ON EXTREMAL QUASICONFORMAL MAPPINGS

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Let W and W' be two closed Riemann surfaces of the same genus $g \geqq 2$. Let $\mathfrak{H}$ be a preassigned homotopy classs of topological mappings $T: W \to W'$ and $\mathfrak{H}(\mathfrak{q})$ be a subclass of $\mathfrak{H}$, each member of which carries a given fixed point $\mathfrak{p}_0$ on W to a point $\mathfrak{q}$ on W'. Let $T(\mathfrak{q}; \mathfrak{p})$ be a unique extremal quasiconformal mapping in $\mathfrak{H}(\mathfrak{q})$ such that its dilatation-quotient $D_{T(\mathfrak{q};\mathfrak{p})}(\mathfrak{p})$ has a constant value $K(\mathfrak{q})$ except a finite number of points on W and is less than the maximal dilatation-quotient of any other member S of $\mathfrak{H}(\mathfrak{q})$. In his heuristic paper [4] Teichmüller had presented this problem and in a subsequent paper [5] he had completely solved it. Recently Ahlfors [1] has succeeded to give another simple proof of it.

The constant $K(\mathfrak{q})$ depends on the variable point $\mathfrak{q}$ on W'. A system of relations among the numbers of the various sorts of (homotopically) critical points of a given functional and the Betti numbers of the basic space has been established by Morse under very general assumptions; cf. [2], [3]. To establish such a Morse-theoretic relation for $K(\mathfrak{q})$ is our aim in this paper. For all the Morse-theoretic terminologies, cf. [2], [3].

For a real positive ε $(R > \varepsilon)$, let $\psi_{\varepsilon, R}(z)$ be an auxiliary quasiconformal mapping on $|z| \leqq R$ defined by a transformation: $(x, y) \to (X, Y)$ such that

$$X = x + \varepsilon - \frac{\varepsilon}{R} r, \qquad Y = y$$

with $z = re^{i\theta} = x + iy$. For a complex $\varepsilon = |\varepsilon| e^{i\alpha}$, we define

$$\psi_{\varepsilon, R}(z) = e^{i\alpha} \psi_{|\varepsilon|, R}(z e^{-i\alpha}).$$

Then its maximal dilatation-quotient is

$$1 + 2\frac{|\varepsilon|}{R} + O\left(\frac{|\varepsilon|^2}{R^2}\right)$$

for any sufficiently small $|\varepsilon|$.

Lemma 1. *$K(\mathfrak{q})$ is a continuous function of $\mathfrak{q}$ on W'.*

Proof. Let z be a local parameter around $\mathfrak{q}$ such that $z = 0$ corresponds to $\mathfrak{q}$. Let ε be the value of the coordinate parameter z corresponding to $\mathfrak{q}'$. For a sufficiently small R, we put

$$T'(\mathfrak{q}; \mathfrak{p}) = \begin{cases} \psi_{\varepsilon, R}(T(\mathfrak{q}; \mathfrak{p})) & \text{in } |z| \leqq R, \\ T(\mathfrak{q}; \mathfrak{p}) & \text{outside of } |z| \leqq R \text{ on } W. \end{cases}$$

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Then evidently $T'(\mathfrak{q};\mathfrak{p})$ belongs to the class $\mathfrak{H}(\mathfrak{q}')$, whence follows that $K(\mathfrak{q}')$ is not greater than the maximal dilatation-quotient of $T'(\mathfrak{q};\mathfrak{p})$ by the definition of $K(\mathfrak{q}')$. Therefore we have

$$K(\mathfrak{q}') \leqq \left(1 + 2\frac{|\varepsilon|}{R} + O\left(\frac{|\varepsilon|^2}{R^2}\right)\right)K(\mathfrak{q})$$

for any point on W. Similarly we have

$$K(\mathfrak{q}) \leqq \left(1 + 2\frac{|\varepsilon|}{R} + O\left(\frac{|\varepsilon|^2}{R^2}\right)\right)K(\mathfrak{q}').$$

Thus we have

$$|K(\mathfrak{q}) - K(\mathfrak{q}')| \leqq A|\varepsilon|.$$

THEOREM 1. *Let M_k be the sum of type numbers of any critical points or sets of index k of $K(\mathfrak{q})$ on W', then there holds a system of inequalities*

$$M_0 \geqq 1,$$
$$M_1 - M_0 \geqq 2g - 1,$$
$$M_2 - M_1 + M_0 = 2 - 2g.$$

Proof. Since W' is a closed Riemann surface, W' is locally connected of orders 1 and 2. By the continuity of $K(\mathfrak{q})$ on W' we have all the necessary conditions for the Morse theory. For these we recommend [2] p. 37 and [2] Theorem 5.2 and [3] §6. Then by Theorem 9.1 in [2] M_k is at least the smaller of the two cardinal numbers, alef-null and the kth connectivity R_k of W'. By Corollary 12.6 in [3] we have the relation in our theorem.

LEMMA 2. *If $\mathfrak{q}' \to \mathfrak{q}$, then $T(\mathfrak{q}';\mathfrak{p})$ tends to $T(\mathfrak{q};\mathfrak{p})$ uniformly on W.*

Proof. Since $\{T(\mathfrak{q}';\mathfrak{p})\}$ for $\mathfrak{q}' \to \mathfrak{q}$ forms a family with bounded dilatation-quotient on W and hence it is an equicontinuous one. Therefore we can select a subsequence $T(\mathfrak{q}_n;\mathfrak{p})$ tending uniformly on W to its limit mapping $T^\infty(\mathfrak{p})$. $T^\infty(\mathfrak{p})$ does not reduce to a constant map and hence $T^\infty(\mathfrak{p}) \in \mathfrak{H}(\mathfrak{q})$. Moreover $D_{T^\infty(\mathfrak{p})}(\mathfrak{p}) = \lim_{n\to\infty} D_{T(\mathfrak{q}_n;\mathfrak{p})}(\mathfrak{p}) = K(\mathfrak{q})$ for any point $\mathfrak{p}$ except only a finite number of points on W. Since the extremal quasiconformal mapping in $\mathfrak{H}(\mathfrak{q})$ is unique, $T^\infty(\mathfrak{q})$ must coincide with $T(\mathfrak{q};\mathfrak{p})$. Since any uniformly convergent subsequence has the same limit mapping $T(\mathfrak{q};\mathfrak{p})$, the original sequence $\{T(\mathfrak{q}';\mathfrak{p})\}$ itself tends uniformly to $T(\mathfrak{q};\mathfrak{p})$ on W for $\mathfrak{q}' \to \mathfrak{q}$.

If $\mathfrak{q}$ and $\mathfrak{q}'$ are sufficiently near, then $T(\mathfrak{q};\mathfrak{p})$ and $T(\mathfrak{q}';\mathfrak{p})$ are sufficiently and uniformly near each other on W, and hence there is an infinitesimal deformation $\delta S(\mathfrak{q}): w \to w + H(w;\varepsilon)$ defined on W' for which $\delta S(\mathfrak{q}) \circ T(\mathfrak{q};\mathfrak{p}) = T(\mathfrak{q}';\mathfrak{p})$ and $\lim_{\varepsilon\to 0} H(w;\varepsilon)/\varepsilon$ exists uniformly. The following is a result proved already by Teichmüller [4]: For any analytic quadratic differential $d\zeta^2(w)$ on W'

$$\iint_{W'} \lim_{\varepsilon \to 0} \frac{H_{\bar{w}}(w\,;\,\varepsilon)}{\varepsilon} \frac{d\zeta^2(w)}{dw^2} dudv = 0$$

with $w = u + iv$. Conversely, if $Bdw^2/|dw|^2$ is invariant and

$$\iint_{W'} \bar{B} \frac{d\zeta^2(w)}{dw^2} dudv = 0$$

holds for any analytic quadratic differential $d\zeta^2(w)$, then there is an invariant $H(w)/dw$ such that $B = \bar{H}_{\bar{w}}/2$ and moreover $H(w)$ defines an infinitesimal deformation $w \to w + \varepsilon H(w) + o(\varepsilon) = w + H(w\,;\,\varepsilon)$.

In a topological view point it is an interesting problem to decide what critical set is isolated or consists of only one point. Now we shall enter in this tendency. Let $\mathfrak{q}_0$ be a point on a connected component of a critical set of $K(\mathfrak{q})$ and moreover $K(\mathfrak{q}) \geqq K(\mathfrak{q}_0)$ hold for any point $\mathfrak{q}$ sufficiently near to $\mathfrak{q}_0$. Let $\mathfrak{q}$ be a critical point sufficiently near to $\mathfrak{q}_0$ such that $K(\mathfrak{q}) = K(\mathfrak{q}_0)$. Let $\delta S(\mathfrak{q}_0)$ be an infinitesimal deformation such that $\delta S(\mathfrak{q}_0) \circ T(\mathfrak{q}_0\,;\,\mathfrak{p}) = T(\mathfrak{q}\,;\,\mathfrak{p})$. We denote it $w \to w + H(w\,;\,\varepsilon)$ with $\varepsilon = \overline{\mathfrak{q}\mathfrak{q}_0}$ and $H(w\,;\,\varepsilon)/\varepsilon = O(1)$ uniformly. Then $\delta S(\mathfrak{q}\,;\,t)$ defined by $w \to w + tH(w\,;\,\varepsilon)$ is also an infinitesimal deformation for $t \in [0, 1]$. Let h_Δ and $h_{\mathfrak{q}\mathfrak{q}_0}$ be defined by

$$h_\Delta = \frac{q_\Delta}{p_\Delta} \qquad \text{with} \quad q_\Delta = \frac{\partial}{\partial \bar{z}} T(\Delta\,;\,\mathfrak{p}), \qquad p_\Delta = \frac{\partial}{\partial z} T(\Delta\,;\,\mathfrak{p})$$

and

$$h_{\mathfrak{q}\mathfrak{q}_0} = \frac{H_{\bar{w}}}{1 + H_w}.$$

Then $K(\Delta) \leqq K(\Delta')$ is equivalent to $|h_\Delta| \leqq |h_{\Delta'}|$ since $K(\Delta)$ is equal to $(1 + |h_\Delta|)/(1 - |h_\Delta|)$. By the assumptions we have, with $p = p_{\mathfrak{q}_0}$,

$$|h_{\mathfrak{q}_0}| = |h_{\mathfrak{q}}| = \left| \frac{h_{\mathfrak{q}_0} + \frac{\bar{p}}{p} h_{\mathfrak{q}\mathfrak{q}_0}}{1 + \bar{h}_{\mathfrak{q}_0} \frac{\bar{p}}{p} h_{\mathfrak{q}\mathfrak{q}_0}} \right| \quad \text{and} \quad |h_{\mathfrak{q}_0}| \leqq \left| \frac{h_{\mathfrak{q}_0} + \frac{\bar{p}}{p} t\, h_{\mathfrak{q}\mathfrak{q}_0}}{1 + \bar{h}_{\mathfrak{q}_0} \frac{\bar{p}}{p} t\, h_{\mathfrak{q}\mathfrak{q}_0}} \right|.$$

By a simple calculation we have

$$-(u \bar{h}_{\mathfrak{q}_0} + \bar{u} h_{\mathfrak{q}_0}) = |u|^2 (1 + |h_{\mathfrak{q}_0}|^2), \qquad u = \frac{\bar{p}}{p} h_{\mathfrak{q}\mathfrak{q}_0}.$$

If $h_{\mathfrak{q}\mathfrak{q}_0} \neq 0$, then for any $t \in [0, 1]$ we have

$$-(tu \bar{h}_{\mathfrak{q}_0} + t\bar{u} h_{\mathfrak{q}_0}) < |tu|^2 (1 + |h_{\mathfrak{q}_0}|^2),$$

whence follows

$$|h_{\mathfrak{q}_0}|^2 > \left| \frac{h_{\mathfrak{q}_0} + tu}{1 + tu \bar{h}_{\mathfrak{q}_0}} \right|^2,$$

which contradicts what we have already shown. Thus $h_{\mathfrak{q}\mathfrak{q}_0}$ should vanish and hence $\delta S(\mathfrak{q}_0)$ is a conformal mapping in an identity homotopy class.

Since there is at most one conformal mapping in each homotopy class when the genus g is greater than 1, $\delta S(\mathfrak{q}_0)$ should be an identity map. Therefore we conclude that $T(\mathfrak{q}_0; \mathfrak{p}) \equiv T(\mathfrak{q}, \mathfrak{p})$. Thus, we can state the following

THEOREM 2. *There is no critical set γ which is a connected continuum containing at least one point $\mathfrak{q}_0$ such that $K(\mathfrak{q}_0) \leqq K(\mathfrak{q})$ always holds for any point $\mathfrak{q}$ lying in a sufficiently small neighborhood of $\mathfrak{q}_0$ on W'. Especially any critical set of index 0, that is, the one giving a relative minimum of $K(\mathfrak{q})$ consists of only one point.*

THEOREM 3. *There is no non-increasing sequence c_n such that each level $K(\mathfrak{q}) = c_n$ contains at least one relative minimum point $\mathfrak{q}_n$.*

Proof of Theorem 3 is quite similar as that of Theorem 2.

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