ON CONVOLUTION OF POWER SERIES

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1. In their recent paper [3] Pólya and Schoenberg have stated a conjecture on power series mapping a circle onto a convex domain. Let $\mathfrak{K}$ be the class of analytic functions which are regular in the unit circle and map it univalently onto convex domains. They then state

Conjecture. *If both functions*

$$f(z)=\sum_{n=1}^{\infty}a_n z^n \quad and \quad g(z)=\sum_{n=1}^{\infty}b_n z^n$$

belong to $\mathfrak{K}$, also the function defined by

$$h(z)=\sum_{n=1}^{\infty}a_n b_n z^n$$

belongs to $\mathfrak{K}$.

In the present note, considering a related class, it will be shown that an analogous proposition is affirmatively verified. Let $\mathfrak{R}$ be the class of analytic functions $\Phi(z)$ regular in the unit circle and characterized by

$$\Re\Phi(z)>0 \quad \text{for} \quad |z|<1 \quad \text{and} \quad \Phi(0)=1.$$

We will then establish the following

Theorem 1. *If both functions*

$$f(z)=1+2\sum_{n=1}^{\infty}a_n z^n \quad and \quad g(z)=1+2\sum_{n=1}^{\infty}b_n z^n$$

belong to $\mathfrak{R}$, also the function defined by

$$h(z)=1+2\sum_{n=1}^{\infty}a_n b_n z^n$$

belongs to $\mathfrak{R}$.

Proof. Based on the integral representation valid for the class $\mathfrak{R}$, we may put

$$g(z)=\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(\theta)$$

where $\mu(\theta)$ is a real-valued function defined for $-\pi<\theta\leqq\pi$ which is increasing and has the total variation equal to unity; cf. e.g. [1]. Consequently,

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the coefficients in the expansion of $g(z)$ are represented by

$$b_n = \int_{-\pi}^{\pi} e^{-in\theta} d\mu(\theta).$$

Hence, for $|z| < 1$, we have

$$h(z) = 1 + 2\sum_{n=1}^{\infty} a_n z^n \int_{-\pi}^{\pi} e^{-in\theta} d\mu(\theta) = \int_{-\pi}^{\pi} f(ze^{-i\theta})\, d\mu(\theta),$$

whence follows

$$\Re h(z) = \int_{-\pi}^{\pi} \Re f(ze^{-i\theta})\, d\mu(\theta) > 0 \qquad (|z| < 1).$$

It may be noted by the way that if we substitute also the integral representation of a_n the function $h(z)$ can be written in the form

$$h(z) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{e^{i(\varphi+\psi)} + z}{e^{i(\varphi+\psi)} - z} d\lambda(\varphi)\, d\mu(\psi)$$

where $\lambda(\theta)$ is such a function associated to $f(z)$ as $\mu(\theta)$ to $g(z)$. It is further transformed into

$$h(z) = \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\rho(\theta)$$

with

$$\rho(\theta) = \int_{-\pi}^{\pi} \lambda(\theta - \psi)\, d\mu(\psi)$$

where $\lambda(\varphi)$ is regarded to be continued beyond the original interval of definition in such a manner that $\lambda(\varphi) - \varphi/(2\pi)$ has the period 2π. Since $\rho(\theta)$ is an increasing function with the total variation equal to unity, the last expression of $h(z)$ gives its proper representation as a member of $\Re$.

2. As an evident consequence of theorem 1 we can formulate the following

THEOREM 2. *If functions $f(z)$ and $g(z)$ defined by*

$$\lg f'(z) = 2\sum_{n=1}^{\infty} \frac{a_n}{n} z^n \quad \textit{and} \quad \lg g'(z) = 2\sum_{n=1}^{\infty} \frac{b_n}{n} z^n$$

belong to $\mathfrak{K}$, also the function $h(z)$ defined by

$$\lg h'(z) = 2\sum_{n=1}^{\infty} \frac{a_n b_n}{n} z^n$$

belongs to $\mathfrak{K}$.

Proof. It is well known that a function $F(z)$ defined by

$$\lg F'(z) = 2\sum_{n=1}^{\infty} \frac{A_n}{n} z^n$$

belongs to $\mathfrak{K}$ if and only if the function

$$\Phi(z) \equiv 1 + \frac{zF''(z)}{F'(z)} = 1 + 2\sum_{n=1}^{\infty} A_n z^n$$

belongs to $\mathfrak{R}$. Hence, theorem 2 results readily from theorem 1.

A proposition equivalent to theorem 2 may be formulated in terms of star-like mapping. Let $\mathfrak{St}$ be the class of analytic functions vanishing at the origin which are regular in unit circle and map it univalently onto domains star-like with respect to the origin. We then have

THEOREM 3. *If functions $f(z)$ and $g(z)$ defined by*

$$\lg \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \frac{a_n}{n} z^n \quad \textit{and} \quad \lg \frac{g(z)}{z} = 2\sum_{n=1}^{\infty} \frac{b_n}{n} z^n$$

belong to $\mathfrak{St}$, also the function $h(z)$ defined by

$$\lg \frac{h(z)}{z} = 2\sum_{n=1}^{\infty} \frac{a_n b_n}{n} z^n$$

belongs to $\mathfrak{St}$.

Proof. In general, $f(z) \in \mathfrak{St}$ is equivalent to $\int^z (f(z)/z)\,dz \in \mathfrak{K}$ or also to $zf'(z)/f(z) \in \mathfrak{R}$.

3. In a previous paper [2] we have dealt with mean distortions for the class $\mathfrak{R} = \{\Phi(z)\}$ of analytic functions which are single-valued and of positive real part in an annulus $(0<)\, q < |z| < 1$ and further normalized by the conditions

$$\Re\Phi(z) = 1 \text{ along } |z| = q \quad \text{and} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(qe^{i\theta})\,d\theta = 1.$$

We can now establish an analogue of theorem 1 for Laurent expansions of functions of this class.

THEOREM 4. *If both functions*

$$f(z) = 1 + 2\sum_{n=-\infty}^{\infty}{}' \frac{a_n}{1-q^{2n}} z^n \quad \textit{and} \quad g(z) = 1 + 2\sum_{n=-\infty}^{\infty}{}' \frac{b_n}{1-q^{2n}} z^n$$

belong to $\mathfrak{R}$, also the function defined by

$$h(z) = 1 + 2\sum_{n=-\infty}^{\infty}{}' \frac{a_n b_n}{1-q^{2n}} z^n$$

belongs to $\mathfrak{R}$. Here the prime means that the summand with $n=0$ is to be omitted.

Proof. We can proceed quite similarly as in the proof of theorem 1. In fact, we have only to replace the Poisson kernel

$$\frac{e^{i\theta}+z}{e^{i\theta}-z}=1+2\sum_{n=1}^{\infty}z^n e^{-in\theta} \qquad (|z|<1)$$

by the Villat kernel

$$\frac{2}{i}\left(\zeta(i\lg z+\theta)-\frac{\eta_1}{\pi}(i\lg z+\theta)\right)=1+2\sum_{n=-\infty}^{\infty}{}'\frac{z^n e^{-in\theta}}{1-q^{2n}} \qquad (q\leqq|z|<1)$$

in which the notations on elliptic functions concern the Weierstrassian theory constructed with the primitive periods $2\omega_1=2\pi$ and $2\omega_3=-2i\lg q$; cf. [2].

REFERENCES

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