ON CONFORMAL MAPPING OF A MULTIPLY-CONNECTED DOMAIN ONTO A CANONICAL COVERING SURFACE

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§1. Introduction.

It is well known that by means of an extremal method we can construct a mapping function which maps conformally a multiply-connected planar domain of finite connectivity whose two or more boundary components are continua, onto an annulus cut along concentric circular slits (cf. [3], [5]).

In this paper we concern ourselves with a conformal mapping of a multiply-connected planar domain of finite connectivity whose each boundary component is a continuum, onto a covering surface of annular type cut along concentric circular slits (cf. §2). This mapping may be regarded as an extension of the above-mentioned one. If a finitely-sheeted covering surface separating 0 from ∞ (cf. §2) is conformally equivalent to a covering surface of annular type cut along concentric circular slits centred at the origin in such a manner that rotation numbers about the origin of corresponding boundary components remain invariant by the mapping, the logarithmic area of the former is not smaller than that of the latter, and further they are equal if and only if the former is obtained from the latter by a dilatation and a rotation about the origin of the basic plane (Theorem 1 in §3). Based on this fact, we obtain a procedure of constructing the mapping function by an extremal method: There exists an analytic function which maps a multiply-connected domain of finite connectivity whose each boundary component is a continuum, onto a covering surface of annular type cut along concentric circular slits. If we indicate a rotation number about the origin of the image of every boundary component of the original domain, the mapping function is determined uniquely except an entire linear transformation on the basic plane of the image (Theorem 2 in §3).

It is well known that an N-ply-connected domain whose each boundary component is a continuum can be mapped conformally onto an N-sheeted disk (cf. [1], [2], [4]). However, according to the above reasoning, a $2N$-ply-connected domain whose each boundary component is a continuum, can not necessarily be mapped onto an N-sheeted annulus. In §4 we shall consider a condition for the possibility of such a mapping.

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The argument in this paper may be applied to the case of finite Riemann surfaces. We will concern ourselves with this case subsequently.

2. Preliminaries.

Let F be a finitely-sheeted covering surface laid on the w-plane whose boundary Γ consists of N continua Γ_j $(j=1, \cdots, N)$. We further suppose that two or more among Γ_j consist of closed curves separating two points a' and a'' each other on the w-plane and there exist no points of $\overline{F}$ on a' or a''. Then we call F *a finitely-sheeted covering surface separating a' and a''.*

Let F be such a covering surface and Γ_j^* $(j=1, \cdots, N)$ be simple analytic closed curves on F homotop to Γ_j, respectively. Then we define *the rotation number of Γ_j about the point a'* by

$$\frac{1}{2\pi}\int_{\Gamma_j^*} d\arg(z-a') = \nu_j \qquad (j=1, \cdots, N), \tag{1}$$

the integration path being always taken in the positive sense with respect to F. The value on the left-hand side of (1) does not depend on a particular choice of a path Γ_j^*. Namely if $\Gamma_j^{*\prime}$ is another simple analytic closed curve homotop Γ_j, we have

$$\frac{1}{2\pi}\int_{\Gamma_j^{*\prime}} d\arg(z-a') = \frac{1}{2\pi}\int_{\Gamma_j^*} d\arg(z-a') = \nu_j.$$

Especially if Γ_j is itself simple and analytic, we have

$$\frac{1}{2\pi}\int_{\Gamma_j} d\arg(z-a') = \int_{\Gamma_j^*} d\arg(z-a') = \nu_j.$$

The rotation number about the point a' of the sum of some boundary components is defined by the sum of their rotation numbers of each boundary component about a'.

Let G be a finitely-sheeted covering surface separating a' and ∞ whose each boundary component Λ_j $(j=1, \cdots, N)$ has as the projection onto the basic plane a concentric circle or a concentric circular slit centred at a'. Then we call G *a covering surface of annular type cut along concentric circular slits centred at a'.*[1]

Let F be a finitely-sheeted covering surface separating 0 and ∞, then we call

$$I(F) = D_F(\lg w(p)) = \iint_F d\lg|w(p)|\, d\arg w(p) = \iint_F \frac{du(p)d\tilde{u}(p)}{|w(p)|^2}$$

the logarithmic area of F, where $w(p) = u(p) + i\tilde{u}(p)$ is a projection map of F onto the basic w-plane.

1) It is permitted that there is no concentric circular slit.

§3. Theorems.

We begin with a fundamental inequality that exposes an extremality for a covering surface of annular type cut along concentric circular slits.

THEOREM 1. *Let F be a finitely-sheeted covering surface separating 0 and ∞, and G a covering surface of annular type cut along concentric circular slits centred at the origin. If F is conformally equivalent to G in such a manner that the rotation numbers about the origin of the corresponding boundary components are equal, then there holds an inequality*

$$I(G) \leqq I(F).$$

Here the equality sign appears if and only if F is obtained from G by a dilatation and a rotation about the origin of the basic plane.

Proof. Let z-plane and w-plane be basic planes of F and G, respectively, and $z = z(p)$ and $w = w(q)$ the projection maps of F and G onto z-plane and w-plane, respectively. Further let $q = \Psi(p)$ be any conformal mapping of F onto G satisfying the condition stated in the theorem. Let

$$Z = Z(p) = \lg z(p), \qquad W = W(q) = \lg w(q),$$
$$X = X(p) = \Re Z(p), \qquad U = U(q) = \Re W(q).$$

Since G is a covering surface of annular type cut along concentric circular slits centred at the origin, U takes a constant value c_j $(j = 1, \cdots, N)$ on each boundary component Λ_j $(j = 1, \cdots, N)$ of G as the boundary value. Let $\{c_{j_\nu}\}_{\nu=1}^{N'}$ be constructed from the set $\{c_j\}_{j=1}^{N}$ by taking the members without repetition and $\{\varepsilon_m\}_{m=1}^{\infty}$ a monotone decreasing sequence consisting of positive numbers which converges to zero. Let ε_1 be chosen sufficiently small such that

$$\varepsilon_1 < \min_{\mu \neq \nu} \frac{|c_{j_\mu} - c_{j_\nu}|}{2}.$$

Let G^m be a subset of G which is obtained by rejecting all portions of G lying on

$$c_{j_\nu} - \varepsilon_m \leqq |w| \leqq c_{j_\nu} + \varepsilon_m \qquad (\nu = 1, \cdots, N'),$$

and ${}^-\Lambda_\nu^m$ (or ${}^+\Lambda_\nu^m$) the whole of boundary components of the set G^m lying on

$$|w| = c_{j_\nu} - \varepsilon_m \ (\text{resp. } |w| = c_{j_\nu} + \varepsilon_m) \qquad (\nu = 1, \cdots, N').^{2)}$$

G^m $(m = 1, 2, \cdots)$ consists of a finite number of subdomains of G and each ${}^\pm\Lambda_\nu^m$ $(\nu = 1, \cdots, N')$ consists of a finite number of closed curves in G whose projections onto the w-plane lie on the circle $|w| = c_{j_\nu} \pm \varepsilon_m$. It is obvious that

2) Here either ${}^-\Lambda_\nu^m$ or ${}^+\Lambda_\nu^m$ may be vacuous for some ν.

$$I(G^m) < I(G) \qquad (m = 1, 2, \cdots)$$

and

$$\lim_{n\to\infty} I(G^m) = I(G).$$

Next, let F^m be the image-set of G^m by the inverse mapping Ψ^{-1} and ${}^{\pm}\Gamma_\nu^m$ the image-curves of ${}^{\pm}\Lambda_\nu^m$ $(\nu = 1, \cdots, N')$. Then we have

$$I(F^m) < I(F) \qquad (m = 1, 2, \cdots)$$

and

$$\lim_{n\to\infty} I(F^m) = I(F).$$

Further ${}^{\pm}\Gamma_\nu^m$ consists of a finite number of analytic closed curves and its rotation number about the origin is equal to that of ${}^{\pm}\Lambda_\nu^m$. This is verified as follows. The boundary of portions ${}^{-}G_\nu^m$ of G on $|w| < c_{j_\nu} - \varepsilon_m$ consists of ${}^{-}\Lambda_\nu^m$ and the boundary components $\Lambda_{k_1}, \cdots, \Lambda_{k_\nu}$ of G on $|w| < c_{j_\nu} - \varepsilon_m$. Obviously the rotation number of $\Lambda_{k_1} + \cdots + \Lambda_{k_\nu} + {}^{-}\Lambda_\nu^m$ about the origin is equal to zero. Thus the rotation number about the origin of the image-curve $\Gamma_{k_1} + \cdots + \Gamma_{k_\nu} + {}^{-}\Gamma_\nu^m$ of $\Lambda_{k_1} + \cdots \Lambda_{k_\nu} + {}^{-}\Lambda_\nu^m$ by Ψ^{-1} is equal to zero too, since the function Ψ^{-1} attains neither 0 nor ∞ on ${}^{-}\overline{G}_\nu^m$. On the other hand, by the assumption of the theorem, the rotation number of $\Lambda_{k_1} + \cdots + \Lambda_{k_\nu}$ about the origin is equal to that of $\Gamma_{k_1} + \cdots + \Gamma_{k_\nu}$. Therefore the rotation number of ${}^{-}\Lambda_\nu^m$ about the origin is equal to that of ${}^{-}\Gamma_\nu^m$. We can also verify the same fact for ${}^{+}\Lambda_\nu^m$ and ${}^{+}\Gamma_\nu^m$ by considering the portions ${}^{+}G_\nu^m$ of G on $|w| > c_{j_\nu} + \varepsilon_m$. Thus let

$$h(q) = X \circ \Psi^{-1}(q) - U(q),$$

then we have

$$(2) \qquad \begin{aligned} \int_{\pm\Lambda_\nu^m} \frac{\partial h}{\partial n} ds &= \int_{\pm\Lambda_\nu^m} \frac{\partial X \circ \Psi^{-1}}{\partial n} ds - \int_{\pm\Lambda_\nu^m} \frac{\partial U}{\partial n} ds \\ &= \int_{\pm\Gamma_\nu^m} \frac{\partial X}{\partial n} ds - \int_{\pm\Lambda_\nu^m} \frac{\partial U}{\partial n} ds = 0 \\ & \qquad (\nu = 1, \cdots, N';\ m = 1, 2, \cdots), \end{aligned}$$

where $\partial/\partial n$ expresses the differentiation along inner normal and ds the line element. Now we have

$$(3) \qquad \begin{aligned} I(F^m) &= D_{F^m}(X) = D_{G^m}(X \circ \Psi^{-1}) = D_{G^m}(U + h) \\ &= D_{G^m}(U) + 2D_{G^m}(U, h) + D_{G^m}(h) \\ &= I(G^m) + 2D_{G^m}(U, h) + D_{G^m}(h) \\ & \qquad (m = 1, 2, \cdots). \end{aligned}$$

By means of Green's formula we have, by (2),

$$(4) \qquad \begin{aligned} D_{G^m}(U, h) &= -\int_{\Lambda^m} U \frac{\partial h}{\partial n} ds \\ &= -\sum_{\nu=1}^{N'} \left\{ (c_{j_\nu} - \varepsilon_m) \int_{-\Lambda_\nu^m} \frac{\partial h}{\partial n} ds + (c_{j_\nu} + \varepsilon_m) \int_{+\Lambda_\nu^m} \frac{\partial h}{\partial n} ds \right\} = 0, \end{aligned}$$

where

$$\Lambda^m = \sum_{\nu=1}^{N'} ({}^+\Lambda_\nu^m + {}^-\Lambda_\nu^m).$$

Then by (3) and (4) we have

$$I(F^m) - I(G^m) = D_{G^m}(h)$$

and hence

$$I(F) - (G) = \lim_{m\to\infty} I(F^m) - \lim_{m\to\infty} I(G^m)$$

$$= \lim_{m\to\infty} D_{G^m}(h) = D_G(h) \geqq 0.$$

The equality in the last inequality appears if and only if

$$h \equiv a \qquad (a \text{ being a real constant}).$$

We then have successively

$$X \circ \Psi^{-1} \equiv U + a,$$

$$\lg z \circ \Psi^{-1} \equiv \lg w + (a + ib) \qquad (b \text{ being a real constant}),$$

$$z \circ \Psi^{-1} \equiv cw \qquad (c = \exp(a + ib)).$$

The last equation shows that F is obtained from G by a dilatation and a rotation about the origin on the basic plane.

Next we state a fundamental theorem showing that there exists an analytic function mapping a multiply-connected domain of finite connectivity onto a covering surface of annular type cut along concentric circular slits.

THEOREM 2. *Let B be a multiply-connected domain of finite connectivity on the z-plane. We suppose that each components C_j $(j = 1, \cdots, N)$ of its boundary C is a continuum. Then B can be conformally mapped onto a covering surface of annular type G cut along concentric circular slits centred at the origin. Further we can indicate the rotation number about the origin of the image of each boundary component arbitrarily under the condition that the sum of the rotation numbers is equal to zero (except the case where the rotation number of each boundary component is equal to zero). If we indicate the rotation number about the origin of the image of each boundary component, the mapping fnnction*

$$w = \Phi(z)\ ^{3)}$$

3) Though Φ is a mapping of B onto G, we regard that Φ assumes values projected onto the w-plane from G so far as a confusion does not arise. For details we should denote it as $w = w \circ \Phi(z)$ where $w = w(q)$ is the projection map of G onto the w-plane.

is uniquely determined under an additive condition $\Phi(z_0)=1$ where z_0 is an arbitrarily indicated point on B.

Proof. Let the rotation number about the origin of the image of C_j $(j=1,\cdots,N)$ be equal to

$$\nu_j \qquad \left(j=1,\cdots,N;\ \sum_{j=1}^{N}\nu_j=0\right).$$

Let B^* be a subdomain of B whose boundary C^* consists of components C_j^* $(j=1,\cdots,N)$ such that C_j^* is a simple analytic closed curve homotop to C_j. Let $w=f(z)$ be an analytic function regular on B which satisfies the conditions

$$\frac{1}{2\pi}\int_{C_j^*} d\arg f(z)=\nu_j \qquad (j=1,\cdots,N),\text{[4]}$$
$$f(z_0)=1$$

and maps B onto a finitely-sheeted covering surface $F(f)$ with finite logarithmic area on the w-plane separating 0 and ∞.[5] Let $\mathfrak{F}=\{f(z)\}$ be the family consisting of such mapping functions. Then $\mathfrak{F}\neq\phi$. In fact, it is readily shown that there exist surely rational functions on the z-plane belonging to $\mathfrak{F}$, by carrying out, if necessary, a mapping of B onto a domain whose each boundary component separates exterior points. Now let

$$I_0\equiv\inf_{f\in\mathfrak{F}} I(F(f))=\inf_{f\in\mathfrak{F}} D_B(\lg f),$$

then we select a sequence of functions $\{f_k\}_{k=1}^{\infty}$ such that

$$f_k\in\mathfrak{F},\qquad \lim_{k\to\infty} D_B(\lg f_k)=I_0.$$

Since each member of $\{f_k\}_{k=1}^{\infty}$ has a bounded logarithmic area and is normalized by $f(z_0)=1$, it forms a normal family. Then it contains a subsequence which converges on B uniformly in the wider sense. Without loss of generality, we may suppose that $\{f_k\}_{k=1}^{\infty}$ does so and let Φ be the limiting function. We have obviously

$$\Phi(z_0)=1$$

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4) The value on the left-hand side does not depend on a particular choice of B^*, i.e. if $B^{*\prime}$ is another admitted subdomain of B and $C_j^{*\prime}$ $(j=1,\cdots,N)$ its boundary components, we have

$$\frac{1}{2\pi}\int_{C_j^{*\prime}} d\arg f(z)=\frac{1}{2\pi}\int_{C_j^*} d\arg f(z) \qquad (j=1,\cdots,N).$$

It is sufficient that we verify it for the case $\overline{B^*}\subset B^{*\prime}$. Since $f'(z)/f(z)$ is regular on a ring domain surrounded by C_j^* and $C_j^{*\prime}$, we get

$$\frac{1}{2\pi}\int_{C_j^{*\prime}} d\arg f(z)-\frac{1}{2\pi}\int_{C} d\arg f(z)=\frac{1}{2\pi}\int_{C_j^{*\prime}-C_j^*}\frac{f'(z)}{f(z)}dz=0 \qquad (j=1,\cdots,N).$$

5) Here we admit the case where there exist boundary points of $F(f)$ on 0 or ∞.

and, since f_k converges to Φ uniformly on C_j^* $(j=1, \cdots, N)$,

$$\int_{C_j^*} d\arg\Phi = \lim_{k\to\infty}\int_{C_j^*} d\arg f_k = 2\pi\nu_j \qquad (j=1, \cdots, N).$$

Further let $\{B^m\}_{m=1}^{\infty}$ be an exhaustion of B. Then we have

$$D_{B^m}(\lg\Phi) = \lim_{k\to\infty} D_{B^m}(\lg f_k) \leqq \lim_{k\to\infty} D_B(\lg f_k) = I_0,$$

since $\{f_k\}_{k=1}^{\infty}$ converges uniformly on B^m for any fixed m. Thus we have

$$I(F(\Phi)) = D_B(\lg\Phi) = \lim_{m\to\infty} D_{B^m}(\lg\Phi) \leqq I_0.$$

Since the opposite inequality is obvious, we have consequently

$$I(F(\Phi)) = I_0.$$

By the above reasoning, we see that $\Phi\in\mathfrak{F}$.

Next we show that the function Φ thus obtained is a desired mapping function. If the projection onto the w-plane of the image Λ_κ by Φ of certain boundary component C_κ of B would not lie on a circle centred at the origin, then

(5) $$U = \lg|\Phi|$$

would not remain constant on C_κ and were moreover not to be constant almost everywhere.

Now a family $\mathfrak{H}=\{u\}$ of harmonic functions u on B with $D_B(u) < +\infty$ forms a Hilbert space by the norm $\|u\| = \sqrt{D_B(u)}$. Let $\mathfrak{H}_1$ be a subclass of $\mathfrak{H}$ consisting of functions which take constant value on each boundary component of B and $\mathfrak{H}_2$ a subclass of $\mathfrak{H}$ consisting of functions which have one-valued conjugate harmonic functions. Then $\mathfrak{H}_1$ forms an orthogonal complement of $\mathfrak{H}_2$ in $\mathfrak{H}$. This fact may be shown as follows. The whole of harmonic measures ω_j of boundary components C_j $(j=1, \cdots, N)$ with respect to B forms a basis of $\mathfrak{H}_1$. Now let

(6) $$D_B(\omega_j, h) = 0 \qquad (j=1, \cdots, N)$$

for $h\in\mathfrak{H}$. Then if we select a sufficiently small positive number δ for any given positive number ε, we see that

$$C_j^\delta = \{z \mid \omega_j = 1-\delta\}$$

is a simple analytic closed curve homotop to C_j $(j=1, \cdots, N)$ and by (6)

$$|D_{B_j}(\omega_j, h)| = |D_{B-B_j}(\omega_j, h)| < \varepsilon$$

where

$$B_j = \{z \mid \omega_j < 1-\delta\}, \qquad (j=1, \cdots, N).$$

Therefore, by using of Green's formula, we get

$$|D_{B_j}(\omega_j, h)| = (1-\delta)\left|\int_{C_j^\delta} \frac{\partial h}{\partial n} ds\right| < \varepsilon \qquad (j=1, \cdots, N).$$

Hence, for all simple analytic closed curves C_j^* on B homotop to C_j, we have

$$\left|\int_{C_j^*} \frac{\partial h}{\partial n} ds\right| < \frac{\varepsilon}{1-\delta} \qquad (j=1, \cdots, N).$$

ε being any positive number, we must have

$$\int_{C_j^*} \frac{\partial h}{\partial n} ds = 0 \qquad (j=1, \cdots, N).$$

That is to say, $h \in \mathfrak{H}_2$ and therefore $\mathfrak{H}_1$ forms an orthogonal complement of $\mathfrak{H}_2$ in $\mathfrak{H}$. By the above reasoning there exists a harmonic function h having a one-valued conjugate harmonic function such that

$$D_B(U, h) \neq 0$$

for U in (5). Especially there exists an h such that $h(z_0)=0$. Next we take a one-valued conjugate harmonic function $\tilde{h}$ of h such that $\tilde{h}(z_0)=0$. Then we can easily see that for any real number ε the function defined by

$$g(z) = \Phi(z) \exp(\varepsilon(h(z) + i\tilde{h}(z)))$$

belongs to $\mathfrak{F}$. Since

$$\begin{aligned} I(F(g)) - I(F(\Phi)) &= D_B(U+\varepsilon h) - D_B(U) \\ &= 2\varepsilon D_B(U, h) + \varepsilon^2 D_B(h), \end{aligned}$$

we have

$$I(F(g)) - I(F(\Phi)) < 0$$

by selecting ε which has a sufficiently small absolute value and has the opposite sign for $D_B(U, h)$. This contradicts the minimality of Φ. Hence we conclude that the projection onto the w-plane of the image Λ_j of each boundary component C_j of B by Φ lies on a circle centred at the origin $(j=1, \cdots, N)$. Further since the image Λ_j^* of C_j^* by Φ is a simple analytic closed curve homotop to Λ_j and

$$\int_{C_j^*} d \arg \Phi = 2\pi\nu_j \qquad (j=1, \cdots, N),$$

the rotation number of Λ_j about the origin is exactly equal to ν_j $(j=1, \cdots, N)$. According to the above argument, Φ is surely a desired function. The uniqueness is obvious by Theorem 1.

§4. Supplement.

Let B be a multiply-connected domain of finite connectivity laid on the z-plane and each component C_j $(j=1, \cdots, N)$ of its boundary C be a continuum.[6] Let ν_j $(j=1, \cdots, N)$ be an arbitrarily given integer such that at

6) In this section we assume for simplicity that all C_j $(j=1, \cdots, N)$ are simple analytic closed curves.

least two among them do not vanish and

$$\sum_{j=1}^{N} \nu_j = 0.$$

Then, as seen in §3, there exists an analytic function mapping B onto a covering surface G of annular type cut along concentric circular slits centred at the origin under the condition

$$(7) \qquad \frac{1}{2\pi}\int_{C_j} d \arg \Phi(z) = \nu_j \qquad (j=1, \cdots, N),$$

and Φ is uniquely determined except a dilatation and a rotation about the origin on the basic plane of G.

Here we shall require an explicit expression of the mapping function $\Phi(z)$. Now, $\lg|\Phi(z)|$ is a harmonic function on B and attains a constant value c_j on each boundary component C_j $(j=1, \cdots, N)$. Therefore, we obtain an expression of the form

$$(8) \qquad \lg|\Phi(z)| = \sum_{j=1}^{N} c_j \omega_j.$$

Further by the condition (7) the relations

$$\frac{1}{2\pi}\int_{C_j} d \arg \Phi(z) = \frac{1}{2\pi}\sum_{k=1}^{N} c_k \int_{C_j} d\tilde{\omega}_j = \nu_j \qquad (j=1, \cdots, N),$$

i.e.

$$(9) \qquad \sum_{k=1}^{N} c_k \int_{C_j} \frac{\partial \omega_k}{\partial n} ds = -2\pi\nu_j \qquad (j=1, \cdots, N)$$

must be satisfied, where $\tilde{\omega}_k$ denotes a harmonic function conjugate to ω_k $(k=1, \cdots, N)$. (9) is a system of linear equations for variables $c_1, \cdots, c_N$, which has surely a solution and whose general solution is of the form

$$(10) \qquad c_1^0 + c, \cdots, c_N^0 + c,$$

where $c_1^0, \cdots, c_N^0$ denotes a particular solution and c is an arbitrary real constant. Conversely, for a solution $c_1, \cdots, c_N$ of (9) an analytic function expressed by

$$(11) \qquad \Phi(z) = \exp\left(\sum_{j=1}^{N} c_j(\omega_j + i\omega_j)\right)$$

is surely a desired mapping function. By observing that a general solution of (9) is given by (10) and that $\tilde{\omega}_j$ $(j=1, \cdots, N)$ is uniquely determined except an arbitrary additive constant, we can again conclude that the mapping function (11) is uniquely determined except a dilatation and a rotation about the origin on the basic plane of G.

Let now the boundary C of B consist of $2N$ components C_j $(j=1, \cdots, 2N)$. We consider in what case B can be mapped onto an N-sheeted annulus G_0 such that $C_1, \cdots, C_N$ correspond to the interior boundary components and $C_{N+1}, \cdots, C_{2N}$ to the exterior boundary components, respectively.

For the simplicity, let

$$\tau_{jk} = \frac{1}{2\pi}\int_{C_j} \frac{\partial \omega_k}{\partial n} ds.$$

First, by using Green's formula, we have

$$\tau_{jk} = \tau_{kj}, \tag{12}$$

and also

$$\sum_{j=1}^{2N} \tau_{jk} = 0, \qquad \sum_{k=1}^{2N} \tau_{jk} = 0. \tag{13}$$

A condition that B can be mapped onto G_0 under the given condition at the beginning can be described by

$$\begin{cases} \sum_{k=1}^{2N} c_k \tau_{jk} = 1 & (j = 1, \cdots, N), \\ \sum_{k=1}^{2N} c_k \tau_{jk} = -1 & (j = N+1, \cdots, 2N), \\ c_1 = \cdots = c_N, & c_{N+1} = \cdots = c_{2N}. \end{cases} \tag{14}$$

Let

$$\mu = \frac{1}{c_1 - c_{N+1}}$$

then by (13) and (14) we have

$$\begin{cases} \sum_{k=1}^{N} \tau_{jk} = \mu & (j = 1, \cdots, N), \\ \sum_{k=1}^{N} \tau_{jk} = -\mu & (j = N+1, \cdots, 2N). \end{cases} \tag{15}$$

Therefore, by taking (12) into account, we obtain

$$\begin{cases} \sum_{k=1}^{N-1} \tau_{jk} = \sum_{\substack{k=1 \\ k \neq j}}^{N} \tau_{Nk} & (j = 1, \cdots, N-1), \\ \sum_{k=1}^{N-1} \tau_{jk} = \sum_{\substack{k=1 \\ k \neq j-N}}^{N} \tau_{N, N+k} & (j = N+1, \cdots, 2N-1). \end{cases} \tag{16}$$

Conversely, if (16) is satisfied, we see, by considering of (13), that there exists a negative number μ satisfying (15). Then the function

$$w = \Phi(z) = \exp \frac{1}{\mu} \sum_{k=1}^{N} (\omega_k + i\tilde{\omega}_k)$$

maps B onto an N-sheeted covering surface lying on

$$e^{1/\mu} < |w| < 1$$

such that $C_1, \cdots, C_N$ correspond to the interior boundary components. Consequently, (16) is a necessary and sufficient condition in order that B can be mapped onto an N-sheeted covering surface such that $C_1, \cdots, C_N$ correspond to the interior boundary components.

By (16) we see that a ring domain can always be mapped onto an annulus (the case $N=1$) and that a quatriply-connected domain B can be mapped onto two-sheeted annulus such that C_1 and C_2 correspond to the interior boundary components if and only if

$$\text{(17)} \qquad \begin{cases} \tau_{11}=\tau_{22}, \\ z_{31}=\tau_{24} \end{cases}$$

(the case $N=2$). The latter case may be reasonable by virtue of the following fact. Let w_1 and w_2 be branch-points of the two-sheeted annulus G_0 on the w-plane.[7] G_0 is mapped by

$$z=\sqrt{\frac{w-w_1}{w-w_2}}$$

onto a quatriply-connected domain B_0 which is symmetric with respect to the origin on the z-plane. Two boundary components of B_0 corresponding to the interior or exterior boundary components become also symmetric each other with respect to the origin. Conversely, if a quatriply-connected domain B_0 is symmetric with respect to certain interior point of B_0, then it can be mapped conformally onto a two-sheeted annulus. Thus a quatriply-connected domain B can be conformally mapped onto a two-sheeted annulus if and only if B is conformally equivalent to a domain such as B_0. It may be noticed that the condition (17) expresses this fact precisely.

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