SOME THEOREMS IN AN EXTENDED RENEWAL THEORY, II

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1. Let X_ν $(\nu=1,2,\cdots)$ be non-negative independent random variables, having finite mean values $E\{X_\nu\}=a_\nu$ $(\nu=2,3,\cdots)$ except X_1. In our previous paper [1], we have proved the following fact: When

$$\lim_{n\to\infty}\frac{1}{n-1}\sum_{\nu=2}^{n}a_\nu=a$$

exists, then

$$\lim_{t\to\infty}\frac{E\{N(t)^\alpha\}}{t^\alpha}=\frac{1}{a^\alpha}\qquad\text{for all }\alpha>0 \tag{1.1}$$

under some further conditions, where $N(t)$ is the number of sums X_1, X_1+X_2, $\cdots$ which are less than t. In the following, we shall begin to note that the condition (1.1) for $\alpha=1,2,\cdots$ is equivalent to each of

$$\lim_{t\to\infty}\frac{1}{t^{\alpha+1}}\sum_{n=1}^{\infty}n^\alpha P\{S_n\leqq t\}=\frac{1}{(\alpha+1)a^{\alpha+1}}\qquad\text{for }\alpha=0,1,2,\cdots, \tag{1.2}$$

and

$$\lim_{T\to\infty}\frac{1}{T^{\alpha+1}}\int_{-\infty}^{T}dt\sum_{n=1}^{\infty}n^\alpha P\{t<S_n\leqq t+h\}=\frac{h}{(\alpha+1)a^{\alpha+1}}\qquad\text{for }\alpha=0,1,2,\cdots, \tag{1.3}$$

where

$$S_n=\sum_{\nu=1}^{n}X_\nu.$$

Thus we have (1.3) under the conditions of Theorem 1 in [1], which has been proved for $\alpha=0$ by Kawata [2] under somewhat different conditions.

Secondly, let X_ν $(\nu=1,2,\cdots)$; Y_ν $(\nu=1,2,\cdots)$; $\cdots$; Z_ν $(\nu=1,2,\cdots)$ be non-negative mutually independent random variables with finite means a_ν $(\nu=1,2,\cdots)$; b_ν $(\nu=1,2,\cdots)$; $\cdots$; c_ν $(\nu=1,2,\cdots)$, respectively and suppose that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n}a_\nu=a,\quad \lim_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n}b_\nu=b,\quad\cdots,\quad\text{and}\quad \lim_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n}c_\nu=c$$

exist. Then defining $M(t)$ as the number of V_1, V_2, $\cdots$ which are less than t, we have

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$$\lim_{t\to\infty}\frac{E\{M(t)^\alpha\}}{t^\alpha}=\frac{1}{\phi(a, b,\cdots, c)^\alpha}\qquad \text{for } \alpha>0 \tag{1.4}$$

under some conditions where

$$V_n=\phi(S_n, T_n,\cdots, U_n),\quad S_n=\sum_{\nu=1}^{n}X_\nu,\quad T_n=\sum_{\nu=1}^{n}Y_\nu,\quad \cdots,\quad \text{and}\quad U_n=\sum_{\nu=1}^{n}Z_\nu.$$

In the case where

$$\phi(x, y,\cdots, z)=\max(x, y,\cdots, z),$$

this fact was stated in [1] with a brief proof. In the latter half of the present paper, we shall prove (1.4) provided V_n is a some more general function of S_n, T_n, $\cdots$, and U_n.

2. THEOREM 1. *Assuming that* X_ν $(\nu=1, 2,\cdots)$ *are non-negative random variables, the following two conditions* (2.1) *and* (2.2) *are equivalent*:

$$\lim_{t\to\infty}\frac{E\{N(t)^\alpha\}}{t^\alpha}=\frac{1}{a^\alpha}\qquad \textit{for } \alpha=1, 2,\cdots; \tag{2.1}$$

$$\lim_{t\to\infty}\frac{1}{t^{\alpha+1}}\sum_{n=1}^{\infty}n^\alpha P\{S_n<t\}=\frac{1}{(\alpha+1)a^{\alpha+1}}\qquad \textit{for } \alpha=0, 1, 2,\cdots. \tag{2.2}$$

Proof.
$$\begin{aligned}\sum_{n=1}^{\infty}n^\alpha P\{N(t)\geqq n\}&=\sum_{n=1}^{\infty}(1^\alpha+2^\alpha+\cdots+n^\alpha)P\{N(t)=n\}\\&<\sum_{n=1}^{\infty}n^{\alpha+1}P\{N(t)=n\}=E\{N(t)^{\alpha+1}\}<+\infty\end{aligned}$$

which is implied in the consideration of $E\{N(t)^\alpha\}$ in (2.1), while

$$\sum_{n=1}^{\infty}n^\alpha P\{N(t)\geqq n\}=\sum_{n=1}^{\infty}n^\alpha P\{S_n<t\},$$

which is also finite for $\alpha=0, 1, 2,\cdots$ because of (2.2). Hence, considering each of the conditions (2.1) and (2.2), we may suppose that

$$\sum_{n=1}^{\infty}n^\alpha P\{N(t)\geqq n\}<+\infty.$$

Now we have

$$\begin{aligned}E\{N(t)^\alpha\}&=\sum_{n=1}^{\infty}n^\alpha P\{N(t)=n\}\\&=\sum_{n=1}^{\infty}n^\alpha[P\{N(t)\geqq n\}-P\{N(t)\geqq n+1\}]\\&=\sum_{n=1}^{\infty}n^\alpha P\{N(t)\geqq n\}-\sum_{n=1}^{\infty}n^\alpha P\{N(t)\geqq n+1\}\end{aligned}$$

$$=\sum_{n=1}^{\infty} n^{\alpha} P\{S_n < t\} - \sum_{n=2}^{\infty} (n-1)^{\alpha} P\{S_n < t\}$$
$$=\sum_{n=1}^{\infty} [n^{\alpha} - (n-1)^{\alpha}] P\{S_n < t\}$$
$$=\alpha \sum_{n=1}^{\infty} n^{\alpha-1} P\{S_n < t\} - \binom{\alpha}{2} \sum_{n=1}^{\infty} n^{\alpha-2} P\{S_n < t\}$$
$$+\cdots+(-1)^{\alpha+1} \sum_{n=1}^{\infty} P\{S_n < t\}$$

from which it will be obvious that (2.1) follows from (2.2).

Conversely since it is easily seen that

$$\sum_{n=1}^{\infty} n^{\alpha-1} P\{S_n < t\}$$

is expressed by means of

$$E\{N(t)^{\beta}\}, \qquad \beta = 0, 1, 2, \cdots, \alpha,$$

we can show that (2.2) follows from (2.1).

COROLLARY. *Assuming that* X_ν $(\nu = 1, 2, \cdots)$ *are non-negative random variables,* (2.1) *is equivalent to*

$$\lim_{t\to\infty} \frac{1}{t^{\alpha+1}} \sum_{n=1}^{\infty} n^{\alpha} P\{S_n \leqq t\} = \frac{1}{(\alpha+1)a^{\alpha+1}} \qquad \textit{for } \alpha = 0, 1, 2, \cdots. \tag{2.3}$$

Proof. For any positive number ε, we have

$$\frac{1}{t^{\alpha+1}} \sum_{n=1}^{\infty} n^{\alpha} P\{S_n < t\} \leqq \frac{1}{t^{\alpha+1}} \sum_{n=1}^{\infty} n^{\alpha} P\{S_n \leqq t\}$$
$$\leqq \left(\frac{t+\varepsilon}{t}\right)^{\alpha+1} \cdot \frac{1}{(t+\varepsilon)^{\alpha+1}} \sum_{n=1}^{\infty} n^{\alpha} P\{S_n < t+\varepsilon\}.$$

Consequently, we know that (2.3) is equivalent to (2.2) and so to (2.1).

THEOREM 2. *When* X_ν $(\nu = 1, 2, \cdots)$ *are not necessarily non-negative random variables, then the condition* (2.3) *is equivalent to the following*:

$$\lim_{T\to\infty} \frac{1}{T^{\alpha+1}} \int_{-\infty}^{T} dt \sum_{n=1}^{\infty} n^{\alpha} P\{t < S_n \leqq t+h\} = \frac{h}{(\alpha+1)a^{\alpha+1}} \qquad \textit{for } \alpha = 0, 1, 2, \cdots. \tag{2.4}$$

Proof. Since

$$\int_{-\infty}^{T} dt \sum_{n=1}^{\infty} n^{\alpha} P\{t < S_n \leqq t+h\} = \sum_{n=1}^{\infty} n^{\alpha} \int_{-\infty}^{T} dt \int_{t}^{t+h} d\sigma_n(\tau),$$

where $\sigma_n(x)$ is the distribution function of S_n, we have

$$\sum_{n=1}^{\infty} n^{\alpha}\int_{-\infty}^{T} d\sigma_n(\tau)\int_{\tau-h}^{\tau} dt \leqq \int_{-\infty}^{T} dt \sum_{n=1}^{\infty} n^{\alpha} P\{t < S_n \leqq t+h\} \leqq \sum_{n=1}^{\infty} n^{\alpha}\int_{-\infty}^{T+h} d\sigma_n(\tau)\int_{\tau-h}^{\tau} dt,$$

i. e.

$$h\sum_{n=1}^{\infty} n^{\alpha} P\{S_n \leqq T\} \leqq \int_{-\infty}^{T} dt \sum_{n=1}^{\infty} n^{\alpha} P\{t < S_n \leqq t+h\} \leqq h\sum_{n=1}^{\infty} n^{\alpha} P\{S_n \leqq T+h\},$$

which proves the theorem.

3. Through this section we set the following assumptions:

(i) X_ν $(\nu=1,2,\cdots)$; Y_ν $(\nu=1,2,\cdots)$; $\cdots$; Z_ν $(\nu=1,2,\cdots)$ are non-negative mutually independent random variables,

(ii) X_ν $(\nu=1,2,\cdots)$; Y_ν $(\nu=1,2,\cdots)$; $\cdots$; Z_ν $(\nu=1,2,\cdots)$ have finite means a_ν $(\nu=1,2,\cdots)$; b_ν $(\nu=1,2,\cdots)$; $\cdots$; c_ν $(\nu=1,2,\cdots)$, respectively and there exists a positive constant L such that $a_\nu \geqq L$, $b_\nu \geqq L, \cdots$, and $c_\nu \geqq L$ for $\nu=1,2,\cdots$,

(iii) there exists a positive constant K such that $\mathrm{Var}(X_\nu) \leqq K$, $\mathrm{Var}(Y_\nu) \leqq K$, $\cdots$, $\mathrm{Var}(Z_\nu) \leqq K$ for $\nu=1,2,\cdots$,

(iv) the limits

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n} a_\nu = a, \quad \lim_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n} b_\nu = b, \quad \cdots, \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n} c_\nu = c,$$

exist,

(v) $\phi(x, y, \cdots, z)$ is monotone non-decreasing,

(vi) there exists a positive constant γ such that

$$\phi(x, y, \cdots, z) \geqq \gamma\cdot\min(x, y, \cdots, z), \qquad \text{for all } x, y, \cdots, z,$$

and

(vii)
$$\lim_{n\to\infty}\frac{1}{n}\phi(xn, yn, \cdots, zn)$$

exists for all $x, y, \cdots, z$ and is equal to a continuous function $\Phi(x, y, \cdots, z)$.

Now we set the following

DEFINITION. $N_X(t)$, $N_Y(t)$, $\cdots$, $N_Z(t)$ and $M(t)$ are integral valued random variables such that

$$S_{N_X(t)} < t \leqq S_{N_X(t)+1},$$
$$T_{N_Y(t)} < t \leqq T_{N_Y(t)+1},$$
$$\cdots\cdots\cdots\cdots,$$
$$U_{N_Z(t)} < t \leqq U_{N_Z(t)+1}$$

and

$$V_{M(t)} < t \leqq V_{M(t)+1},$$

where

$$S_n = \sum_{\nu=1}^{n} X_\nu, \quad T_n = \sum_{\nu=1}^{n} Y_\nu, \quad \cdots, \quad U_n = \sum_{\nu=1}^{n} Z_\nu$$

and

$$V_n = \phi(S_n, T_n, \cdots, U_n).$$

$N_X(t)$, $N_Y(t)$, $\cdots$, $N_Z(t)$ and $M(t)$ can be defined uniquely and are finite with probability 1 by the conditions (i), (ii), (iii), (v) and (vi) and we have the following lemma which have been proved in Theorem 1 in [1].

LEMMA 1. *Under the conditions* (i)—(iv), *we have*

$$\lim_{t\to\infty} \frac{E\{N_X(t)^\alpha\}}{t^\alpha} = \frac{1}{a^\alpha} < +\infty,$$
$$\lim_{t\to\infty} \frac{E\{N_Y(t)^\alpha\}}{t^\alpha} = \frac{1}{b^\alpha} < +\infty,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

and

$$\lim_{t\to\infty} \frac{E\{N_Z(t)^\alpha\}}{t^\alpha} = \frac{1}{c^\alpha} < +\infty \qquad \textit{for } \alpha > 0.$$

THEOREM 3. *Under the conditions* (i)—(vii), *we have*

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{\Phi(a, b, \cdots, c)} \qquad (a.\ s.) \tag{3.1}$$

and

$$\lim_{t\to\infty} \frac{E\{M(t)^\alpha\}}{t^\alpha} = \frac{1}{\Phi(a, b, \cdots, c)^\alpha} \qquad \textit{for all } \alpha > 0. \tag{3.2}$$

Proof. We know by the law of large numbers that

$$(a-\varepsilon)n < S_n < (a+\varepsilon)n,$$
$$(b-\varepsilon)n < T_n < (b+\varepsilon)n,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$(c-\varepsilon)n < U_n < (c+\varepsilon)n$$

for sufficient large n with probability 1, ε being an arbitrary positive number. Since

$$M(t) \to \infty \qquad \text{as } t \to \infty \qquad \text{(a. s.)}, \tag{3.3}$$

we have

$$\begin{aligned}&(a-\varepsilon)M(t)<S_{M(t)}<(a+\varepsilon)M(t),\\&(b-\varepsilon)M(t)<T_{M(t)}<(b+\varepsilon)M(t),\\&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\&(c-\varepsilon)M(t)<U_{M(t)}<(c+\varepsilon)M(t)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{M(t)}\phi((a-\varepsilon)M(t),(b-\varepsilon)M(t),\cdots,(c-\varepsilon)M(t))\\&<\frac{t}{M(t)}\\&\leqq\frac{t}{M(t)}\phi((a+\varepsilon)(M(t)+1),(b+\varepsilon)(M(t)+1),\cdots,(c+\varepsilon)(M(t)+1))\end{aligned}$$

for sufficient large t with probability 1, which give (3.1) with (vii) and (3.3). On the other hand, we have

$$t>\phi(S_{M(t)},T_{M(t)},\cdots,U_{M(t)})\geqq\gamma\cdot\min(S_{M(t)},T_{M(t)},\cdots,U_{M(t)})$$

which implies

$$M(t)\leqq\max\left(N_X\left(\frac{t}{\gamma}\right),N_Y\left(\frac{t}{\gamma}\right),\cdots,N_Z\left(\frac{t}{\gamma}\right)\right)$$

and

$$M(t)^\alpha\leqq N_X\left(\frac{t}{\gamma}\right)^\alpha+N_Y\left(\frac{t}{\gamma}\right)^\alpha+\cdots+N_Z\left(\frac{t}{\gamma}\right)^\alpha\qquad\text{for }\alpha>0$$

and so we see by Lemma 1 that

$$\overline{\lim_{t\to\infty}}\frac{E\{M(t)^\alpha\}}{t^\alpha}<+\infty\qquad\text{for }\alpha>0.$$

Therefore we get

$$\overline{\lim_{t\to\infty}}E\left\{\left(\frac{M(t)^\alpha}{t^\alpha}\right)^2\right\}<+\infty\qquad\text{for all }\alpha>0,$$

that is, these second moments of $M(t)^\alpha/t^\alpha$ are bounded at $t=\infty$, which implies that (3.1) can be integrated term by term, giving (3.2).

REMARK. We can prove by the similar way the following theorem, which is a more general extension of Theorem 3. First of all, we set an assumption:

(viii) There exist positive numbers γ and μ such that

$$\phi(x,y,\cdots,z)\geqq\gamma\cdot(\min(x,y,\cdots,z))^\mu$$

for sufficiently large $x, y, \cdots, z$ and

$$\lim_{n\to\infty} \frac{1}{n^{\mu}} \phi(xn, yn, \cdots, zn)$$

exists and is equal to a continuous function $\Phi(x, y, \cdots, z)$.

THEOREM 4. *Under the conditions* (i)—(v) *and* (viii), *we have*

$$\lim_{t\to\infty} \frac{M(t)^{\mu}}{t} = \frac{1}{\Phi(a, b, \cdots, c)} \qquad (a.\ s.) \tag{3.4}$$

and

$$\lim_{t\to\infty} \frac{E\{M(t)^{\alpha\mu}\}}{t^{\alpha}} = \frac{1}{\Phi(a, b, \cdots, c)^{\alpha}} \qquad \text{for all } \alpha > 0. \tag{3.5}$$

The argument analogous to the proofs of Theorems 1 and 2 in the preceding section gives the following

THEOREM 5. *Under the notations of this section, the following three conditions* (3.6), (3.7) *and* (3.8) *are equivalent:*

$$\lim_{t\to\infty} \frac{E\{M(t)^{\alpha}\}}{t^{\alpha}} = \frac{1}{\Phi(a, b, \cdots, c)^{\alpha}} \qquad \text{for } \alpha = 1, 2, \cdots; \tag{3.6}$$

$$\lim_{t\to\infty} \frac{1}{t^{\alpha+1}} \sum_{n=1}^{\infty} n^{\alpha} P\{V_n \leqq t\} = \frac{1}{(\alpha+1)\Phi(a, b, \cdots, c)^{\alpha+1}} \qquad \text{for } \alpha = 0, 1, 2, \cdots; \tag{3.7}$$

$$\lim_{T\to\infty} \frac{1}{T^{\alpha+1}} \int_{-\infty}^{T} dt \sum_{n=1}^{\infty} n^{\alpha} P\{t < V_n \leqq t+h\} = \frac{h}{(\alpha+1)\Phi(a, b, \cdots, c)^{\alpha+1}} \qquad \text{for } \alpha = 0, 1, 2, \cdots. \tag{3.8}$$

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REFERENCES

[1] H. HATORI, Some theorems in an extended renewal theory, I. Kōdai Math. Sem. Rep. 11 (1959), 139–146.

[2] T. KAWATA, A renewal theorem. Journ. Math. Soc. Japan 8 (1956), 118–126.

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