KŌDAI MATH. SEM. REP.
Nos. 5-6, Dec., 1949.

FABER'S POLYNOMIALS.

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(Communicated by Y. Komatu.)

§1. Fundamental Identities.

The following method in §1 can be proceded verbatin for more general and even for multiply-connected domains, but in this Note we suppose the boundary of domain is the unit circle in order to apply our results for the coefficient problem.

Let $g(z)$ be a meromorphic, schlicht and non-vanishing function in the exterior of the unit circle $|z| > 1$, and whose Laurent expansion about the point at infinity is of the form

$$(1) \qquad g(z) = z + \sum_{\nu=0}^{\infty} \frac{c_\nu}{z^\nu}.$$

Then $f(z) \equiv 1/g(1/z)$ is regular and schlicht in the unit circle $|z| < 1$ and, about the origin, it can be expanded in the form

$$(2) \qquad f(z) = z + \sum_{\nu=2}^{\infty} a_\nu z^\nu.$$

Let $P_n(z)$ $(n = 1, 2, \cdots)$ be polynomial of z of degree n, which satisfies the condition

$$(3) \qquad P_n(g(z)) = z^n + \sum_{\nu=1}^{\infty} \frac{\alpha_\nu^{(n)}}{z^\nu}.$$

Then, $P_n(z)$ is called the "Faber's polynomial" of degree n with respect to $g(z)$.[1] By means of the Cauchy's integral formula, we have

$$(4) \qquad P_n(w) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{P_n(\zeta)}{\zeta - w}\, d\zeta,$$

where w is an arbitrary point in the circle $|\zeta| < r$. Making the change of variable $\zeta = g(z)$, we get, for sufficiently large r,

$$(5) \qquad P_n(w) = \frac{1}{2\pi i}\int_{|z|=r} P_n(g(z))\, d \lg (g(z) - w).$$

On the other hand, we can easily prove

$$(6) \qquad \begin{aligned} 0 &= \frac{1}{2\pi i}\int_{|z|=r} \frac{P_n(g(z))}{z}\, dz \\ &= \frac{1}{2\pi i}\int_{|z|=r} P_n(g(z))\, d \lg z, \end{aligned}$$

by virtue of (3). Hence, substracting (6) from (5), we have

$$(7) \qquad P_n(w) = \frac{1}{2\pi i}\int_{|z|=r} P_n(g(z))\, d \lg \frac{g(z) - w}{z}$$

Now, putting

$$(8) \qquad \lg \frac{g(z) - w}{z} = -\sum_{\nu=1}^{\infty} \frac{Q_\nu(w)}{\nu} \frac{1}{z^\nu}$$

and substituting (8) into (7), we obtain

$$(9) \qquad P_n(w) = Q_n(w).$$

Since (9) holds for infinitely many values of w if we take a sufficiently large r, also does (9) hold good identically. After all, we have the following fundamental relation:[2]

$$(10) \qquad \lg \frac{g(z) - w}{z} = -\sum_{\nu=1}^{\infty} \frac{P_\nu(w)}{\nu} \frac{1}{z^\nu}$$

for an arbitrary w, the logarithm always denoting the branch which vanishes for $w = 0$ and $z = \infty$.

Putting $\zeta = 1/z$, $g(z) = 1/f(\zeta)$, and comparing the coefficients of both sides of (10), we have

$$(11) \qquad P_n(z) = n \sum_{\mu=1}^{n} \Big(\sum_{n_1 + \cdots + n_\mu = n} a_{n_1} \cdots a_{n_\mu} \Big) \frac{z^\mu}{\mu} + P_n(0),$$

and in particular $P_n(0) = n a_n$. Differentiating (10) with respect to z and making use of the same reason as above, we obtain

$$(12) \qquad \begin{aligned} &P_1(z) = z - c_0, \\ &P_{n+1}(z) + (c_0 - z)P_n(z) + \sum_{\mu=1}^{n-1} c_\mu P_{n-\mu}(z) + (n+1)c_n = 0 \\ &\qquad (n = 2, 3, \cdots) \end{aligned}$$

§2. Some Applications to the Distortion Theorems.

Putting

$$(13) \qquad F(z) = \begin{cases} f(z)/z & (z \neq 0), \\ 1 & (z = 0), \end{cases}$$

we get, from (10),

$$(14) \qquad \lg F(z) = \sum_{\nu=1}^{\infty} \frac{P_\nu(0)}{\nu} z^\nu.$$

If we consider a family of schlicht

functions $\{f(z)\}$, satisfying $|P_n(0)| \leqq 2$ $(n=1,2,\cdots)$, we can prove the following distortion theorems

$$(15) \quad |F^{(n)}(z)| \leqq \frac{(n+1)!}{(1-|z|)^{n+2}}, \quad |f^{(n)}(z)| \leqq \frac{n!\,(n+|z|)}{(1-|z|)^{n+2}},$$

where in each relation the equality sign holds only for the extremum functions of the form $f(z) = z/(1-e^{i\alpha}z)^2$

(α: real). Especially, if we put $z = 0$, we obtain

$$|a_n| \leqq n$$

for our family, which is the conjecture of Bieberbach.

Now, we consider the function $f(z)$ star-shaped in the unit circle $|z|<1$. Then, $\lg F(z)$ is expressed by the formula of Herglotz:

$$(16) \quad \lg F(z) = -2\int_0^{2\pi} \lg \frac{e^{i\theta}}{e^{i\theta}-z}\, d\mu(\theta), \qquad d\mu(\theta) \geqq 0, \quad \int_0^{2\pi} d\mu(\theta) = 1,$$

and the inequalities

$$(17) \quad |P_n(0)| \leqq 2 \quad (n=1,2,\cdots)$$

hold good. Hence, for our family we get the distortion inequalities (15). From the last inequalities, we obtain $|a_n| \leqq n$ by putting $z = 0$.[4]

If $f(z)$ is convex in $|z|<1$, we can represent it as before:

$$(18) \quad \lg f'(z) = -2\int_0^{2\pi} \lg \frac{e^{i\theta}}{e^{i\theta}-z}\, d\mu(\theta), \qquad d\mu(\theta) \geqq 0, \quad \int_0^{2\pi} d\mu(\theta) = 1;$$

and we get the following relations:

$$(19) \quad \begin{aligned} &|\lg f'(z)| \leqq 2\lg \frac{1}{1-|z|}, \\ &|P_n(0)| \leqq 2 \end{aligned} \quad (n=1,2,\cdots), \qquad \begin{aligned} &|F^{(n)}(z)| \leqq \frac{n!}{(1-|z|)^{n+1}}, \\ &|f^{(n)}(z)| \leqq \frac{n!}{(1-|z|)^{n+1}} \end{aligned} \quad (n=0,1,2,\cdots).$$

Each equality sign holds only for functions of the form $f(z) = z/(1-e^{i\beta}z)$, β : real. Putting $z=0$, we can establish the coefficient theorem $|a_n| \leqq 1$ for convex functions already proved by various ways.[5]

At the same time, we can, by means of (18), derive the relation, obtained by Marx,[6]

$$\Re \sqrt{f'(z)} \geqq \frac{1}{1+|z|},$$

which is sharpened than that obtained by himself.

(*) Received September 30, 1949.

(1) H.Grunsky: Koeffizientenbedingungen fuer schlicht abbildende meromorphe Funktionen. Math. Zeit. 45(1939), 29-61; especially, § 3.

(2) Recently I found that M.Schiffer has proved the relation (10), but I had obtained another proof of it. Cf. M.Schiffer: Faber's polynomials in the theory of univalent functions. Bull. Amer. Math. Soc. 54(1948), 503-517.

(3) L.Bieberbach: Ueber die Koeffizienten diejenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreis vermitteln. Sitzungsb. preuss. Akad. Wiss. Berlin (1916), 940-945.

(4) Cf. R.Nevanlinna: Ueber die konforme Abbildung von Sterngebieten. Oeversikt av Finska Vetenakaps-Soc. Foerh. (A) 63 (1920-1), 1-21.

(5) E.g. K.Loewner: Untersuchungen ueber die Verzerrung bei konformen Abbildungen des Einheitskreises $|z|<1$, die durch Funktionen mit nicht verschwindenden Ableitung geliefert werden. Leipziger Berichte. 69 (1917), 89-106.

(6) A.Marx: Untersuchung ueber schlichte Abbildungen. Math. Ann. 107 (1932), 40-67.

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