ON FUCHS' RELATION FOR THE LINEAR DIFFERENTIAL EQUATION WITH ALGEBRAIC COEFFICIENTS

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1. In the theory of linear differential equations of Fuchsian type with rational coefficients, it is well known that the sum of the roots of indicial equations of an equation of this type is equal to

$$n(n-1)\left(\frac{m}{2}-1\right)$$

where n is the order of the equation and m is the number of its singular points. The purpose of this paper is to generalize this Fuchs' relation for the case when the coefficients of the equation are algebraic.

The equation we are to investigate is

$$(1) \quad \frac{d^n v}{dx^n} + p_1 \frac{d^{n-1} v}{dx^{n-1}} + \cdots + p_n v = 0$$

where the coefficients $p_1, p_2, \dots, p_n$ are all supposed to be one-valued and meromorphic functions on a Riemann surface $\mathfrak{F}$ of an algebraic function $y = \varphi(x)$ with genus p and consisting of r sheets.

We denote by

$$\begin{aligned} q_1 &= (\alpha_1, \beta_1), \quad \dots, \quad q_s = (\alpha_s, \beta_s); \\ r_1 &= (\infty, \gamma_1), \quad \dots, \quad r_r = (\infty, \gamma_r); \\ p_1 &= (a_1, b_1), \quad \dots, \quad p_m = (a_m, b_m) \end{aligned}$$

the branch points of $\varphi(x)$, the points at infinity on $\mathfrak{F}$, and the singular points of the equation (1) respectively, where the notation (α, β) stands for the point of $\mathfrak{F}$ such that $x = \alpha, y = \beta$. For simplicity's sake, we assume that

$$(2) \quad \begin{aligned} p_j &\ni q_k, & \text{for } j=1, \dots, m \text{ and } k=1, \dots, s, \\ p_j &\ni r_k, & \text{for } j=1, \dots, m \text{ and } k=1, \dots, r, \\ q_j &\ni r_k, & \text{for } j=1, \dots, s \text{ and } k=1, \dots, r. \end{aligned}$$

2. We suppose that the equation (1) is of Fuchsian type. To derive the generalized Fuchs' relation for this equation, we must investigate the behaviour of the coefficient p_1 on $\mathfrak{F}$.

First, p_1 must have a pole of order at most 1 at $p_1, \dots, p_m$. Accordingly it is expanded, in the neighbourhood of each p_j , in the form

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$$p_1 = (x - \alpha_j)^{-1} [A_{j0} + A_{j1}(x - \alpha_j) + \dots].$$

Next, we consider the equation (1) in the neighbourhood of q_j . If it is a branch point of order $m_j - 1$ ($m_j > 1$), we can take

$$\tau = (x - \alpha_j)^{1/m_j}$$

as a local uniformizing variable at q_j . Then, as can easily be shown by induction, we have

$$\frac{d^k v}{dx^k} = \left(\frac{1}{m_j}\right)^k \tau^{k-km_j} \frac{d^k v}{d\tau^k} + \frac{k(k-1)}{2} \left(\frac{1}{m_j}\right)^{k-1} \left(\frac{1}{m_j} - 1\right) \tau^{k-1-km_j} \frac{d^{k-1} v}{d\tau^{k-1}} + \dots,$$

$k=1, 2, \dots$

Substituting these relations in the left-hand side of the equation (1), we obtain

$$\begin{aligned} & \frac{d^n v}{dx^n} + p_1 \frac{d^{n-1} v}{dx^{n-1}} + \dots + p_n v \\ &= \left(\frac{1}{m_j}\right)^n \tau^{n-nm_j} \frac{d^n v}{d\tau^n} + \frac{n(n-1)}{2} \left(\frac{1}{m_j}\right)^{n-1} \left(\frac{1}{m_j} - 1\right) \tau^{n-1-nm_j} \frac{d^{n-1} v}{d\tau^{n-1}} + \dots \\ & \quad + p_1 \left(\frac{1}{m_j}\right)^{n-1} \tau^{n-1-(n-1)m_j} \frac{d^{n-1} v}{d\tau^{n-1}} + \dots \end{aligned}$$

Therefore, in this new variable τ , the equation (1) will be written as

$$\frac{d^n v}{d\tau^n} + Q_1 \frac{d^{n-1} v}{d\tau^{n-1}} + \dots + Q_n v = 0,$$

where

$$Q_1 = m_j \tau^{-1} \left[\frac{n(n-1)}{2} \left(\frac{1}{m_j} - 1\right) + \tau^{m_j} p_1 \right].$$

q_j being a regular point of the equation (1), Q_1 must be regular at $\tau = 0$. Hence we must have

$$p_1 = \tau^{-m_j} \left[-\frac{n(n-1)}{2} \left(\frac{1}{m_j} - 1\right) + B_{j1} \tau + \dots \right].$$

Consequently, in the neighbourhood of each q_j , p_1 is expanded in the form

$$p_1 = (x - \alpha_j)^{-1} [B_{j0} + B_{j1}(x - \alpha_j)^{1/m_j} + \dots],$$

where

$$B_{j0} = -\frac{n(n-1)}{2} \left(\frac{1}{m_j} - 1\right).$$

Finally, we must determine the behaviour of p_1 at q_j . For that purpose, however, it suffices to replace m_j by -1 and $\tau = (x - \alpha_j)^{1/m_j}$ by $\tau = x^{-1}$ in above discussions. Thus, in the neighbourhood of each r_j , p_1 is expanded in the form

$$p_1 = x^{-1} [C_{j0} + C_{j1} x^{-1} + \dots],$$

where

$$(4) \quad C_{j0} = n(n-1).$$

Evidently p_1 should be regular elsewhere on $\mathfrak{F}$.

3. Let

$$\begin{aligned} &\Gamma_1, \Gamma_2, \dots, \Gamma_m; \\ &\Gamma'_1, \Gamma'_2, \dots, \Gamma'_s; \\ &\Gamma''_1, \Gamma''_2, \dots, \Gamma''_r \end{aligned}$$

be small circuits around

$$\begin{aligned} &\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_m; \\ &\mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_s; \\ &\mathfrak{r}_1, \mathfrak{r}_2, \dots, \mathfrak{r}_r, \end{aligned}$$

respectively. Evidently we have

$$\int_{\Gamma_j} p_1 dx = \int_{\Gamma_j} (x - \alpha_j)^{-1} [A_{j0} + A_{j1}(x - \alpha_j) + \dots] dx = 2\pi i A_{j0},$$

where the integration is to be carried out counterclockwise. Also, by putting $\tau = (x - \alpha_j)^{1/m_j}$, we have

$$\begin{aligned} \int_{\Gamma_{j'}} p_1 dx &= \int_{\Gamma_{j'}} (x - \alpha_j)^{-1} [B_{j0} + B_{j1}(x - \alpha_j)^{1/m_j} + \dots] dx \\ &= \int_{\Gamma_{j'}} \tau^{-m_j} [B_{j0} + B_{j1}\tau + \dots] m_j \tau^{m_j-1} d\tau = 2\pi i m_j B_{j0}. \end{aligned}$$

Similarly, putting $\tau = x^{-1}$, we have

$$\begin{aligned} \int_{\Gamma_{j''}} p_1 dx &= \int_{\Gamma_{j''}} x^{-1} [C_{j0} + C_{j1}x^{-1} + \dots] dx \\ &= \int_{\Gamma_{j''}} \tau [C_{j0} + C_{j1}\tau + \dots] (-1)\tau^{-2} d\tau = -2\pi i C_{j0}. \end{aligned}$$

Since p_1 is one-valued and the differential $p_1 dx$ is regular elsewhere, we have

$$\sum_{j=1}^m \int_{\Gamma_j} p_1 dx + \sum_{j=1}^s \int_{\Gamma_{j'}} p_1 dx + \sum_{j=1}^r \int_{\Gamma_{j''}} p_1 dx = 0,$$

whence follows the relation

$$(5) \quad \sum_{j=1}^m A_{j0} + \sum_{j=1}^s m_j B_{j0} - \sum_{j=1}^r C_{j0} = 0.$$

4. Fuchs' relation for the equation (1) can be derived immediately from relation (5).

An indicial equation (1) at $\mathfrak{p}_j$ is given by

$$\lambda(\lambda-1)\cdots(\lambda-n+1) + A_{j_0} \cdot \lambda(\lambda-1)\cdots(\lambda-n+2) + \cdots = 0,$$

which can be rewritten in the form

$$\lambda^n + [-1 - 2 - \cdots - (n-1) + A_{j_0}] \lambda^{n-1} + \cdots = 0.$$

Therefore, if $\lambda_{j_1}, \dots, \lambda_{j_n}$ are its roots, we have

$$\begin{aligned} \sum_{k=1}^n \lambda_{jk} &= 1 + 2 + \cdots + (n-1) - A_{j_0} \\ &= \frac{n(n-1)}{2} - A_{j_0}. \end{aligned}$$

From this follows that

$$\sum_{j=1}^m \sum_{k=1}^n \lambda_{jk} = \frac{mn(n-1)}{2} - \sum_{j=1}^m A_{j_0}.$$

Now, from (3), (4) and (5), we get

$$\begin{aligned} \sum_{j=1}^m A_{j_0} &= - \sum_{j=1}^s m_j B_{j_0} + \sum_{j=1}^r C_{j_0} \\ &= - \frac{n(n-1)}{2} \sum_{j=1}^s (m_j - 1) + rn(n-1). \end{aligned}$$

Since a well-known formula of Hurwitz

$$\sum_{j=1}^s (m_j - 1) = 2(r + p - 1)$$

holds, we have

$$\begin{aligned} \sum_{j=1}^m A_{j_0} &= -n(n-1)(r+p-1) + rn(n-1) \\ &= -n(n-1)(p-1). \end{aligned}$$

Thus will follow the generalized Fuchs' relation:

$$\sum_{j=1}^m \sum_{k=1}^n \lambda_{jk} = n(n-1) \left(\frac{m}{2} + p - 1 \right).$$

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