory of differential equations. Thus, we shall illustrate two examples of difference-differential equations whose solutions are not uniquely determined.

EXAMPLE 1. We consider the following difference-differential equation

$$\frac{dx(t)}{dt} = 2x(t-1)\sqrt{x(t)} \tag{1.10}$$

with the conditions

$$x(t-1)=1 \quad \text{for} \quad 0 \leqq t < 1, \qquad \phi(1-0)=1, \quad \text{and} \quad x(0)=0.$$

It is apparent that the function $y\sqrt{x}$ is continuous for $x \geqq 0$ and $y \geqq 0$. Then, we define the following functions:

$$x_1(t) = \begin{cases} 1 & \text{for} \quad -1 \leqq t < 0, \\ t^2 & \text{for} \quad 0 \leqq t \leqq 1, \end{cases} \qquad x_1(-0)=1,$$

and

$$x_2(t) = \begin{cases} 1 & \text{for} \quad -1 \leqq t < 0, \\ 0 & \text{for} \quad 0 \leqq t < \infty, \end{cases} \qquad x_2(-0)=1.$$

We can easily continue the function $x_1(t)$ so that it is to be continuous and satisfies (1.10) for $0 \leqq t < \infty$. Hence, we obtain two solutions with the same conditions.

REMARK 2. By means of the lemmas proved before, it follows that the solutions of (1.1) with (1.2) and (1.3) are continuous for $0 \leqq t \leqq t_1$. However, the continuity of derivatives of solutions can not always be guaranteed at $t=1$ as the above example shows when $t_1 > 1$.

It will easily be expected that the discontinuity of derivatives at $t=1$ is caused by the discontinuity of the initial conditions, that is, $\phi(1-0) \neq x(0)$.

The following example shows that the solutions and its derivatives are continuous for $0 \leqq t < \infty$.

EXAMPLE 2.

$$\frac{dx(t)}{dt} = 2(x(t-1)+1)\sqrt{x(t)}$$

with the conditions $x(t-1)=0$ for $-1 \leqq t < 0$, $x(-0)=0$, and $x(0)=0$.

§2. Method of Prolonging Interval.

The method of prolonging the interval is applicable for (1.1). However, as an illustration, we shall consider the difference-differential equation

$$\frac{dx(t)}{dt} = f\left(t,\ x(t),\ x(t-1),\ \frac{dx(t-1)}{dt}\right). \tag{1.11}$$

It is necessary to impose on (1.11) the following conditions that

$$x(t-1) = \phi(t) \quad \text{and} \quad x'(t-1) = \phi'(t) \quad (0 \leqq t < 1),^{1)} \quad x(0) = x_0. \tag{1.12}$$

Since it seems to be not easy to apply for (1.11) the topological method, that is, it is hard to find out a theorem in C^1 class which plays a similar role as that due to Tychonoff in C^0 class, we shall make use of the method of prolonging the interval of existence of solutions.

Let $f(t, x, y, z)$ satisfy the following conditions:

(i) $f(t, x, y, z)$ is continuous for

$$0 \leqq t \leqq t_0, \quad |x - x_0| \leqq K, \quad |y - x_0| \leqq K, \quad |z| \leqq M, \tag{1.13}$$

where $t_0 > 1$;

(ii) $\quad |f(t, x, y, z)| \leqq M \quad$ for (1.13);

(iii) $\phi(t)$ and $\phi'(t)$ are continuous for $0 \leqq t < 1$, $\phi(1-0)$ and $\phi'(1-0)$ exist, $|\phi(t) - x_0| \leqq K$, and $|\phi'(t)| \leqq M$.

Then, if we put

$$F(t, x(t)) \equiv f(t, x(t), \phi(t), \phi'(t))$$

for $0 \leqq t < 1$, it is sufficient to consider the ordinary differential equation

$$\frac{dx(t)}{dt} = F(t, x(t)) \qquad (0 \leqq t < 1), \tag{1.14}$$

1) It is sufficient to assume the existence of right and left derivatives at $t=0$ and 1 respectively.

where $F(t, x)$ is continuous for $0 \leqq t < 1$ and $|x - x_0| \leqq K$. It is well known that there exists a continuous solution of (1.14) with the initial condition $x(0) = x_0$ for $0 \leqq t \leqq t_1 = \min(1, K/M)$.

By the same reason as in the proof of Theorem 1, we shall consider the present problem for $K \geqq M$. We denote by $x_1(t)$ a solution for $0 \leqq t \leqq 1$ and consider the equation

$$\frac{dx(t)}{dt} = f(t, x(t), x(t-1), x'(t-1))$$

for $1 \leqq t \leqq t_1$ with the conditions

$$(1.12)' \quad x(t-1) = x_1(t-1), \quad x'(t-1) = x_1'(t-1) \quad \text{for} \quad 1 \leqq t \leqq 2, \quad x(1) = x_1(1)$$

instead of (1.12). Then, it is sufficient to consider the ordinary differential equation

$$\frac{dx(t)}{dt} = F(t, x(t))$$

for $1 \leqq t \leqq 2$ with the initial condition $x(1) = x_1(1)$, where

$$F(t, x) \equiv f(t, x(t), x_1(t-1), x_1'(t-1))$$

is continuous for $1 \leqq t \leqq 2$ and $|x - x_0| \leqq K$.

It follows by the existence theorem that there exists a continuous solution for $0 \leqq t \leqq t_1 = \min(2, K/M)$. The process is ceased if $t_1 \leqq 2$. Otherwise, we have to continue the process for $2 < t \leqq t_1$, and so on.

If we continue the process and obtain a continuous solution for $0 \leqq t \leqq [t_1]$, [ ] being Gauss' symbol, we denote the solution by $x_{[t_1]}(t)$ and consider the ordinary differential equation

$$\frac{dx(t)}{dt} = F(t, x(t))$$

for $[t_1] \leqq t \leqq t_0$, where

$$F(t, x) \equiv f(t, x, x_{[t_1]}(t-1), x_{[t_1]}'(t-1)).$$

Then, it follows that there exists a continuous solution $x(t)$ for $0 \leqq t \leqq t_1 = \min(t_0, K/M)$. Thus, we obtain the following

THEOREM 2. *In the equation* (1.11) *with* (1.12), *we suppose that* $f(t, x, y, z)$ *satisfies the conitions* (i), (ii), (iii) *stated above.*

Then, there exists a continuous solution of (1.11) *with* (1.12) *for* $0 \leqq t \leqq t_1 = \min(t_0, K/M)$.

REMARK. As in the preceding section, the following example shows that the uniquess of solutions is not always guaranteed even if the function $f(t, x, y, z)$ is supposed to be continuous.

EXAMPLE.

$$\frac{dx(t)}{dt} = 2\left(x(t-1) + \frac{dx(t-1)}{dt}\right)\sqrt{x(t)}$$

with the conditions $x(t-1)=1$, $x'(t-1)=0$ for $0 \leqq t < 1$, and $x(0)=0$.

Chapter 2. Uniqueness of Solutions. (Summary of Results.)

As has been already seen by two examples in Chapter 1, the uniqueness of solutions is not always guaranteed even if $f(t, x, y)$ is continuous. For the practical applications, however, it is useful to obtain some criteria by which the solutions of difference-differential equations are determined uniquely. There are a lot of methods guaranteeing the uniqueness of solutions of differential equations. It is fortunate that some of them can be applied for the theory of difference-differential equations, if we modify slightly. Thus, we intend, in this chapter, to summarize the results without any proofs.

§ 1. Successive Approximation.

In the equation

$$\frac{dx(t)}{dt} = f(t, x(t), x(t-1)) \tag{2.1}$$

with the conditions

$$x(t-1) = \phi(t) \qquad (0 \leqq t < 1) \tag{2.2}$$

and

$$x(0) = x_0, \tag{2.3}$$

we suppose that the following conditions are satisfied:

(i) $f(t, x, y)$ is continuous for

$$0 \leqq t \leqq t_0, \qquad |x - x_0| \leqq K, \qquad |y - x_0| \leqq K; \tag{2.4}$$

(ii) $|f(t, x, y)| \leqq M$ for (2.4);

(iii) $f(t, x, y)$ satisfies Lipschitz condition, that is, for any x_1, x_2, y_1, y_2 and t in (2.4):

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leqq a(t)\,|x_1 - x_2| + b(t)\,|y_1 - y_2|,$$

where $a(t)$ is continuous for $0 \leqq t \leqq t_0$ and $b(t)$ for $0 \leqq t \leqq t_0 + 1$;

(iv) $\phi(t)$ is a given function continuous for $0 \leqq t < 1$, $\phi(1-0)$ exists, and $|\phi(t) - x_0| \leqq K$.

Then, the following theorem can be proved by a well known method of successive approximation.

THEOREM 3. *In the equation* (2.1) *with* (2.2) *and* (2.3), *there exists a unique solution with* (2.2) *and* (2.3) *for* $0 \leqq t \leqq t_1$ *under the hypotheses* (i), (ii), (iii), *and* (iv), *where*

$$t_1 = \min(t_0, K/M).$$

REMARK. If Lipschitz condition is satisfied by two constants L_1 and L_2 instead of two functions $a(t)$ and $b(t)$ respectively, it is able to improve the estimation for t_1, that is, t_1 may be determined by

$$t_1 = \min\left(t_0, \frac{M}{L_1+L_2}\log\left(1+\frac{(L_1+L_2)K}{M_0}\right)\right),$$

where $|f(t, x, y)| \leqq M_0 (\leqq M)$. This corresponds to a remark due to Lindelöf.

§ 2. Comparison Method.

In order to obtain some results affirming the uniqueness of solutions, it is useful to define the major and minor functions as in the theory of ordinary differential equations. In [1], the case where the functions are represented by vectors is treated by making use of the so-called "Kamke's S-function", and in [2], the classical method is discussed in detail. In the following, we shall deal with the scalar functions for the sake of simplicity. The results obtained here, however, can be extended to vector functions.

DEFINITION. Let $f_i(t)$ $(i=1, 2, 3, 4)$ be continuous for $0 \leqq t \leqq t_1$ and have the right and left derivatives. If the inequalities[2]

$$D^+f_1(t) > f(t, f_1(t), f_1(t-1)), \qquad D^-f_2(t) > f(t, f_2(t), f_2(t-1)) \tag{2.5}$$

are satisfied for $0 \leqq t \leqq t_1$ with the conditions that $f_i(t-1) = \phi(t)$ for $0 \leqq t < 1$, $f_i(-0)$ exists, and $f_i(0) = x_0$, we call $f_1(t)$ and $f_2(t)$ the right and left major functions respectively of the difference-differential equation

$$\frac{dx(t)}{dt} = f(t, x(t), x(t-1)) \tag{2.6}$$

with the conditions

$$x(t-1) = \phi(t) \quad (0 \leqq t < 1), \quad \text{and} \quad x(0) = x_0, \tag{2.7}$$

where $\phi(t)$ is a given function continuous for $0 \leqq t < 1$, and $\phi(1-0)$ exists.

If the second inequality of (2.5) is also satisfied by $f_1(t)$, we simply call $f_1(t)$ the major function of (2.6).

If the inequalities

$$D^+f_3(t) < f(t, f_3(t), f_3(t-1)), \qquad D^-f_4(t) < f(t, f_4(t), f_4(t-1)) \tag{2.8}$$

are satisfied for $0 \leqq t \leqq t_1$ with the conditions that $f_i(t-1) = \phi(t)$ for $0 \leqq t < 1$, $f_i(-0)$ exists, and $f_i(0) = x_0$, we call $f_3(t)$ and $f_4(t)$ the right and left minor functions respectively of (2.6) with (2.7).

If the second inequality of (2.8) is also satisfied by $f_3(t)$, we simply call $f_3(t)$ the minor function of (2.6) with (2.7).

Let $\underline{\omega}(t)$ and $\bar{\omega}(t)$ be continuous functions for $0 \leqq t \leqq t_1$ such that the inequality $\underline{\omega}(t) \leqq \bar{\omega}(t)$ holds good, and $\underline{\omega}(0) = \bar{\omega}(0) = \phi(1-0)$. If $f(t, x, y)$ is defined and continuous in the domain

$$0 \leqq t \leqq t_1, \quad \underline{\omega}(t) \leqq x \leqq \bar{\omega}(t), \quad \underline{\omega}(t-1) \leqq y \leqq \bar{\omega}(t-1), \tag{2.9}$$

where $\underline{\omega}(t-1) = \bar{\omega}(t-1) = \phi(t)$ $(0 \leqq t < 1)$ and $\underline{\omega}(-0) = \bar{\omega}(-0) = \phi(1-0)$, we extend the function $f(t, x, y)$ into the space $0 \leqq t \leqq t_1$, $-\infty < x < \infty$, $-\infty < y < \infty$ as follows:

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2) $D^+f(t)$ and $D^-f(t)$ represent the right and left derivaties of $f(t)$ respectively.

$$f(t, x, y)=\begin{cases} f(t, \bar{\omega}(t), \bar{\omega}(t-1)) & \text{for } x>\bar{\omega}(t) \text{ and } y>\bar{\omega}(t-1), \\ f(t, \bar{\omega}(t), y) & \text{for } x>\bar{\omega}(t) \text{ and } \underline{\omega}(t-1)\leqq y\leqq \bar{\omega}(t-1), \\ f(t, \bar{\omega}(t), \underline{\omega}(t-1)) & \text{for } x>\bar{\omega}(t) \text{ and } y<\underline{\omega}(t-1), \\ f(t, x, \bar{\omega}(t-1)) & \text{for } \underline{\omega}(t)\leqq x\leqq \bar{\omega}(t) \text{ and } y>\bar{\omega}(t-1), \\ f(t, x, \underline{\omega}(t-1)) & \text{for } \underline{\omega}(t)\leqq x\leqq \bar{\omega}(t) \text{ and } y<\underline{\omega}(t-1), \\ f(t, \underline{\omega}(t), \bar{\omega}(t-1)) & \text{for } x<\underline{\omega}(t) \text{ and } y>\bar{\omega}(t-1), \\ f(t, \underline{\omega}(t), y) & \text{for } x<\underline{\omega}(t) \text{ and } \underline{\omega}(t-1)\leqq y\leqq \bar{\omega}(t-1), \\ f(t, \underline{\omega}(t), \underline{\omega}(t-1)) & \text{for } x<\underline{\omega}(t) \text{ and } y<\underline{\omega}(t-1), \end{cases}$$

with the conditions $\underline{\omega}(t-1)=\bar{\omega}(t-1)=\phi(t)$ for $0\leqq t<1$, $\underline{\omega}(-0)=\bar{\omega}(-0)=\phi(1-0)=\underline{\omega}(0)=\bar{\omega}(0)$. Thus, the extended function $f(t, x, y)$ is continuous and bounded for $0\leqq t\leqq t_1$, $-\infty<x<\infty$, $-\infty<y<\infty$. If we make use of the function defined above, we can prove the following

THEOREM 4. *Suppose that in the equation* (2.6) *with* (2.7), $f(t, x, y)$ *is continuous for* (2.9) *and monotone increasing with respect to* y. *Furthermore, we suppose that* $\underline{\omega}(t)$ *and* $\bar{\omega}(t)$ *are the left minor and right major functions respectively. Then, for any solution* $x(t)$ *of* (2.6) *we have the relation*

$$\underline{\omega}(t)\leqq x(t)\leqq \bar{\omega}(t)$$

for $0\leqq t\leqq t_1$.

If the solutions are not uniquely determined, there exist two solutions $\underline{\varphi}(t)$ *and* $\bar{\varphi}(t)$ *of* (2.6) *with* (2.7) *such that any other solution* $x(t)$ *of* (2.6) *with* (2.7) *lies between* $\underline{\varphi}(t)$ *and* $\bar{\varphi}(t)$, *that is, the inequality*

$$\underline{\varphi}(t)\leqq x(t)\leqq \bar{\varphi}(t)$$

remains valid for $0\leqq t\leqq t_1$.

The solutions $\underline{\varphi}(t)$ and $\bar{\varphi}(t)$ are called the maximal and minimal solutions of (2.6) with (2.7) respectively.

By using this theorem, we can prove the coincidence of $\underline{\varphi}(t)$ with $\bar{\varphi}(t)$, if $f(t, x, y)$ satisfies the Lipschitz condition, that is, the uniqueness of solutions is established.

The following theorem is more practically applicable than that above.

THEOREM 5. *We suppose that the following conditions are satisfied:*

(i) $f(t, x, y)$ *is continuous for* $0\leqq t\leqq t_1$, $|x-x_0|\leqq K$, $|y-x_0|\leqq K$;

(ii) $F(t, u, v)$ *is continuous for* $0\leqq t\leqq t_1$, $0\leqq u\leqq M$, $0\leqq v\leqq M$, *and monotone increasing with respect to* v;

(iii) $f(t, x_1, y_1)-f(t, x_2, y_2)\leqq F(t, |x_1-x_2|, |y_1,-y_2|)$;

(iv) *there exist no solutions of*

$$\frac{du(t)}{dt}=F(t, u(t), u(t-1))$$

with $u(t-1)=0$ *for* $0\leqq t\leqq 1$ *other than* $u(t)\equiv 0$ *for* $0\leqq t\leqq t_1$.

Then, solution of (2.6) *with* (2.7) *is uniquely determined.*

As the applications of the above theorem, we consider the special types of $F(t, u, v)$, which are very similar to those of differential equations (Cf. [1], [2]).

EXAMPLE 1. $$F(t, u, v) = L_1 u + L_2 v.$$

EXPMPLE 2. $$F(t, u, v) = F(u),$$

where $F(u)$ fulfills the following conditions:

(i) *$F(u)$ is continuous for $0 \leqq u \leqq K$;*

(ii) *$F(u) > 0$ for $0 < u \leqq K$, and $F(0) = 0$;*

(iii) $$\lim_{\varepsilon \to +0} \int_{\varepsilon}^{K} \frac{du}{F(u)} = +\infty.$$

EXAMPLE 3. $$F(t, u, v) = \frac{1}{t}(L_1 u + L_2 v).$$

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