# REMARKS ON HAAR MEASURE

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**1.** In this note we make simple remarks on Haar measures, which give an alternating expression of their uniqueness.

Let $G$ be a locally compact topological group and $K$, $Z$ be closed subgroups of $G$ into which $G$ splits, i.e. $G=K\cdot Z$, $K\cap Z=e$.

Denote by $dg$, $dk$, $dz$ left invariant Haar measures on $G$, $K$, $Z$ respectively, where $k\in K$, $z\in Z$, then it is known that relations $dg=\varDelta(g)dg^{-1}$, $dz=\delta(z)dz^{-1}$, $dk=\delta'(k)dk^{-1}$ hold in which $\varDelta(g)$, $\delta(z)$, $\delta'(k)$ are continuous one dimensional representations of $G$, $Z$, $K$, respectively.

**2.** Our first remark is the following:

*For any* $f\in L^1(G)$ *we have*

$$\int f(g)\,dg=\iint f(kz)\frac{\varDelta(z)}{\delta(z)}dz\,dk=\iint f(zk)\frac{\varDelta(k)}{\delta'(k)}dk\,dz$$

*when measures* $dg$, $dk$ *and* $dz$ *are properly normalized.*

*Proof*. Let $f(g)$ be a continuous function on $G$ which has a compact carrier. We prove the above relations for such functions since we lose no generalities by this restriction. Put

$$\bar{f}(k)=\int f(kz)\,\omega(z)\,dz$$

where $\omega(z)=\varDelta(z)/\delta(z)$, then $\bar{f}(k)$ is continuous on $K$ and of carrier compact. Conversely, any such function $\varphi(k)$ on $K$ i.e. continuous on $K$ and of carrier $C$ compact ($C\subset K$), can be represented as an $\bar{f}$ for some $f(g)$. In fact, let $C'$ be an open set of $G$ containing $C$ and whose closure is compact. If we take a function $f_1(g)$, continuous on $G$, of carrier compact, non-negative and $f_1(g)\neq 0$ for $g\in C'$, then the function $f(g)$ defined by

$$f(g)=f(kz)=\begin{cases}\varphi(k)f_1(kz)\Big/\int f_1(kz)\,\omega(z)\,dz, & \text{if the denominator} \neq 0,\\ 0, & \text{otherwise,}\end{cases}$$

is continuous, of carrier compact and satisfies the relation $\bar{f}(k)=\varphi(k)$. Hence, for any $\varphi(k)$, there exists at least one $f(g)$, for which $\bar{f}=\varphi$. Consider the correspondence

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$$\varphi \rightarrow \int f(g)\, dg$$

where $f(g)$ stands for $\varphi(k)$ in the above situation, i.e. $\bar{f}=\varphi$. Then, by the additivity, this correspondence can be uniquely determined if $\varphi=0$ implies

$$\int f(g)\, dg = 0.$$

We show that the last is true for the sake of completeness. Firstly, the condition

$$\bar{f} = \int f(kz)\, \omega(z)\, dz = 0$$

is equivalent to

$$\int f(gz)\, \omega(z)\, dz = 0 \qquad \text{for any} \quad g \in G.$$

For, if we decompose $g = k'z'$ then

$$\int f(gz)\, \omega(z)\, dz = \int f(k'z'z)\, \omega(z)\, dz = \int f(k'z)\, \omega(z')^{-1}\, \omega(z)\, dz = \omega(z')^{-1} \int f(k'z)\, \omega(z)\, dz.$$

Thus, if $\bar{f}=0$, then

$$\int u(g)\Big(\int f(gz)\, \omega(z)\, dz\Big)\, dg = 0$$

for any measurable $u(g)$. But

$$\int u(g)\Big(\int f(gz)\, \omega(z)\, dz\Big) dg = \int \omega(z)\, dz \int f(gz)\, u(g)\, dg$$

$$= \int \omega(z)\, dz \Big(\int f(g)\, u(gz^{-1})\, \Delta(z)^{-1}\, dg\Big) = \int f(g)\Big(\int u(gz^{-1})\, \delta(z)^{-1}\, dz\Big)\, dg$$

$$= \int f(g)\Big(\int u(gz)\, dz\Big)\, dg.$$

On the other hand, the function $u(g)$ can be taken such that

$$\int u(gz)\, dz = 1$$

for $g$ belonging to the carrier of $f(g)$. Hence we obtain

$$\int f(g)\, dg = 0.$$

Above correspondence

$$\varphi \rightarrow \int f(g)\, dg = I(\varphi)$$

is obviously left invariant, i.e.

$$I(\varphi(k_0{}^{-1}k)) = I(\varphi(k))$$

and hence

$$I(\varphi) = \int \varphi(k)\, dk.$$

Thus

$$\int f(g)\, dg = \iint f(kz)\, \omega(z)\, dz\, dk.$$

By interchanging $K$ and $Z$, we have

$$\int f(g)\, dg = \iint f(zk)\, \omega'(k)\, dk\, dz.$$

**3.** Our second remark is the following:

*Let $d'g$ be a measure on $G$ having the same field of measurable sets as that of $dg$, i.e. every continuous function of compact carrier is measurable with respect to $d'g$. If $d'kg = d'g$ and $d'gz = \Delta(z)\, d'g$ where $k \in K$, $z \in Z$, then $d'g$ is equal to $dg$ up to a constant factor.*

*Proof.* The proof proceeds in the same way as in the proof of uniqueness of Haar measure. Let $\varphi(g)$, $\theta(g)$ be any two continuous functions of compact carriers. We consider the integral

$$I = \int \varphi(g)\, dg \int \theta(h)\, d'h = \int \theta(h) \Big( \int \varphi(g)\, dg \Big)\, d'h.$$

In the inner integral of the last member, we replace $g$ by $k'^{-1} g z'^{-1}$ where $h = k'z'$, then we get

$$I = \int \theta(h) \Big( \int \varphi(k'^{-1} g z'^{-1})\, \Delta(z'^{-1})\, dg \Big)\, d'h.$$

Uniqueness and continuity of the decomposition $h = k'z'$ imply that both elements $k' = \kappa(h)$ and $z' = \zeta(h)$ are continuous in $h$, and thus $\varphi(k'^{-1} g z'^{-1})\, \Delta(z'^{-1})$ is a continuous function in $h$. By the theorem of Fubini, we can change the order of integration and we get

$$I = \int \Big( \Delta(z')^{-1} \theta(h)\, \varphi(k'^{-1} g z'^{-1})\, d'h \Big)\, dg.$$

Again, in the inner integral, we replace $h$ by $khz$ where $g = kz$, then $k' = \kappa(h)$ and $z' = \zeta(h)$ are replaced by $\kappa(khz) = k\kappa(h) = kk'$ and $\zeta(khz) = \zeta(h)z = zz'$, respectively. From our assumption for $d'h$

$$I = \int \Big( \int \Delta(z')^{-1} \Delta(z)^{-1}\, \theta(khz)\, \varphi(k'^{-1} z'^{-1}) \Delta(z)\, d'h \Big)\, dg$$

$$= \int \Delta(z')^{-1}\, \varphi(k'^{-1} z'^{-1}) \Big( \int \theta(khz)\, dg \Big)\, d'h.$$

By our first remark this is equal to

$$= \int \Delta(z')^{-1} \varphi(k'^{-1}z'^{-1}) \left( \iint \theta(khz)\, \omega(z)\, dz\, dk \right) d'h.$$

Finally we replace in the above integral $z$ by $z'^{-1}z$ and $k$ by $kk'^{-1}$, then

$$I = \int \Delta(z')^{-1} \varphi(k'^{-1}z'^{-1}) \left( \iint \theta(kz)\, \omega(z')^{-1}\, \omega(z)\, \delta'(k')^{-1}\, dz\, dk \right) d'h$$

$$= \int \theta(g)\, dg \int \Delta(z')^{-1}\, \omega(z')^{-1}\, \delta'(k')^{-1}\, \varphi(k'^{-1}z'^{-1})\, d'h.$$

Thus we have

$$\int \varphi(g)\, dg \int \theta(h)\, d'h = \int \theta(g)\, dg \int \Delta(z')^{-1}\, \omega(z')^{-1}\, \delta'(k')^{-1}\, \varphi(k'^{-1}z'^{-1})\, d'h.$$

If we take as $\theta$ a fixed function $\theta_0$ for which

$$\int \theta_0(g)\, dg \neq 0,$$

then we have

$$\int \Delta(z')^{-1}\, \omega(z')^{-1}\, \delta'(k')^{-1}\, \varphi(k'^{-1}z'^{-1})\, d'h = c \int \varphi(g)\, dg$$

where

$$c = \int \theta_0(g)\, d'g \Big/ \int \theta_0(g)\, dg = \text{const.}$$

Now put

$$\varphi(g) = \varphi(kz) = \Delta(z)^{-1}\, \omega(z)^{-1}\, \delta'(k)^{-1}\, \psi(k^{-1}z^{-1})$$

for any continuous function $\psi(g)$ with a compact carrier, then we have

$$\int \psi(g)\, d'g = c \int \Delta(z)^{-1}\, \omega(z)^{-1}\, \delta'(k)^{-1}\, \psi(k^{-1}z^{-1})\, dg$$

$$= c \int \Delta(z)^{-1}\, \delta'(k)^{-1}\, \psi(k^{-1}z^{-1})\, dz\, dk = c \int \psi(g)\, dg.$$

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