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SOME GAP THEOREMS

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1. If a trigonometric series with gaps

$$(1.1) \quad \sum (a_k \cos n_k x + b_k \sin n_k x),$$

$$\frac{n_{k+1}}{n_k} \geq \lambda > 1$$

is a Fourier series of a periodic bounded function, then the series converges absolutely, that is

$$\sum (|a_k| + |b_k|) < \infty.$$

This theorem is due to S.Sidon[1] and is well known. We shall prove in § 2, that the similar theorem is also valid for non-harmonic trigonometric series, which seems us to be not yet published. The idea of proof is to use an elegant devise which is used by Hartman[2] in proving the divergence of (1.1)

when $\sum (a_k^2 + b_k^2) = \infty.$

We consider the more general series

$$(1.2) \quad \sum_{n=1}^{\infty} c_n \varphi(\lambda_n x),$$

where $\varphi(x)$ is a periodic function with period 2π and satisfies some continuity conditions. Mean convergence and almost everywhere convergence of (1.2) were discussed by several writers.

M.Kac[3] has proved that if $\{\lambda_n\}$ is an increasing sequence of integers which satisfies Hadamard gap condition

$$(1.3) \quad \lambda_{n+1}/\lambda_n \geq \lambda > 1,$$

and

$$(1.4) \quad \sum (a_n^2 + b_n^2) < \infty,$$

then the series (1.2) is convergent almost everywhere in $(-\infty, \infty)$, and is L_2 -mean convergent in every finite interval, provided that $\varphi(x)$ satisfies the Lipschitz condition:

$$|\varphi(x) - \varphi(x')| = O(|x-x'|^{\alpha}), \quad 0<\alpha<1,$$

uniformly.

We shall prove in § 3, this theorem also holds even if the integral character of λ_n is not supposed. In the proof of mean convergence, we shall again make use of the Hartman's device above mentioned, and the almost everywhere convergence then can be proved in quitely similar manner.

2. __Theorem 1.__ __Let the series__

$$(2.1) \quad \sum_{k=1}^{\infty} (a_k \cos \lambda_k x + b_k \sin \lambda_k x),$$

$$\lambda_{k+1}/\lambda_k \geq \lambda > 1$$

__be a Fourier series of a bounded__ S^2-a.p. __function__ $f(x)$, __then__

$$(2.2) \quad \sum (|a_k| + |b_k|)$$

is convergent.

Proof. As usual, we may restrict ourselves to the case of purely cosine or sine Fourier series. We now treat, for example, the purely cosine series. Let

$$(2.3) \quad \sigma_h(E) = \frac{1}{\pi} \int_E \frac{\sin^2(ht/2)}{ht^2/2} dt, \quad h>0$$

for any measurable set E. Then for fixed $h>0$, $\sigma_h(E)$ is a non-negative completely additive set function of Lebesque measurable set on the x -axis and moreover

$$(2.4) \quad \int_{-\infty}^{\infty} \cos \lambda x \, d\sigma_h(x) = \left(1 - \frac{|\lambda|}{h}\right), \quad |\lambda| < h,$$
$$= 0, \quad |\lambda| > h,$$

$$\int_{-\infty}^{\infty} \sin \lambda x \, d\sigma_h(x) = 0, \quad \text{for every } \lambda.$$

Thus $\{\sqrt{2} \cos \lambda_k x , \sqrt{2} \sin \lambda_k x \}$ is an ortho-normal set of functions in $(-\infty, \infty)$ with respect to the measure $\sigma_h(E)$ when $|\lambda_{k+1} - \lambda_k| \geq h, \ \lambda_k \geq h.$

Since (2.1) is an almost periodic Fourier series, we have

$$\sum |a_k^2| < \infty.$$

So by Riesz-Fisher Theorem on orthogonal series, there exists a function $f^*(x) \in L_2$, L_2 meaning here the class of functions squarily integrable with respect to $\sigma_h(E)$ in $(-\infty, \infty)$, such that

$$(2.5) \quad \lim_{n \to \infty} \int_{-\infty}^{\infty} | f^*(x) - \sum_{k=1}^{n} a_k \cos \lambda_k x |^2 d\sigma_h$$

$$= \lim_{n\to\infty} \sum_{k=n+1}^{\infty} |a_k|^2 = 0.$$

From (2.5) we can prove that

$$\underset{t}{\text{u.b}} \int_t^{t+\alpha} |f^*(x) - \sum_{k=1}^{n} a_k \cos \lambda_k x|^2 dx \to 0,$$

as $n \to \infty$,

where $\alpha = 2\pi/h$ [4]. Hence if we assume $2\pi \geq h$, then $f^*(x)$ is an S^2. a. p. function and we conclude that $f^*(x) = f(x)$ almost everywhere. Thus we get from (2.5)

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} | f(x) - \sum_{k=1}^{n} a_k \cos \lambda_k x |^2 d\sigma_h(x) = 0,$$

$$(2\pi \geq h)$$

from which it results that

$$\int_{-\infty}^{\infty} f(x) \cos \lambda_m x \, d\sigma_h$$
$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} \sum_{k=1}^{n} a_k \cos \lambda_k x \cos \lambda_m x \, d\sigma_h$$
$$= a_m \int_{-\infty}^{\infty} \cos^2 \lambda_m x \, d\sigma_h,$$

or

$$(2.7) \qquad a_m = \frac{2}{\pi} \int_{-\infty}^{\infty} f(x) \cos \lambda_m x \, d\sigma_h(x).$$

Then we can follow after Sidon's well-known method to prove our theorem. Let

$$(2.8) \qquad P_l(x) = 1 + \sum_j A_j^{(l)} \cos jx,$$

where $A_j^{(l)}$ vanishes except the case indices $j \geq 0$ is of the form

$$\pm \lambda_{k_1} \pm \lambda_{k_2} \pm \cdots \pm \lambda_{k_m}.$$

Returning to the series (7), take an integer r such that $\lambda^r > 3$,

put

$$\mu_k^{(s)} = \lambda_{kr+s}, \quad k = 1, 2, \ldots, \quad 0 \leq s \leq r-1,$$

and let

$$P_l^{(s)}(x) = \prod_{k=1}^{l} (1 + \varepsilon_k \cos \mu_k^{(s)}),$$

where $\varepsilon_k = \text{sign } a_{kr+s}$. Then since $\mu_{k+1}^{(s)} / \mu_k^{(s)} \geq \lambda^r > 3$, we obtain

$$(2.9) \qquad \sum_{k=1}^{l} |a_{kr+s}| = \int_{-\infty}^{\infty} f(x) P_l^{(s)}(x) \, d\sigma_h$$
$$\leq M \int_{-\infty}^{\infty} | P_l^{(s)}(x) | \, d\sigma_h = M \int_{-\infty}^{\infty} P_l^{(s)}(x) \, d\sigma_h(x)$$
$$= M.$$

Making $l \to \infty$, we get

$$\sum_{k=1}^{\infty} |a_{kr+s}| \leq M, \quad (s = 0, 1, \cdots r-1)$$

which results, summing up with respect to s,

$$\sum_{n=1}^{\infty} |a_n| < \infty.$$

This completes the proof.

<u>3.</u> Let $\varphi(x)$ be a periodic function with period 2π, and suppose that $\varphi(x)$ satisfies the Lipschitz condition of order α, $0 < \alpha \leq 1$ which we denote

$$\varphi(x) \in \text{Lip}\,\alpha, \quad 0 < \alpha \leq 1.$$

We consider the series

$$(3.1) \qquad \sum_{n=1}^{\infty} c_n \varphi(\lambda_n x).$$

When λ_n are integers, having Hadamard gaps (1.3), (3.1) is convergent in mean L_2 in (0.2π) and further (3.1) is convergent almost everywhere. This is due to M.Kac [5]. We shall prove the theorem for the case $\{\lambda_n\}$ is not necessarily a sequence of integers.

<u>Theorem 2. If $\varphi(x)$ is a periodic function which has the mean 0 or $\int_0^{2\pi} \varphi(x)\,dx = 0$ and belongs to Lip α $(0 < \alpha \leq 1)$ and $\{\lambda_k\}$, $k = 1, 2, \cdots$ is a sequence of positive numbers satisfying</u>

$$(3.2) \qquad \lambda_{k+1}/\lambda_k \geq \lambda > 1,$$

<u>then the convergence of the series</u>

$$(3.3) \qquad \sum_{k=1}^{\infty} c_k^2$$

<u>implies the convergence in mean L_2 of</u>

$$(3.4) \qquad \sum_{k=1}^{\infty} c_k \varphi(\lambda_k x)$$

<u>in every finite interval and also implies the convergence of (3.4) for almost all x in $(-\infty, \infty)$.</u>

<u>For the proof, we need the following lemma.</u>

<u>Lemma 1. Under the conditions of Theorem 2, we have</u>

$$(3.5) \qquad \left| \int_{-\infty}^{\infty} \varphi(\lambda_j x) \varphi(\lambda_k x) \, d\sigma(x) \right| \leq A \lambda^{-\alpha|j-k|},$$

$$j, k = 1, 2, \cdots$$

<u>where</u>

$$\sigma(x) = \frac{1}{\pi} \int_{-\infty}^{x} \frac{\sin^2(t/2)}{t^2/2} \, dt,$$

<u>and A is a constant independent of j and k.</u>

Let $\varphi(x) \sim \sum_1^\infty (a_n \cos nx + b_n \sin nx)$ let $S_n(x)$ be its n-th partial sum, and let $\sigma_n(x)$ denote the n-th Fejér mean. Then by (2.6), we have

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} | \varphi(\lambda_k x) - S_n(\lambda_k x) |^2 d\sigma(x) = 0$$

for fixed k, from which we have

$$(3.6)\quad \int_{-\infty}^{\infty}\varphi(\lambda_j x)\,\varphi(\lambda_k x)\,d\sigma(x)$$

$$= \lim_{m\to\infty}\int_{-\infty}^{\infty} S_m(\lambda_j x)\,S_m(\lambda_k x)\,d\sigma(x).$$

The right hand side is now equal to

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\sum_{n=1}^{m}\{a_n\cos\lambda_j nx + b_n\sin\lambda_j nx\}\cdot\sum_{n=1}^{m}(a_n\cos\lambda_k nx + b_n\sin\lambda_k nx)\,d\sigma x,$$

$$= \sum_{|r\lambda_j - s\lambda_k|<1}(\theta_{rs}a_r a_s + \eta_{rs}b_r b_s),$$

where $|\theta_{rs}|\leqq 1$, $|\eta_{rs}|\leqq 1$.
Let $j<k$. For given integer s, there is at most one integer r which satisfies the inequality $|r\lambda_j - s\lambda_k|<1$. Consequently we have, by the Schwarz inequality

$$\Big|\sum_{|r\lambda_j - s\lambda_k|<1}(\theta_{rs}a_r a_s + \eta_{rs}b_r b_s)\Big|$$

$$\leqq C\Big[\sum_{r\geqq\lambda_k/\lambda_j}(a_r^2+b_r^2)\Big]^{\frac12}\cdot\Big[\sum_{s=1}^{\infty}(a_s^2+b_s^2)\Big]^{\frac12}$$

$$(3.7)\quad \leqq C'\Big[\sum_{r\geqq\lambda_k/\lambda_j}(a_r^2+b_r^2)\Big]^{1/2}.$$

But

$$\sum_{r=n+1}^{\infty}(a_r^2+b_r^2)\leqq\sum_{r=1}^{n}\frac{r}{n+1}(a_k^2+b_r^2)+\sum_{r=n+1}^{\infty}(a_r^2+b_r^2)$$

$$=\int_{-\pi}^{\pi}\{\varphi(x)-\sigma_n(x)\}^2dx,$$

which is, by Bernstein's result that $\varphi(x)-\sigma_n(x)=O(n^{-\alpha})$ holds uniformly,

$$\leqq D n^{-2\alpha},$$

D being a constant independent of n.
Hence we have by (3.7)

$$\Big|\int_{-\infty}^{\infty}\varphi(\lambda_j x)\,\varphi(\lambda_k x)\,d\sigma(x)\Big|\leqq A\,(\lambda_k/\lambda_j)^{-\alpha}$$

$$\leqq A\,\lambda^{-\alpha(j-k)}$$

which proves the lemma.

We shall now prove Theorem 2. We have, by Lemma 1,

$$\int_{-\infty}^{\infty}\Big|\sum_{k=n}^{m}c_k\,\varphi(\lambda_k x)\Big|^2d\sigma(x)$$

$$\leqq\sum_{j,k=n}^{m}|c_j c_k|\Big|\int_{-\infty}^{\infty}\varphi(\lambda_j x)\,\varphi(\lambda_k x)\,d\sigma(x)\Big|$$

$$\leqq A\sum_{j,k=n}^{m}\frac{|c_j||c_k|}{\lambda^{\alpha|j-k|}}\leqq\frac12 A\sum_{j,k=n}^{m}\frac{|c_j|^2+|c_k|^2}{\lambda^{\alpha|j-k|}}$$

$$\leqq A\sum_{j=n}^{m}|c_j|^2\sum_{k=n}^{m}\frac{1}{\lambda^{\alpha|j-k|}}$$

$$(3.8)\quad\leqq 2A\sum_{n}^{m}|c_j|^2\sum_{r=1}^{\infty}\frac{1}{\lambda^{\alpha r}}$$

which is tend to zero as $m\to\infty$, $n\to\infty$ by the assumption that $\sum|c_j|^2<\infty$.

Similarly every positive integer h, we have

$$\lim_{m,n\to\infty}\int_{-\infty}^{\infty}\Big|\sum_{k=n}^{m}c_k\,\varphi(\lambda_k x)\Big|^2d\sigma(hx)=0,$$

that is,

$$(3.9)\quad\lim_{m,n\to\infty}\int_{-\infty}^{\infty}\Big|\sum_{n}^{m}c_k\varphi(\lambda_k x)\Big|^2\frac{\sin^2hx}{h^2x^2}dx=0$$

Now we take any two numbers a and b $(a<b)$, and we choose h such that $|ha|,|hb|<\pi/2$. Then

$$\frac{\sin^2hx}{h^2x^2}\geqq\Big(\frac{2}{\pi}\Big)^2\quad\text{for}\quad a<x<b.$$

We have

$$\int_{-\infty}^{\infty}\Big|\sum_{n}^{m}c_k\,\varphi(\lambda_k x)\Big|^2\frac{\sin^2hx}{h^2x^2}dx$$

$$\geqq\int_{a}^{b}\Big|\sum_{n}^{m}c_k\varphi(\lambda_k x)\Big|^2\frac{\sin^2hx}{h^2x^2}dx$$

$$\geqq\Big(\frac{2}{\pi}\Big)^2\int_{a}^{b}\Big|\sum_{n}^{m}c_k\,\varphi(\lambda_k x)\Big|^2dx$$

By (3.8), we get

$$(3.9)\quad\lim\int_{a}^{b}\Big|\sum_{n}^{m}c_k\,\varphi(\lambda_k x)\Big|^2dx=0,$$

in other words the series (3.1) converges in mean L_2 in every finite interval.

The almost everywhere convergence of (3.1) can be proved in quitely similar manner as in Kac's paper by using the fact (3.8), and the proof is omitted here.

(*) Received December 20, 1949.

(1) See, A.Zygmund, Trigonometrical Series, Warsaw, 1935, p.139.
(2) P.Hartman, Divergence of non-harmonic gap theorems. Duke Math. Jour., 9 (1942).
(3) M.Kac, Convergence of certain gap series, Ann. of Math., 44 (1943).
(4) F. ex. T.Kawata, A gap theorem for the Fourier series of an almost periodic function. Tohoku Math. Jour., 43 (1937).
(5) M.Kac, ibid.

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For the proof, we need the following lemma.