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# ALGEBRAIC FUNCTIONS*

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## PREFACE.

The whole thesis consists of three chapters. In chapter I, we deal with the structure of Rational Functions at various places of the Riemann - Surface of an Algebraic Function and deduce some new results. It also serves as an introduction to the rest of the chapters.

The second chapter consists of three parts; the first one gives three theorems concerning the structure of the branching of the Riemann - Surface of the Fundamental Equation. The second one deals with the investigation of the differential coefficient of an Algebraic Function. This produces a result which is an improvement over the result already published by Beatty. The third part is merely to show how to extend these results to all algebraically closed fields of characteristic zero.

Chapter III consists of two main parts. The first part is a proof of the Riemann - Roch Theorem, and the second is its applications. A new method of proof for the Riemann - Roch Theorem based mostly on the ideas of analysis is given. In doing so important new theorems are introduced. In the second part it is demonstrated that some of the well-known results in the Algebraic Function Theory are easily deduced by the application of the new method.

References to various chapters are given at the end of the thesis in the Bibliography.

## CHAPTER I.

### THEOREMS ON THE STRUCTURE OF RATIONAL ALGEBRAIC FUNCTIONS.

1. Let $f(z,u) \equiv f_0 u^n + f_1 u^{n-1} + \cdots + f_n = 0$ be an irreducible algebraic equation ( $f_s$ are rational functions in $z$ with coefficients in the field of complex numbers $k$ ), defining the field of rational functions $k(z,u)$. If $a, b \in k$ is a solution of

$$f(z,u)=0$$

then there exists a formal power series, solution of

$$f(z,u)=0$$

in the form

$$(a)\quad \begin{cases} z-a = t^{r} \\ u-b = t^{\sigma}(a_\sigma + a_{\sigma+1}t + \cdots) \end{cases} \qquad a_\sigma \neq 0$$

where $r, \sigma$ are integers and $\sigma > 0$. Such a pair of functions (a) is called a place-representation of the Riemann-Surface of the Algebraic Function.

2. <u>Value of a Rational Function at a Place.</u>

Let a Rational Function $R(z,u) \in k(z,u)$. Let $\pi$ be given by a place-representation (a). In virtue of the substitution (a), we have at $\pi$,

$$R(z,u) = t^{\rho}(a_0 + a_1 t + \cdots)$$

where

$$a_0 \neq 0.$$

If $\rho > 0$, then $R(\pi)$ is said to have zero of order $\rho$ at the place $\pi$, and $\rho < 0$ is said to have a pole of order $-\rho$ at the place, and $\rho = 0$, $R(\pi)$ is regular.

3. At every place of the Riemann-Surface of the Algebraic Function, any rational function $R(z,u)$ has either a pole, or a zero of some definite order or is regular in the sense of paragraph 2. Also every rational function $\eta$ has a unique divisor except for a constant. This can be represented symbolically as

$$\eta \backsim \frac{P_1^{\gamma_1}\cdots P_t^{\gamma_t}}{Q_1^{s_1}\cdots Q_\ell^{s_\ell}}$$

where $P_1 \cdots P_t$ are places at which the rational function $\eta$ has zeros of order $\gamma_1, \cdots, \gamma_t$ and $Q_1, \cdots Q_\ell$ are places at which it has poles of order $s_1, \cdots, s_\ell$

4. At every cycle $Q$ of $Q^s$ of the denominator, the expansion for the rational function $\eta$ has the form,

$$\eta = \frac{\beta}{t^s} + \cdots\cdots$$

where $\beta$ is a constant different from zero. At other cycles this expansion has the form,

$$\eta = \alpha + \beta t^{\gamma} + \cdots$$

where $\beta$ is again different from zero and $\alpha$ is a constant which vanishes at the factors of the numerator of the divisor of $\eta$. If $\alpha = 0$ then $\eta$ has zero of order $\gamma$ at $P$. In such cases we have the relation that the sum of the orders of a rational function $\eta$ is zero. That is

4. (i) $\sum_{\gamma} \gamma_i - \sum_{\delta} \delta_i = 0$

the first summation is over zero places, and the second over poles.

In cases where

4. (ii) $\eta = \alpha + a_{\sigma} t^{\sigma} + \cdots$

where $\alpha \neq 0$ and $\sigma$ is a positive number not equal to one, nothing is known so far about the nature of $\sigma \delta$. First of all we note that there are only a finite number of places of the type 4. (ii). For $\eta$ we shall denote the sum

$$\sum_{\sigma} (\sigma - 1) \equiv S_{\eta}$$

THEOREM I.

For every rational algebraic function $\eta$ we have,

$$S_{\eta} \equiv \sum_{\sigma} (\sigma - 1) \geqq 2p$$

where $\sigma$ is as defined in 4. (ii), and the summation extends to all such places, and $p$ is the genus of the algebraic equation I.

PROOF:-

Applying the Invariant property of the genus number.[1], we have,

4. (iii) $\sum_{\delta}(\delta_i - 1) + \sum_{\gamma}(\gamma_i - 1) + \sum_{\sigma}(\sigma - 1) - 2\sum_{\delta} \delta_i = I$

where $I$ is an invariant and is equal to $2p-2$, and the other summations have already been defined. On simplification, by using the property 4. (i), 4. (iii) becomes,

4. (iv) $\sum_{\sigma}(\sigma - 1) = 2p - 2 + \ell + t$

where $\ell$ is the number of factors of the denominator and $t$ that of the numerator of the divisor of $\eta$. We also know that $p \geqq 0$ and that if there should exist any $\eta$ at all then,

$$\ell \geqq 1,$$
$$t \geqq 1.$$

Hence we have,

$$\sum(\sigma - 1) \geqq 2p$$

COROLLARY I.

$\sum(\sigma - 1) = 0$, if and only if $p = 0$, then $\ell = t = 1$

COROLLARY II.

$\sum(\sigma - 1) = \ell + t$, if and only if $p = 1$.

COROLLARY III.

If there exists a rational function $\eta$ with a single simple pole, then the value of the genus $p$ must be zero.

PROOF:-

If $\eta$ has a single pole, then for every constant $\alpha$ the difference $\eta - \alpha$ has a single simple zero. The expansion of $\eta$ at a place where it is finite has the form, $\eta = \eta_0 + ct + \cdots$ and the constant $c$ is different from zero, since otherwise $\eta - \eta_0$ would have a double zero. At the pole of $\eta$ we have,

$$\eta = \frac{d}{t} + \cdots \qquad d \neq 0$$

Hence,

$$S_{\eta} \equiv \sum(\sigma - 1) = 0$$

and

$$\ell = t = 1.$$

Applying the result 4. (iv) we have, $p = 0$.

5. THEOREM II.

For an adjoint function of a polynomial algebraic equation of degree $n$ which has single sheets at infinity,

$$\sum_{\sigma}(\sigma - 1) > 2p + n - 2.$$

PROOF:-

The divisor on which an adjoint function is built is $D^2/X$, where $D$ is the divisor which is the product of the cycles at $z = \infty$ and $X = \prod P^{\nu - 1}$ the divisor of the branch cycles, $P$ is a branch cycle, and $\nu$ is the number of roots $u_i$ of the equation $f(z\ u) = 0$ furnished by it and the product is taken for all of the branch cycles. If $\eta$ is an adjoint function then, $\eta \backsim \frac{D^2}{X} P_1^{\lambda_1} \cdots P_t^{\lambda_t}$

where $\lambda$s are positive integers. We know that applying 4. (i) we have,

5. (i) $\sum_{\lambda} \lambda + 2n - \sum_{\nu}(\nu - 1) = 0$

and

5. (ii) $n + \sum_{\lambda}(\lambda - 1) + \sum_{\nu}(\nu - 2) + \sum_{\sigma}(\sigma - 1) - 2\sum_{\nu}(\nu - 1)$
$$= 2p - 2$$

$\sum_{\sigma}(\sigma-1)$ has the same meaning as in 4. (ii). Simplifying 5 (ii) using the result 5 (i), we have

$$\sum(\sigma-1) = 2p-2+n+t+\sum(\nu-1)-\sum(\nu-2)$$

where $t$ is an integer or zero, and

$$\sum(\nu-1) > \sum(\nu-2).$$

Therefore

$$\sum_{\sigma}(\sigma-1) > 2p-2+n \ .$$

6. <u>THEOREM III.</u>

If the divisor of an algebraic function $\eta$ is increased by introducing in the denominator and numerator factors $P$ and $Q$ either different from or equal to those of $\eta$, such that the sum of the indices of the extra factors so introduced in the numerator and the denominator is $n$ each, and if there exists a rational function $\xi$ for the new divisor, then

$$S_\xi - S_\eta \leqq 2n \ .$$

<u>PROOF:-</u>

6. (i) Let

$$\eta \backsim \frac{P_1^{\lambda_1}\cdots P_t^{\lambda_t}}{Q_1^{\mu_1}\cdots Q_\ell^{\mu_\ell}}$$

where

$$\sum_{\lambda}\lambda = \sum_{\mu}\mu$$

6. (ii) and $S_\eta \equiv \sum_{\sigma}(\sigma-1) = 2p-2+\ell+t.$

(Refer result 4. (iv))

Let

$$\xi \backsim \frac{P_1^{\lambda_1'}\cdots P_{t'}^{\lambda_{t'}'}}{Q_1^{\mu_1'}\cdots Q_{\ell'}^{\mu_{\ell'}'}}$$

where

$$\lambda_i' \geqq \lambda_i \quad (\text{for } i=1,\cdots,t),$$
$$\mu_i' \geqq \mu_i \quad (\quad " \quad i=1,\cdots,\ell)$$

and

$$t' \leqq t+n,$$
$$\ell' \leqq \ell+n$$

and

$$\sum_1^{t'}\lambda_i' = \sum_1^{t}\lambda_i + n,$$
$$\sum_1^{\ell'}\mu_i' = \sum_1^{\ell}\mu_i + n \ .$$

Following 6. (ii) we have for $\xi$,

6. (iii) $S_\xi \equiv \sum_{\sigma'}(\sigma'-1) = 2p-2+\ell'+t'.$

From 6. (iii) and 6. (ii) we have,

$$S_\xi - S_\eta = (\ell'-\ell)+(t'-t)$$

and since

$$\ell'-\ell \leqq n,$$
$$t'-t \leqq n,$$

we have,

$$S_\xi - S_\eta \leqq 2n.$$

<u>COROLLARY I.</u>

6. (iv) If $P_1\cdots P_t.P_{t+1}$ are all different and distinct from $Q_1\cdots Q_\ell.Q_{\ell+1}$ which are also different

and if $\xi \backsim \dfrac{P_1^{\lambda_1}\cdots P_t^{\lambda_t}P_{t+1}}{Q_1^{\mu_1}\cdots Q_\ell^{\mu_\ell}Q_{\ell+1}}$

then $S_\xi - S_\eta = 2$ .

<u>COROLLARY II.</u>

If the conditions 6. (iv) are satisfied except

$$P_{t+1} = P_i \qquad i \leqq t,$$
$$\text{or} \quad Q_{\ell+1} = Q_i \qquad i \leqq \ell$$

then $S_\xi - S_\eta = 1$ .

<u>COROLLARY III.</u>

If conditions 6. (iv) are satisfied except

$$P_{t+1} = P_i \qquad i \leqq t,$$
$$Q_{\ell+1} = Q_i \qquad i \leqq \ell$$

then $S_\xi - S_\eta = 0$

<u>COROLLARY IV.</u>

If conditions 6. (iv) are satisfied except

$$P_{t+1} = Q_i \qquad i \leqq \ell,$$
$$Q_{\ell+1} = P_i \qquad i \leqq t$$

then $S_\xi - S_\eta = 0$

<u>COROLLARY V.</u>

If conditions 6. (iv) are satisfied except

$$P_{t+1} = Q_i \qquad i \leqq \ell$$

then $S_\xi - S_\eta = 1$ .

## CHAPTER II. MISCELLANEOUS THEOREMS

### PART I.

### I. SOME THEOREMS CONCERNING THE STRUCTURE OF THE BRANCHING OF THE FUNDAMENTAL EQUATION.

#### INTRODUCTION

Let $u$ be an integral algebraic function of $z$ defined by an irreducible polynomial equation, $f(z,u)=0$ of degree $n$ in $u$.

Denote by $f_{\beta}^{\alpha}(a,b)$ the value of $\left[\frac{\partial^{\alpha+\beta} f(z,u)}{\partial z^{\alpha}\partial u^{\beta}}\right]_{\substack{z=a\\u=b}}$ at the place $P(a,b)$ of the Riemann-Surface, and

$$f^{\alpha}(a,b)=\left[\frac{\partial^{\alpha}}{\partial z^{\alpha}} f(z,u)\right]_{\substack{z=a\\u=b}},\quad f_{\beta}(a,b)=\left[\frac{\partial^{\beta}}{\partial u^{\beta}} f(z,u)\right]_{\substack{z=a\\u=b}}.$$

Let ${}^{m}C_{r,\sigma}$ be the notation for the conditions to be satisfied by $f(z,u)=0$ in order to have the following place representation at the place $P$.

II (i)
$$z-a=t^{r}$$
$$u-b=a_{\sigma}t^{\sigma}+a_{\sigma+1}t^{\sigma+1}+\cdots \qquad (a_{\sigma}\neq 0)$$

where $t$ is the local parameter and $m$ is a positive integer denoting the $m$-ple root of $f(z,u)=0$.

From $f(z,u)=0$, calculate the differential coefficient of $u$ with respect to $z$ at the place representation given by II (i).

Then,
$$\frac{du}{dz}=\frac{1}{r\cdot t^{r-1}}\left(\sigma a_{\sigma}t^{\sigma-1}+(\sigma+1)a_{\sigma+1}t^{\sigma}+\cdots\right).$$

If $\sigma<r$, then $\frac{du}{dz}$ will have a principal part at the place $P$, and is a function of the local parameter '$t$'. Let $D^{m}_{r,\sigma}(z-a,u-b)$ be the principal part of $\frac{du}{dz}$ at the place representation given by II (i) where $f(a,u)=0$ has a $m$-ple has a $m$-ple root. The principal part is therefore a rational function of the base elements $(z-a)$ and $(u-b)$.

### THEOREM I.

Let $f(a,u)=0$ have roots of multiplicity $\nu$ and let $(s_1,r_1),(s_2,r_2)\cdots(s_l,r_l)$ be $l$ different sets of positive integers such that the two numbers in each set are prime[3] to each other; further let them satisfy the following conditions:-

II (ii) $$\frac{s_1}{r_1}\leqq\frac{s_2}{r_2}\leqq\cdots\leqq\frac{s_{l-1}}{r_{l-1}}\leqq\frac{s_l}{r_l},$$

II (iii) $$r_1+r_2+\cdots+r_{l-1}+r_l=\nu$$

then the necessary and sufficient conditions that $f(z,u)=0$ should have the scheme of branching as in II (ii) at $z=a$ of the Riemann-Surface viz.,

$$z-a=t^{r_i}$$
$$u-b=a_{s_i}t^{s_i}+\cdots \qquad (i=1,\cdots,l;\ a_{s_i}\neq 0)$$

are

II (iv) $f_{\beta}^{\alpha}=0$ for all integral values of $\alpha$ and $\beta$ (including $0$ values) such that
$$r_i\alpha+s_i\beta<\nu s_i-(r_1+\cdots+r_{i-1})s_i+(s_1+\cdots+s_{i-1})r_i$$
$$\text{for } (i=1,\cdots,l).$$

Of these some of the conditions may be repeated.

II (v) And
$$f_{\beta_i}^{\alpha_i}\neq 0 \quad \text{for} \quad \begin{aligned}\alpha_i&=s_1+\cdots+s_i\\ \beta_i&=\nu-(r_1+\cdots+r_i)\\ i&=0,1,\cdots,l.\end{aligned}$$

Proof:-

(a) We shall first prove the theorem for II (ii) inequalities and then extend it to equalities.

Suppose $\frac{s_1}{r_1}<\frac{s_2}{r_2}<\cdots<\frac{s_l}{r_l}$.

The conditions are necessary, for suppose $f(z,u)=0$ has the scheme of branching as in II (ii) at $z=a$, $u=b$ then by Newton's Polygon Theorem it must be capable of being represented as a polygon with vertices ($\alpha_i,\beta_i$) given by

$$\alpha_i=s_1+\cdots+s_i,$$
$$\beta_i=\nu-(r_1+\cdots+r_i).$$

Let $(\alpha_0=0,\ \beta_0=\nu),(\alpha_1,\beta_1),\cdots,(\alpha_l,\ \beta_l=0)$ be the vertices of the Newton's Polygon in the $\alpha,\beta$ plane, we have then the following relations:-

II (vi)
$$r_1=\nu-\beta_1,$$
$$r_2=\beta_1-\beta_2,$$
$$\cdots=\cdots,$$
$$r_{l-1}=\beta_{l-2}-\beta_{l-1},$$
$$r_l=\beta_{l-1}-\beta_l$$
and

II (vii)
$$\begin{aligned} s_1 &= \alpha_1, \\ s_2 &= \alpha_2 - \alpha_1, \\ \cdots &= \cdots\cdots, \\ s_{\ell-1} &= \alpha_{\ell-1} - \alpha_{\ell-2}, \\ s_\ell &= \alpha_\ell - \alpha_{\ell-1}. \end{aligned}$$

From II (vi) and II (vii) we have,

$$\alpha_i = s_1 + \cdots + s_i,$$
$$\beta_i = \nu - (r_1 + \cdots + r_i) \qquad i = 0, 1, \cdots, \ell.$$

Equation to the line joining the two points $(\alpha_{i-1}, \beta_{i-1})$, $(\alpha_i, \beta_i)$ in the $\alpha, \beta$ plane is,

II (viii)
$$r_i\alpha + s_i\beta = \nu s_i - (r_1 + \cdots + r_{i-1})s_i + (s_1 + \cdots + s_{i-1})r_i$$

The conditions[3] that there may be no other points $(\alpha_k, \beta_k)$ between the axes and the Polygon are that

$f_\beta^\alpha(a, b) = 0$ where $\alpha, \beta$ are positive integers such that,

$$r_i\alpha + s_i\beta < \nu s_i - (r_1 + \cdots + r_{i-1})s_i + (s_1 + \cdots + s_{i-1})r_i \qquad i = 1, \cdots \ell$$

And the existence of the Polygon ensures the existence of the vertices. Hence the vertices of the Polygon are $\alpha, \beta$ for which

$$f_\beta^\alpha(a, b) \neq 0$$

where
$$\alpha_i = s_1 + \cdots + s_i,$$
$$\beta_i = \nu - (r_1 + \cdots + r_i)$$
$$i = 0, 1, \cdots, \ell$$

(It is clear that $r_0 = 0 = s_0$).

Sufficiency of the conditions is easily proved for with the given conditions the polynomial equation will have Newton's Polygon of the required type.

(b) Consider the theorem when

$$\frac{s_1}{r_1} = \frac{s_2}{r_2} = \cdots = \frac{s_{\ell-1}}{r_{\ell-1}} = \frac{s_\ell}{r_\ell}$$

other conditions remaining the same as in the theorem. In this case the Newton's Polygon degenerates into one straight line with all the vertices situated on it. Then the conditions II (iv) become $f_\beta^\alpha = 0$ for all integral values of $\alpha, \beta$ (including zero values) such that

II (ix) $\quad r_\ell\alpha + s_\ell\beta < \nu \cdot s_\ell$

$$f_\beta^\alpha \neq 0 \quad \text{for} \quad \begin{aligned} \alpha_i &= s_1 + \cdots + s_i, \\ \beta_i &= \nu - (r_1 + \cdots + r_i) \end{aligned} \qquad i = 0, 1, \cdots, \ell.$$

PROOF:-

Put $i = \ell$ in the conditions II (iv) then,

II (x) $\quad r_\ell\alpha + s_\ell\beta < \nu s_\ell - (r_1 + \cdots + r_{\ell-1})s_\ell + (s_1 + \cdots + s_{\ell-1})r_\ell.$

Since

$$\frac{s_1}{r_1} = \frac{s_2}{r_2} = \cdots = \frac{s_{\ell-1}}{r_{\ell-1}} = \frac{s_\ell}{r_\ell} = \frac{s_1 + \cdots + s_{\ell-1}}{r_1 + \cdots + r_{\ell-1}}$$

we have,
$$s_\ell(r_1 + \cdots + r_{\ell-1}) = r_\ell(s_1 + \cdots + s_{\ell-1})$$

hence II (x) on substitution of this result becomes

$$r_\ell\alpha + s_\ell\beta < \nu \cdot s_\ell.$$

This is evident from the equation to the stright line joining the two extreme points of the Newton's degenerated Polygon viz., $(\alpha_0 = 0, \beta_0 = \nu)$ and $(\alpha_\ell = s_1 + \cdots + s_\ell, \beta_\ell = 0)$

II (xi) $\quad \nu\alpha + \alpha_\ell\beta = \nu\alpha_\ell.$

But $\alpha_\ell = s_1 + \cdots + s_\ell.$

II (xii) and

$$\frac{s_1}{r_1} = \frac{s_2}{r_2} = \cdots = \frac{s_\ell}{r_\ell} = \frac{s_1 + \cdots + s_\ell}{r_1 + \cdots + r_\ell} = \frac{\alpha_\ell}{\nu}.$$

since
$$r_1 + r_2 + \cdots + r_\ell = \nu.$$

Hence
$$\alpha_\ell r_\ell = \nu \cdot s_\ell \qquad \text{from II (xii)}$$

also from II (xi)

$$\cancel{\nu}\cdot\alpha + \cancel{\nu}\,\frac{s_\ell \cdot \beta}{r_\ell} = \cancel{\nu}\cdot\nu\frac{s_\ell}{r_\ell}$$
since $\nu \neq 0.$

Therefore,
$$r_\ell\alpha + s_\ell\beta = \nu s_\ell$$

Hence the result.

COROLLARY I.

If $\nu$ is a prime number and is the multiplicity of the roots of $f(a, u) = 0$

and $\quad \begin{aligned} z - a &= t^\nu \\ u - b &= a_\sigma t^\sigma + a_{\sigma+1}t^{\sigma+1} + \cdots \end{aligned}$

where $a_\sigma \neq 0$ and $\sigma < \nu$ then $f(z, u) = 0$ should satisfy at $(a, b)$, $\left[\frac{1}{2}\sigma(\sigma+1) + \nu + 1\right]$ conditions namely,

II (xiii) $\quad f_s(a, b) = 0 \qquad s = 0, 1, \cdots, (\nu - 1).$

$$f_\nu(a, b) \neq 0.$$

$$f^1 = 0, f^2 = 0, \cdots, f^{(\sigma-1)} = 0, f^{(\sigma)} \neq 0,$$

$$f_1' = 0, \quad \cdots \quad f_1^{(\sigma-1)} = 0,$$
$$\cdots\cdots$$
$$f'_{(\sigma-1)} = 0.$$

PROOF:-

At $z = a$, $u = b$

II (xiv) $$0 = f(z,u) \equiv f(a,b) + f'\cdot(z-a) + f_1\cdot(u-b) + \frac{1}{\underline{|2}}\left[f^2\cdot(z-a)^2 + 2f_1'(z-a)(u-b) + f_2\cdot(u-b)^2\right] + \cdots + \frac{1}{\underline{|p}}\left[f^p\cdot(z-a)^p + {}^pC_1 f_1^{p-1} \times (z-a)^{p-1}(u-b) + \cdots + {}^pC_r f_r^{p-r}(z-a)^{p-r}(u-b)^r + \cdots + f_p\cdot(u-b)^p\right].$$

Impose the conditions II (xiii) on the equation and construct the Newton's Polygon with the remaining equation. Then it will be seen that the Newton's Polygon will have a single side. Hence the result.

As in the beginning of this chapter let ${}^mC_{r,\sigma}$ be the symbol for the conditions to be satisfied by $f(z,u)=0$ in order to have the following place representation at the place $p$ of the Riemann-Surface of $f(z,u)=0$

$$z - a = t^r,$$
$$u - b = a_\sigma t^\sigma + a_{\sigma+1} t^{\sigma+1} + \cdots$$

where $t$ is the local parameter and $m$ is a positive integer denoting the $m$-ple root of $f(a,u)=0$.

## THEOREM II.

SOME RESULTS NOT COVERED BY THEOREM I ARE GIVEN BELOW:-

$${}^3C_{2,1} \longrightarrow \begin{cases} f_s(a,b)=0 & \text{for } s=0,1,2 \\ f_3(a,b)\neq 0 \\ f'(a,b)=0, \; f_1'(a,b)\neq 0 \end{cases}$$

$${}^4C_{4,1} \longrightarrow \begin{cases} f_s(a,b)=0 & \text{for } s=0,1,2,3 \\ f_4(a,b)\neq 0 \\ f'\neq 0 \end{cases}$$

$${}^4C_{4,2} \longrightarrow \begin{cases} f_s(a,b)=0 & \text{for } s=0.1.2.3 \\ f_4(a,b)\neq 0 \\ f'=0 \;.\; f_1'=0 \;.\; f^2\neq 0 \end{cases}$$

$${}^4C_{3,1} \longrightarrow \begin{cases} f_s(a,b)=0 & \text{for } s=0.1.2.3 \\ f_4(a,b)\neq 0 \\ f'=0 \;,\; f_1'(a,b)\neq 0 \end{cases}$$

$${}^5C_{4,1} \longrightarrow \begin{cases} f_s(a,b)=0 & \text{for } s=0.1,\cdots.4 \\ f_5(a,b)\neq 0 \\ f'=0 \;,\; f_1'(a,b)\neq 0 \end{cases}$$

$${}^5C_{4,2} \longrightarrow \begin{cases} f_s(a,b)=0 & \text{for } s=0.1,\cdots.4 \\ f_5(a,b)\neq 0 \\ f_2'=0 \;.\; f_1'=0 \;.\; f_1^2(a,b)\neq 0 \end{cases}$$

$${}^5C_{3,1} \longrightarrow \begin{cases} f_s(a,b)=0 & \text{for } s=0.1,\cdots.4 \\ f_5(a,b)\neq 0 \\ f_1'=0 \;,\; f'=0 \;.\; f_2'\neq 0 \end{cases}$$

A proof for one of the results, namely, ${}^3C_{2,1}$ is given below. The proofs for the rest of the results follow exactly the same lines of arguement. i.e. conrest of the results follow exactly the same lines of arguement. i.e. conditions that $f(z,u)=0$ may have two sheets in one cycle and each sheet an expansion beginning with the local parameter $t$, that is $\begin{matrix} z-a=t^2 \\ u-b=a_1t+a_2t^2+\cdots \end{matrix}$ $a_1\neq 0$

and $f(a,u)=0$ has 3 -ple root.

II (xv) ${}^3C_{2,1}$ are
$$f_s(a,b)=0 \quad \text{for } s=0,1,2$$
$$f_3(a,b)\neq 0$$
$$f'(a,b)=0, \quad f_1'\neq 0.$$

PROOF:-

Consider the equation $f(z,u)=0$ at $z=a$, and $u=b$ of the Riemann-Surface and get the expansion for $f(z,u)=0$ as in II (xiv). Impose the conditions as stated in II (xv) on the coefficients of the equation II (xiv). With the remaining equation construct Newton's Polygon. We find that it will give rise to the place representation

$$z-a=t^2,$$
$$u-b=a_1t+a_2t^2+\cdots \qquad a_1\neq 0.$$

## THEOREM III.

At a place representation $p$ of the Riemann-Surface $\begin{matrix} z-a=t^{r_k}, \\ u-b=a_{s_k}t^{s_k}+\cdots \end{matrix}$ $(a_{s_k}\neq 0 \quad k=1,\cdots,l)$
the terms of the lowest order in the expansion of $f(z,u)=0$ at $z=a$, $u=b$, which has the branching arrangement as given in II (ii) and II (iii)

are $(z-a)^{\alpha_{k-1}}(u-b)^{\beta_{k-1}}$ and $(z-a)^{\alpha_k}(u-b)^{\beta_k}$

with appropriate coefficients, where $(\alpha_i, \beta_i)$ in $\alpha, \beta$ plane are the vertices of the corresponding Newton's Polygon

satisfying conditions

$$\alpha_i = s_1 + \cdots + s_i,$$
$$\beta_i = \nu - (r_1 + \cdots + r_i)$$

PROOF:-

Construct Newton's Polygon for $f(z,u)=0$ at $z=a,\ u=b$. Suppose $(\alpha_i, \beta_i)$ is the $i$-th vertex of the Newton's Polygon, then the order of the corresponding term $A(z-a)^{\alpha_i}(u-b)^{\beta_i}$ at the given place is $r_k\alpha_i + s_k\beta_i$.

Put $$C_i = r_k\alpha_i + s_k\beta_i.$$

Since $$\alpha_i = s_1 + \cdots + s_i,$$
$$\beta_i = \nu - (r_1 + \cdots + r_i),$$

$$C_i = r_k(s_1 + \cdots + s_i) + s_k(\nu - \overline{r_1 + \cdots + r_i}).$$

Therefore $C_i - C_{i-1} = r_k s_i - s_k r_i$.

(a) Using the inequalities II (ii) among the $r$s and $s$s we have

$$C_1 > C_2 > \cdots > C_{k-1} = C_k < C_{k+1} < \cdots < C_\ell.$$

Hence the theorem.

(b) Using the equalities of II (ii) among the $r$s and $s$s we have,

$$C_i = C_{i-1} \text{ for all values } i=1,\cdots,\ell.$$

Hence all the terms in question at $z=a,\ u=b$ will begin with the lowest order.

## PART II.

## THE DERIVATIVES OF AN ALGEBRAIC FUNCTION

### INTRODUCTION

Let $u$ be an integral algebraic function of $z$ defined by an irreducible polynomial equation $f(z,u)=0$ of degree $n$ in $u$ and $\rho$ in $z$. The coefficient of $u^n$ in $f(z,u)$ is obviously independent of $z$ and

$$f(z,u)/z^{n\rho} \equiv g\left(\tfrac{1}{z}, \tfrac{u}{z^\rho}\right)$$
where $g(\xi,\eta)$

is of degree $n$ in $\eta$ and the coefficient of $\eta^n$ is independent of $\xi$.

If the equations

II (xvi) $\quad f(a,u)=0$ for all finite $a$,

and

II (xvii) $\quad g(0,\eta)=0$ have roots of multiplicity not greater than two, Beatty in the Transaction Royal Society, Canada Section III (1931) has shown that

II (xviii) $$\frac{du}{dz} \equiv u' = \sum_a \sum_b \frac{2\sigma f(a,u)}{f_2(a,b)\cdot(u-b)(z-a)} + P(z)$$

where $P(z)$ is a polynomial of degree $(\rho-1)$ at most, $b$ runs through the finite multiple roots of II(xvi), $a$ through the associated values of $z$ and $\sigma$ is the first power of $(z-a)$ to have different coefficients in the expansions of the several branches of $(u-b)$ in terms of $(z-a)$.

The object of this part is to extend the above result to the case where the multiplicity of the roots of equations II (xvi) and II (xvii) is not greater than 3. For this an entirely new method is adopted.

At $z=a$ let $f(a,u)=0$ have 3-ple root $u=b$. And let $t$ be the local parameter on the Riemann-Surface. The following branching may be possible at $z=a$.

<u>One branch of 3 sheets, viz.,</u>

II (xix) $z-a=t^3$,
$$u-b=a_1t+a_2t^2+a_3t^3+\cdots$$

where either $a_1=0$
or $a_1=a_2=0$
or $a_1=a_2=a_3=0$ etc.

II (xx) <u>One Branch of 2 sheets and another Branch of one sheet, viz.,</u>

$$z-a=t^2,$$
$$u-b=a_1t+a_2t^2+\cdots$$
where $a_1=0$ or $a_1=a_2=0$ etc.

$$z-a=t,$$
$$u-b=a_1t+a_2t^2+\cdots$$
$a_1=0$ or $a_1=a_2=0$, etc.

II (xxi) Or again at $z=a$, $f(a,u)=0$ may have 2-ple roots and if $t$ is the local parameter on the Riemann-Surface, the following branching may be possible:

$$z-a=t^2$$
$$u-b=a_1t+a_2t^2+\cdots$$

where $a_1=0$ or $a_1=a_2=0$, etc.

II (xxii) At $z=a$, $f(a,u)=0$ may have single roots and if $t$ is the local parameter then,

$$z-a=t,$$
$$u-b=a_1 t+a_2 t^2+\cdots$$

where $a_1=0$

or

$$a_1=a_2=0 \quad etc.$$

We have exhausted all the possible branching at $z=a$, where $f(a,u)=0$ may have roots of multiplicity not greater than 3.

Let us find the principle part of $\frac{du}{dz}$, supposing that all the cases II (xix) to II (xxii) exist on the Riemann-Surface. We note that the following will not contribute anything towards the principle part of $\frac{du}{dz}$. For example all expansions of $(u-b)$ in II (xix) which begin with orders in $t \geqq 3$. Also all cases in II (xx) of $(u-b)$ which begin with orders $t \geqq 2$ etc. do not contribute towards the principle part of $\frac{du}{dz}$.

Find all the principle parts of $\frac{du}{dz}$ in other cases at all multiple points $(a,b)$ in the finite part of the plane. Then the following is true:-

$\frac{du}{dz}$ — all the principle parts in the finite part of the plane = a rational function regular everywhere in the finite part of the plane = an integral rational function i.e. $I(z,u)$.

Hence $\frac{du}{dz}$ = principle part + $I(z,u)$.

Suppose we denote by $D^3_{3,1}(z-a,\, u-b)$ the principle part at $z=a$, $u=b$ of $\frac{du}{dz}$, at the place representation given by II (i) and similarly all other cases II (xix) to II (xxii) which contribute principle parts. Then we shall prove the following

**THEOREM**

If the roots of II (xvi) and II (xvii) be of multiplicity not greater than three we have,

$$\frac{du}{dz}=\sum\sum D^3_{3,1}+\sum\sum D^3_{3,2}+\sum\sum D^3_{2,1}+\sum\sum D^2_{2,1}$$
$$+I(z,u)$$

where the first summation extends to all cycles of three sheets or less at all points $(a,b)$, and the second summation to all the multiple points in the finite part of the plane, and $I(z,u)$ is an integral rational function in $(z,u)$ and is of the form according to Beatty, in the Journal of the London Mathematical Society, Vol. 4, Part I, (1928).

$$I(z,u)=P_0(z)+P_1(z)\cdot u+\cdots+P_{\lambda-1}(z)u^{\lambda-1}$$
$$+P_\lambda(z)U_\lambda(z,u)+\cdots+P_{n-1}(z)U_{n-1}(z,u).$$

where $P$s are polynomials in $z$, $U$s and $\lambda$ are defined in the paper referred to above;

and
$$D^3_{3,1}(z-a,u-b)=\frac{2f(a,u)}{f_3(a,b)(u-b)^2(z-a)}\left[1-3^{-1}\left\{\frac{f_4}{f_3}-\frac{f_1'}{f'}\right\}\times(u-b)\right],$$

$$D^3_{3,2}(z-a,u-b)=\frac{4f(a,u)}{f_3\cdot(u-b)^3(z-a)},$$

$$D^3_{2,1}(z-a,u-b)=\frac{3f(a,u)}{f_3\cdot(u-b)^2(z-a)},$$

$$D^2_{2,1}(z-a,u-b)=\frac{f(a,u)}{f_2\cdot(u-b)(z-a)},$$

$$D^3_{3,3}=D^3_{2,2}=D^2_{2,2}=\cdots=0$$

**PROOF:-**

Take the total differential of $f(z,u)=0$, we have $f_z(z,u)+f_u(z,u)\frac{du}{dz}=0$. Hence $\frac{du}{dz}=-f_z/f_u$.

$$\text{II (xxiii)}\quad \frac{du}{dz}=-\frac{(z-a)f_z}{(z-a)f_u}.$$

At the place $P(z-a,\,u-b)$, the following expansions hold good:-

$$\text{II (xiv)}\quad 0=f(z,u)\equiv f_1'(z-a)+\frac{1}{\lfloor 2}\left[f^2(z-a)^2\right.$$
$$\left.+2f_1'(u-b)(z-a)+f_2(u-b)^2\right]+\cdots$$

and at $z=a$

$$\text{II (xxiv)}\quad f'(z,u)=f'(a,u)+f^2(a,u)\cdot(z-a)$$
$$+\frac{1}{\lfloor 2}f^3(a,u)\cdot(z-a)^2+\cdots$$

$$\text{II (xxv)}\quad 0=f(z,u)\equiv f(a,u)+f'(a,u)\cdot(z-a)$$
$$+\frac{1}{\lfloor 2}\cdot f^2(a,u)\cdot(z-a)^2+\cdots$$

Hence from II (xxiv) we have,

$$\text{II (xxvi)}\quad (z-a)f'(z,u)=(z-a)\left[f'(a,u)+f^2(a,u)\times(z-a)+\frac{1}{\lfloor 2}f^3(a,u)(z-a)^2+\cdots\right]$$
$$=(z-a)f'(a,u)+\frac{f^2(a,u)}{\lfloor 1}(z-a)^2$$
$$+\cdots$$

From II (xxv) $-(z-a)f'(a,u)=f(a,u)+\frac{1}{\lfloor 2}f^2(a,u)(z-a)^2+\cdots.$

Substituting this value in II (xxvi) we have,

$$-(z-a)f'(z,u)=\left[f(a,u)+\frac{1}{\lfloor 2}f^2(a,u)(z-a)^2+\cdots\right]$$
$$-\left[\frac{1}{\lfloor 1}f^2(a,u)(z-a)^2+\frac{1}{\lfloor 2}f^3(a,u)(z-a)^3+\cdots\right].$$

II (xxvii) $-(z-a)f'(z,u)=f(a,u)$

$$+(z-a)^2\sum_{s=2}^{\infty}\left(\frac{1}{\lfloor s}-\frac{1}{\lfloor s-1}\right)f^s(a,u)(z-a)^{s-2}.$$

Again at $z=a$,

$$f_1(z,u)=f_1(a,u)+f_1'(a,u)(z-a)+\cdots$$

Hence

II (xxviii) $(z-a)f_1(z,u)=(z-a)f_1(a,u)$

$$+(z-a)^2\sum_{s=1}^{\infty}\frac{(z-a)^{s-1}}{\lfloor s}f_1^s(a,u).$$

Hence from substituting the values of II (xxvi) and II (xxviii) in $\frac{du}{dz}=-\frac{(z-a)f_z}{(z-a)f_u}$ we have,

II (xxix)

$$(z-a)\frac{du}{dz}=\frac{f(a,u)+(z-a)^2\sum_{s=2}^{\infty}\left(\frac{1}{\lfloor s}-\frac{1}{\lfloor s-1}\right)f^s(a,u)(z-a)^{s-2}}{f_1(a,u)+(z-a)\sum_{s=1}^{\infty}\frac{(z-a)^{s-1}}{\lfloor s}f_1^s(a,u)}.$$

Consider $\frac{du}{dz}$ at $z=a$ for cycles of order 3 beginning with $(z-a)^{\frac{1}{3}}$

i.e. $z-a=t^3$
$u-b=a_1t+a_2t^2+\cdots \qquad a_1\neq 0$

This case will occur when

$$f=0,\ f_1=0,\ f_2=0,\ f_3\neq 0,\ f'\neq 0.$$

From II (xxix) we have,

$$(z-a)u'=\frac{f(a,u)+(z-a)^2\sum_{s=2}^{\infty}\left(\frac{1}{\lfloor s}-\frac{1}{\lfloor s-1}\right)f^s(a,u)(z-a)^{s-2}}{f_1(a,u)+(z-a)\sum_{s=1}^{\infty}\frac{(z-a)^{s-1}}{\lfloor s}f_1^s(a,u)}$$

Since

$$f_1(a,u)=f_1(a,b)+f_2(u-b)+\cdots$$
$$f_1=f_2=0,\ \text{and}\ f_3\neq 0.$$

We have

$$f_1(a,u)=\frac{f_3}{\lfloor 2}(u-b)^2+\frac{f_4}{\lfloor 3}(u-b)^3+\cdots$$

$$(z-a)\sum_{s=1}^{\infty}(z-a)^{s-1}f_1^s(a,u)=(z-a)f_1'(a,u)+(z-a)^2\times\sum(z-a)^{s-2}f_1^s(a,u).$$

Hence the denominator of II (xxix) can be written as

II (xxx) $\frac{f_3}{\lfloor 2}(u-b)^2+\frac{f_4}{\lfloor 3}(u-b)^3+\cdots+(z-a)\big(f_1'+f_2'\cdot(u-b)+\cdots\big)+(z-a)^2\sum_{s=2}^{\infty}\frac{(z-a)^{s-2}}{\lfloor s}f_1^s(a,u).$

II (xxxi) From II (xxvii)

$$-(z-a)f'(z,u)=\frac{1}{6}f_3\cdot(u-b)^3+\text{terms}$$

containing orders higher than 3 in $t$.

II (xxx) because of II (xxxi) will become

$$\frac{f_3}{2}(u-b)^2+\frac{f_4}{6}(u-b)^3-\frac{f_1'\cdot f_3}{6f'}(u-b)^3+\text{terms}$$

containing orders higher than 3 in $t$.

Or

$$\frac{f_3}{2}(u-b)^2+\frac{1}{6}(u-b)^3\left[f_4-\frac{f_1'\cdot f_3}{f'}\right]+\text{terms}$$

containing orders higher than 3 in $t$.

The numerator of II (xxix)

$=f(a,u)+$ terms containing orders higher than 6 in $t$.

Hence $(z-a)u'$

$$=\frac{f(a,u)+\text{terms containing orders higher than 6 in } t}{\frac{f_3}{2}(u-b)^2+\frac{1}{6}(u-b)^3\left[f_4-\frac{f_1'\cdot f_3}{f'}\right]+\text{terms containing oders higher than 3 in } t}$$

$$=\frac{2\left[f(a,u)+A_6t^6+A_7t^7+\cdots\right]}{f_3\cdot(u-b)^2\left[1+\frac{1}{3}\left(\frac{f_4}{f_3}-\frac{f_1'}{f'}\right)(u-b)+B_2t^2+\cdots\right]}$$

$$=\frac{2\left[f(a,u)+A_6t^6+A_7t^7+\cdots\right]}{f_3\,(u-b)^2}\left[1+\frac{1}{3}\left(\frac{f_4}{f_3}-\frac{f_1'}{f'}\right)(u-b)+\cdots\right]^{-1}$$

$$=\frac{2f(a,u)\left[1-\frac{1}{3}\left(\frac{f_4}{f_3}-\frac{f_1'}{f'}\right)(u-b)+C_2t^2+\cdots\right]}{f_3\cdot(u-b)^2}$$

In the numerator, neglecting all terms whose orders are equal to or greater than 5 in $t$ we have,

$$(z-a)u' = \frac{2f(a,u)\left[1-\frac{1}{3}\left(\frac{f_4}{f_3}-\frac{f_1'}{f_1'}\right)(u-b)\right]}{f_3\cdot(u-b)^2}$$

Hence

$$D^3_{3,1} = \frac{2f(a,u)\left[1-\frac{1}{3}\left(\frac{f_4}{f_3}-\frac{f_1'}{f_1'}\right)(u-b)\right]}{f_3\cdot(u-b)^2(z-a)}$$

Since $f(a,u)=\frac{1}{6}f_3(u-b)^3+$ powers of ( $u-b$ ) higher than 4, $D^3_{3,1}$ is finite at all other cycles of the multiple point and is also finite at all other multiple points in the finite part of the plane. Similarly $D^3_{3,2}$ can be obtained from

$$(z-a)u' = \frac{f(a,u)}{\frac{1}{2}f_3\cdot(u-b)^2}\left[1+\frac{1}{3}\cdot\frac{f_4}{f_3}(u-b)\right]$$

On simplification we have,

$$D^3_{3,2} = \frac{4f(a,u)}{f_3\cdot(z-a)(u-b)^2}.$$

Similar results for $D^3_{2,1}$ , etc. are obtained. It is to be noted that the principle parts $D^3_{3,3}=D^3_{2,2}=D^2_{2,2}=\cdots=0$.

Hence

$$u'-\sum\sum D^3_{3,1}-\sum\sum D^3_{3,2}-\sum\sum D^3_{2,1}-\sum\sum D^2_{2,1}$$

is a rational function finite everywhere in the finite part of the plane $=$ a rational Integral function $I(z,u)$.

NOTE:-

Similar results for roots of $f(a,u)=0$ of multiplicity greater than 3 may be derived by using the theorems I to III.

## PART III

### EXTENSION OF THE RESULTS OF PARTS I AND II TO THE ALGEBRAICALLY CLOSED FIELDS

These results do not add anything new to those already discovered in the case of Algebraically Closed Fields with characteristic zero, but the method of deriving the results from Parts I and II is new. It is for this purpose that we add a note to this part which will be helpful in applying the results of Parts I and II to all Algebraically Closed Fields.

In the field $k(x)$ we can define the derivative of a polynomial

$$f(x)\equiv a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n$$

as

$$f'(x)=na_0x^{n-1}+\cdots+a_{n-1}.$$

Thus $f'(x)$ is the coefficient of $h$ in the expansion of $f(x+h)$ in powers of $h$ i.e.

$$f(x+h)=f(x)+hf'(x)+\cdots+h^na_0$$

This definition is therefore equivalent to $\left[\frac{f(x+h)-f(x)}{h}\right]_{h=0}$

The operation of derivation as defined above is easily seen to satisfy the usual relations,

$$\{f(x)+g(x)\}'=f'(x)+g'(x)$$
$$\{f(x)\cdot g(x)\}'=f'(x)\cdot g(x)+g'(x)\cdot f(x).$$

Defining the derivatives of $\frac{f(x)}{g(x)}$ as

$$\left\{\frac{1}{h}\left[\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}\right]\right\}_{h=0}$$

we have

$$\left\{\frac{f(x)}{g(x)}\right\}'=\left[\frac{g(x)\{f(x+h)-f(x)\}-f(x)\{g(x+h)-g(x)\}}{h\cdot g(x)g(x+h)}\right]_{h=0}$$

$$=\frac{g(x)\cdot f'(x)-f(x)g'(x)}{\{g(x)\}^2}$$

Also we have,

$$[f(g(x))]'=\left[\frac{f[g(x+h)]-f[g(x)]}{h}\right]_{h=0}$$

$$=\left[\frac{f[g(x+h)]-f[g(x)]}{g(x+h)-g(x)}\cdot\frac{g(x+h)-g(x)}{h}\right]_{h=0}$$

$$=f'[g(x)]\cdot g'(x).$$

Again we can also deduce from this definition Taylor's theorem for polynomials viz.,

$$f(x+h)=f(x)+hf'(x)+\cdots$$

Newton's Polygon could also be constructed without any difficulty for the new Algebraic Field. Deducing results similar to Parts I and II follows as a matter of routine.

## CHAPTER III.

## RIEMANN - ROCH THEOREM AND ITS APPLICATIONS.

### INTRODUCTION

The Arithmetic Theory of Algebraic Functions of one variable gave rise to a great number of variety of proofs for the famous Riemann-Roch Theorem. Most of them are long and involve great details. The theories differ greatly in detail but have in common as central features the construction and analysis of the rational functions which are the integrands of Abelian Integrals. The object of the present chapter is to give a simple and direct proof based on the theory of analysis. No elaborate appeal is therefore made to divisors and their properties as in Bliss's 'Algebraic Functions'.

Let $f(z,u) \equiv u^n + f_1 u^{n-1} + \cdots + f_n = 0$ be an irreducible monic algebraic equation ( $f$s are polynomials in $z$ with coefficients in the field of complex numbers $k$ ) defining the field of rational functions.

### I. FUNDAMENTALS AND NOTATION.

If $a, b \in k$ is a solution of $f(z,u)=0$ then there exists a formal power series, solution of $f(z,u)=0$ in the form

$$z-a = t^{\nu},$$
$$u-b = t^{\sigma}(a_\sigma + a_{\sigma+1} t + \cdots)$$
$$a_\sigma \neq 0.$$

Such a pair of functions we shall call a place representation of the Riemann-Surface of the algebraic function. This was already stated in Chapter I.

### VALUE OF A RATIONAL FUNCTION AT A PLACE.

Let $R(z,u) \in k(z,u)$.

Let a place $\pi$ be given by a place representation

$$z = z_0 + t^{\nu},$$
$$u = u_0 + \Sigma a_i t^{k_i}$$

In virtue of the substitution of this place representation we have,

$$R(z,u) = t^{\rho} E(t)$$

where $E(t)$ is a power series in $t$ and $E(0) \neq 0$. We call $\rho/\nu$ the order of $R(z,u)$ at the place in question. We assign uniquely the value of $R(z,u)$ at the place $\pi$ as follows:-

$$R(\pi) = \begin{cases} 0 & \text{if } \rho > 0 \\ \infty & \text{if } \rho < 0 \\ E(0) & \text{if } \rho = 0. \end{cases}$$

If $\nu = 1$ the place is called regular, otherwise singular. For any given algebraic curve there are only a finite number of such singularities.

It is seen that for all the branches in a cycle a rational function has the same order. Let there be $h$ cycles of $\nu_1, \cdots, \nu_h$ sheets each at $z=a$ of the Riemann-Surface. Take $h$ numbers $\tau_1, \cdots, \tau_h$ of the type $\rho/\nu$ and denote this set by $(\tau)$. Such sets assigned at different places of the Riemann-Surface are denoted by $((\tau))$ . The orders of the rational function $\frac{\partial f}{\partial u}$ at the cycles of the Riemann-Surface at $z=a$ are denoted by $\mu_1, \cdots, \mu_h$ respectively. Complementary order-basis at $z=a$ are numbers $(\bar{\tau})$ such that

III (1)
$$\tau + \bar{\tau} = \mu - 1 + \tfrac{1}{\nu} \text{ at finite places } z=a$$
$$\tau + \bar{\tau} = \mu + 1 + \tfrac{1}{\nu} \text{ at } z = \infty$$

Given an order-basis $((\tau))$ at points of the Riemann-Surface, in general there always exist* rational functions $R(z,u) \in k(z,u)$ which have orders equal to or greater than the given $((\tau))$ order-basis at all places in question and greater than or equal to zero everywhere else. Denote by $N_\tau$ the maximum number of linearly independent rational functions of the set, and by $N_{\tau - \frac{1}{\nu}}$ the maximum number of linearly independent rational functions constructed on $((\tau))$ everywhere and $(\tau - \frac{1}{\nu})$ at one cycle.

Similarly we define $N_{\bar{\tau}}$ and $N_{\bar{\tau}+\frac{1}{\nu}}$ where $((\bar{\tau}))$ is the complementary order-basis to $((\tau))$ .

### II. PRELIMINARY THEOREMS LEADING UPTO RIEMANN-ROCH THEOREM.

#### THEOREM I.

The maximum number of linearly independent rational functions built on a negative order-basis $((\tau))$ is $\geq -\Sigma\Sigma\tau\nu$ .

PROOF:-

Let the negative order-basis at a multiple point M be $\tau_1, \cdots, \tau_h$ and let $\nu_1, \cdots, \nu_h$ be the cycles. We can represent the order-basis by means of a divisor Q in the sense of Bliss as $Q = P_1^{\tau_1\nu_1} \cdots P_h^{\tau_h\nu_h}$ or putting $\Delta_i = \tau_i \nu_i$ ,

$$Q = P_1^{\Delta_1} \cdots P_n^{\Delta_n}$$

Let the multiples constructed on this order-basis be

$$\eta_1, \eta_2, \eta_3, \cdots$$

Take any one of the multiples say $\eta_1$ and out of the remaining multiples select a multiple $\eta_2$ such that

$$c_1\eta_1 + c_2\eta_2 \neq 0$$

( $c_1$ and $c_2 \in k$ and not all zero).

if $c_1\eta_1 + c_2\eta_2 = 0$, then the maximum number of linearly independent rational functions is 1 - then out of the remaining multiples choose $\eta_3$ such that

$$c_1\eta_1 + c_2\eta_2 + c_3\eta_3 \neq 0.$$

(If $c_1\eta_1 + c_2\eta_2 + c_3\eta_3 = 0$ then the maximum number of linearly independent rational functions is 2, $c$s $\in k$ and not all zero.) Continuing this process of selection suppose we come to the stage where $c_1\eta_1 + c_2\eta_2 + \cdots + c_{\lambda-1}\eta_{\lambda-1} \neq 0$ and $c_1\eta_1 + c_2\eta_2 + \cdots + c_\lambda\eta_\lambda = 0$ ($c$s $\in k$ and not all zero). It is to find the value of $\lambda$. First of all we shall note that if $\eta_1, \eta_2, \cdots, \eta_\lambda$ are multiples then $\eta = c_1\eta_1 + \cdots + c_\lambda\eta_\lambda$ is also a multiple, where $c$s are constants ($\in k$).

Let the following expansions of the multiples in terms of the local parameter $t$ at the various places $P_1, \cdots P_n$ of the multiple point M be considered.

At $P_i$

$$\eta_k = a_{M,k,\Delta_i} t^{\Delta_i} + a_{M,k,\Delta_i+1} t^{\Delta_i+1} + \cdots, \quad k = 1, \cdots, \lambda, \quad i = 1, \cdots, n.$$

then at $P_i$

$$c_k\eta_k = c_k\, a_{M,k,\Delta_i} t^{\Delta_i} + \cdots$$

and

$$\eta = \sum_k c_k\eta_k = \sum_k c_k a_{M,k,\Delta_i} t^{\Delta_i} + \cdots.$$

In order that $\eta$ may have no orders $< 0$ at $P_i$ we have,

$$\sum_k c_k\, a_{M,k,\Delta_i+\ell} = 0 \qquad k = 1, \cdots, \lambda, \quad \ell = 0, 1, \cdots, (-\Delta_i + 1)$$

Conditions that $\eta$ may have no orders $< 0$ at all cycles $P_1, \cdots, P_n$ of the place are,

$$\sum_k c_k\, a_{M,k,\Delta_i+\ell} = 0 \qquad \begin{cases} k = 1, \cdots, \lambda, \\ \ell = 0, 1, \cdots, (-\Delta_i + 1), \\ i = 1, 2, \cdots, n. \end{cases}$$

Hence the total number of conditions that $\eta$ may have no orders $< 0$ at all cycles of the place is $-\sum_i \Delta_i$ $i = 1, \cdots, n$.

Similar conditions exist at all the other multiple points M of the Riemann-Surface. Hence the conditions that $\eta$ may have no orders $< 0$ at all multiple points of the Riemann-Surface are

III (ii)

$$\sum_k c_k\, a_{M_\ell\ k,\Delta_i+\ell} = 0 \qquad \begin{cases} k = 1, \cdots, \lambda \\ \ell = 0, \cdots, (-\Delta_i + 1) \\ i = 1, 2, \cdots, n \\ M_\ell \text{ runs through all} \\ \text{the multiple points} \end{cases}$$

Therefore the total number of conditions that $\eta$ may have no orders $< 0$ at all the multiple points of the Riemann-Surface is $-\sum_{M_\ell}\sum_i \Delta_i$.

The least number of conditions imposed on $c$s that $\eta$ may be zero is $-\sum\sum \Delta_i + 1$. The number of constants $c$s in III (ii) is $\lambda$. In order that it may be possible to have values for $c$s from equation III (i), not all of them zero, it is necessary that

$$\lambda \geq -\sum_{M_\ell}\sum \Delta_i + 1$$

But the maximum number of linearly independent multiples is $(\lambda - 1)$.

Hence $N_\tau \geq -\sum_{M_\ell}\sum_i \Delta_i$.

<u>THEOREM II.</u>

$$(N_{\tau-\frac{1}{\nu}} - N_\tau) + (N_{\bar\tau} - N_{\bar\tau+\frac{1}{\nu}}) = 1$$

<u>PROOF:-</u>

If the number of linearly independent rational functions built on $((\tau))$ order-basis is $\ell$, then the number of linearly independent rational functions built on $((\tau - \frac{1}{\nu}))$ is either equal to $\ell$ or $\ell + 1$. That is $N_\tau$ and $N_{\tau-\frac{1}{\nu}}$ differ by one at most. For suppose

III (iii)

$$\phi_1(z,u), \phi_2(z,u), \cdots, \phi_\ell(z,u), \phi_{\ell+1}(z,u)$$

are $(\ell+1)$ linearly independent rational functions built on $((\tau - \frac{1}{\nu}))$ then we can always choose one out of $(\ell+1)$ functions, which is not built on $((\tau))$. Let it be $\phi_{\ell+1}$.

It is possible always to choose $\ell$ constants $c_1, c_2, \cdots, c_\ell$ (not all zero belonging to $k$, the field of complex numbers) such that

$$\phi_1 + c_1\phi_{\ell+1}, \cdots, \phi_\ell + c_\ell\phi_{\ell+1}$$

are functions built on $((\tau))$. If these $\ell$ functions are linearly dependent, then it follows that the $(\ell+1)$ functions in III (iii) are also linearly dependent, which is a contradiction. Hence it follows that if $N_\tau = \ell$ then

$N_{\tau-\frac{1}{\nu}}$ is at most equal to $(\ell+1)$.

Similarly $N_{\bar\tau} - N_{\bar\tau+\frac{1}{\nu}} = 0$ or $1$.

Hence

III (iv)

$$(N_{\tau-\frac{1}{\nu}} - N_\tau) + (N_{\bar\tau} - N_{\bar\tau+\frac{1}{\nu}}) = 0, 1, \text{ or } 2$$

Case 1.

$$(N_{\tau-\frac{1}{\nu}} - N_\tau) + (N_{\bar\tau} - N_{\bar\tau+\frac{1}{\nu}}) \neq 2$$

For if III (iv) were equal to 2 then

$$(N_{\tau-\frac{1}{\nu}} - N_\tau) = 1$$

and

$$(N_{\bar\tau} - N_{\bar\tau+\frac{1}{\nu}}) = 1.$$

This is possible only if there exist rational functions constructed on

$((\tau-\frac{1}{\nu}))$ having orders exactly $\tau-\frac{1}{\nu}$

at the excepted cycle, and those constructed on $((\bar\tau))$, having orders exactly $\bar\tau$ at the excepted cycle.

Let $R_{\tau-\frac{1}{\nu}}$ and $R_{\bar\tau}$ be any two such rational functions which have exact orders $\tau-\frac{1}{\nu}$ and $\bar\tau$ at the excepted cycle respectively. Consider the residue of the rational function

$$F(z,u) = \frac{R_{\tau-\frac{1}{\nu}} \cdot R_{\bar\tau}}{f_u(z,u)}, \quad [F(z,u) \in k(z,u)].$$

The expansion of $F(z,u)$ in terms of $(z-a)$ at finite places are,

III (v) $$F(z,u) = a_{-1+\frac{1}{\nu}}(z-a)^{-1+\frac{1}{\nu}} + a_{-1+\frac{2}{\nu}}(z-a)^{-1+\frac{2}{\nu}} + \cdots$$

for the order of $F(z,u)$ are $\tau+\bar\tau-\mu = -1+\frac{1}{\nu}$ from III (i).

at infinity,

III (vi) $$F(z,u) = b_{1+\frac{1}{\nu}}\left(\frac{1}{z}\right)^{1+\frac{1}{\nu}} + b_{1+\frac{2}{\nu}}\left(\frac{1}{z}\right)^{1+\frac{2}{\nu}} + \cdots$$ (Refer III (i)).

at the excepted cycle,

III (vii)

$$F(z,u) = C_\nu (z-a)^{-1} + C_{\nu+\frac{1}{\nu}}(z-a)^{-1+\frac{1}{\nu}} + \cdots \qquad C_\nu \neq 0$$

(Refer III (i))

The residue from III (v) is 0, and the residue from III (vi) is 0. $F(z,u)$ gives rise to a residue $C_\nu$ (Refer expansion III (vii)) at the excepted cycle and if the cycle contains $\nu$ branches, then the sum of the residues of $F(z,u)$ is $\nu \cdot C_\nu = 0$. Since $\nu \neq 0$, $C_\nu = 0$, which is a contradiction contradicting III (vii).

III (viii) Case 2.

$$(N_{\tau-\frac{1}{\nu}} - N_\tau) + (N_{\bar\tau} - N_{\bar\tau+\frac{1}{\nu}}) \neq 0 \quad \text{for } ((\bar\tau)) \leqq ((0)).$$

(a) This result III (viii) is easily proved for $((\bar\tau)) = ((0))$, since

$N_{\bar\tau} = 1$, $N_{\bar\tau+\frac{1}{\nu}} = 0$ $(N_{\tau-\frac{1}{\nu}} - N_\tau) =$ not negative.

(b) Result III (viii) for $((\bar\tau)) < ((0))$.

From the set of adjoint orders to the given set of orders we may pass by a series of steps each individual one which involves an addition to the order of coincidence of the function with the branches of one and of only one of the cycles, the addition to the order being $1/\nu$ in case the cycle in question be the one of order $\nu$. Every step in the process just described implies a further condition on the coefficients of the function, and only one further condition as is evident, for the order of a rational function of $(u, z)$ with the branches of a cycle of order $\nu$ is always measured by an integral multiple of $1/\nu$. For this explanation in Case 2 (b) I am indebted to J.C.Fields. He makes use of this idea as the very foundation for his book 'Algebraic Function of One Variable', almost at the very beginning of the book. Making use of this result we have III (vii) for $((\bar\tau)) < ((0))$. Hence III (iv) $\neq 0$. Therefore from Cases 1 and 2

$$(N_{\tau-\frac{1}{\nu}} - N_\tau) + (N_{\bar\tau} - N_{\bar\tau+\frac{1}{\nu}}) = 1 \quad \text{for } ((\bar\tau)) \leqq ((0)).$$

THEOREM III.

To show that any rational function can be made to have order-basis $((\bar{\tau}))$ $\leq ((0))$.

METHOD OF GETTING THE DESIRED FUNCTION.

Let $R_{\bar{t}_1}$ be a rational function built on any given order-basis $((\bar{t}_1))$. Then particular values can be ascribed to the $N_{\bar{t}_1}$ arbitrary constants in $R_{\bar{t}_1}$, in such a way that the resulting specific function $R(z,u)$ is not zero identically. But the orders of the specific function form an order-basis $((\sigma))$ such that $\bar{\tau} = (\bar{t}_1 - \sigma)$ are either zero or negative. The general rational function built on the basis $((\bar{\tau}))$ is $\frac{R_{\bar{t}_1}}{R}$. It is also seen that $N_{\bar{t}_1} = N_{\bar{\tau}}$.

Hence Theorem II.

III. RIEMANN - ROCH THEOREM.

We know from Theorem II that

$$(N_{\tau-\frac{1}{\nu}} - N_{\tau}) + (N_{\bar{\tau}} - N_{\bar{\tau}+\frac{1}{\nu}}) = 1.$$

Applying it successively we have,

III (ix)

$$(N_{t} - N_{\tau}) + (N_{\bar{\tau}} - N_{\bar{t}}) = \Sigma\Sigma(\tau - t)\nu$$

where

$$t + \bar{t} = \begin{cases} \mu - 1 + \frac{1}{\nu} & \text{at finite places} \\ \text{and} & \\ \mu + 1 + \frac{1}{\nu} & \text{at infinite place.} \end{cases}$$

Put $t = \bar{\tau}$, $\bar{t} = \tau$ in the equation III (ix) then,

$$(N_{\bar{\tau}} - N_{\tau}) + (N_{\bar{\tau}} - N) = \Sigma\Sigma(\tau - \bar{\tau})\nu$$

Hence the Riemann-Roch Theorem,

$$N_{\tau} + \tfrac{1}{2}\Sigma\Sigma\tau\nu = N_{\bar{\tau}} + \tfrac{1}{2}\Sigma\Sigma\bar{\tau}\nu .$$

IV. APPLICATIONS OF THEOREM II.

1. To demonstrate the existence of Abelian Integrals of the 2nd and 3rd kind in a simple way.

Suppose $((\bar{\tau})) = ((0))$. Then the orders $((\tau))$ are adjoint.

Therefore $N_{\tau} = p$

( $p$ is the genus of the fundamental curve).

Also $N_{\bar{\tau}} = 1$

and $N_{\bar{\tau}+\frac{1}{\nu}} = 0$

Applying Theorem II we get,

Hence $N_{\tau-\frac{1}{\nu}} = p$

THEOREM IV (i)

Decrease in the adjoint order-basis at a place by a minimum order quantity does not affect the number of linearly independent adjoint rational functions.

THEOREM IV (ii)

Existence of the Abelian Integrals of the 2nd kind.

PROOF:-

Change the order at a place by twice the minimum quantity.

Theorem II then gives

$$(N_{\tau-\frac{2}{\nu}} - N_{\tau}) + (N_{\bar{\tau}} - N_{\bar{\tau}+\frac{2}{\nu}}) = 2$$

But

$$N_{\tau} = p,\ N_{\bar{\tau}} = 1,\ \text{and}\ N_{\bar{\tau}+\frac{2}{\nu}} = 0$$

Therefore

$$N_{\tau-\frac{2}{\nu}} = p + 1.$$

There exist rational functions $R(z,u) \in \mathfrak{K}(z,u)$ which have exactly $\tau - \frac{2}{\nu}$ order at the excepted cycle. Its expansion in terms of the element $(z-a)$ is

$$R(z,u) = A_0(z-a)^{-1-\frac{1}{\nu}} + A_1(z-a)^{-1} + \text{higher powers of } (z-a).$$

$A_1 = 0$ since it is the only residue of the rational function and $A_0 \neq 0$

Now $\int R\,dz = \frac{A_0}{-\frac{1}{\nu}}(z-a)^{-\frac{1}{\nu}} + \text{higher powers of } (z-a).$

This integral has poles only and no logarithms; hence it is the Abelian Integral of the 2nd kind.

**THEOREM IV (iii).**

Existence of the Abelian Integrals of the 3rd kind.

PROOF:-

Reduce $((\tau))$ at two different places $C_1$ and $C_2$ the orders by a minimum quantity. Applying Theorem II we have,

$$\left(N_{\tau-\frac{1}{\nu}-\frac{1}{\nu_1}} - N_{\tau}\right) + \left(N_{\bar{\tau}} - N_{\bar{\tau}+\frac{1}{\nu}+\frac{1}{\nu_1}}\right) = 2.$$

Since

$$N_{\tau} = p, \quad N_{\bar{\tau}} = 1, \text{ and } \quad N_{\bar{\tau}+\frac{1}{\nu}+\frac{1}{\nu_1}} = 0$$

Hence $$N_{\tau-\frac{1}{\nu}-\frac{1}{\nu_1}} = p + 1$$

There exist rational functions $R(z,u)$ which have exactly the prescribed orders at the two excepted cycles. Their expansions at these excepted places are at $C_1$

$$R(z,u) = A_0(z-a)^{-1} + A_1(z-a)^{-1+\frac{1}{\nu}} + \text{powers of } (z-a) \text{ higher than } -1+\frac{1}{\nu}, \quad A_0 \neq 0$$

at $C_2$

$$R(z,u) = -A_0(z-a)^{-1} + B_1(z-a)^{-1+\frac{1}{\nu_1}} + \dots$$

and their integrals are,

$$\int_{C_1} R\,dz = A_0 \log(z-a) + \nu A_1 (z-a)^{\frac{1}{\nu}} + \cdots,$$

$$\int_{C_2} R\,dz = -A_0 \log(z-a) + \nu_1 B_1 (z-a)^{\frac{1}{\nu_1}} + \cdots.$$

These are therefore Abelian integrals of the third kind as they have no poles but logarithms.

2. A method for investigating the reducibility of the fundamental equation.

Evaluate the expression,

$$\text{III (x)} \quad \left(N_{\tau-\frac{1}{\nu}} - N_{\tau}\right) + \left(N_{\bar{\tau}} - N_{\bar{\tau}+\frac{1}{\nu}}\right)$$

for any order-basis. The fundamental equation is reducible or irreducible according to the expression III (x) $\equiv 1$. The proof follows the lines we have already indicated.

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(*) M.Sc.Thesis in the University of Cambridge, 1946.

1) Algebraic Functions, G.A.Bliss.

2) The theorem is still true if instead of $s_i$ and $r_i$ being prime to each other are such that the highest common factor of $r_i$ and all the exponents $s_i$ in the series for $(u-b)$, is one.

3) The perpendicular distance $p$ from $(0,0)$ on the line

$$r_i\alpha + s_i\beta = \nu s_i - (r_1 + \cdots + r_{i-1})s_i + (s_1 + \cdots + s_{i-1})r_i$$

is given by

$$\frac{1}{\sqrt{r_i^2 + s_i^2}}\left[\nu s_i - (r_1 + \cdots + r_{i-1})s_i + (s_1 + \cdots + s_{i-1})r_i\right].$$

Let $(\alpha_1, \beta_1)$ be any point in the $(\alpha, \beta)$ plane and $p'$ be the perpendicular distance from $(\alpha_1, \beta_1)$ on the given line then

$$p' = \frac{r_i\alpha_1 + s_i\beta_1}{\sqrt{r_i^2 + s_i^2}}$$

The condition that $(\alpha_1, \beta_1)$ may be on the same side of the straight line as the origin is $p' < p$ i.e.

$$r_i\alpha + r_i\beta < \nu s_i - (r_1 + \cdots + r_{i-1})s_i + (s_1 + \cdots + s_{i-1})r_i$$

Hence the conditions stated above.

4) For example if the given order-basis $((\tau))$ at points of the Riemann-Surface is positive or its sum is positive then no rational function $R(z,u) \in k(z,u)$ exists. Or the only function in this case is zero.

## BIBLIOGRAPHY.

### CHAPTER I.

1. Memiographed notes of Lefschetz Algebraic Geometry 1936-37, Theorem 8.2, page 14.
2. F.K.Schmidt, Math. Zeitschrift, 41. (1936) page 415.
3. G.A.Bliss, Algebraic Functions, page 44.
4. G.A.Bliss, Algebraic Functions, Theorem 23.1, page 63.

### CHAPTER II.

1. S.Beatty, The Derivatives of an Algebraic Function , Transactions, Royal Society of Canada, Section 111, 1931, pp. 79-82.
2. S.Beatty, University of Toronto Studies, Mathematical Series No. 1.
3. G.A.Bliss, Algebraic Functions, A.M.S.Coll. Publications (1933).
4. S.Beatty, Integral Bases for an Algebraic Function Field , J.L.M. Soc. Vol. 4, Part I, pp. 13-17.
5. H.Hasse, and F.K.Schmidt, Noch eine Begründung der Theorie der höheren Differential-Quotienten , Crelles Journal, 177, S.215 (1937).
6. O.Teichmüller, Differential-Rechnung bei Characteristik p , Crelles Journal, 175, S 89, (1936).
7. H.Hasse, Theorie der Differentiale in Algebraichen Funktionen-Körpern mit Vollkommenen Konstanter Körper , Crelles Journal, 172, S 55, (1935).
8. S.Lefschetz, Algebraic Geometry Vols. 1 and 2, Princeton Notes 1936-37, 1937-38.

### CHAPTER III.

1. Dedekind and Weber, Theorie der Algebraischen Funktionen einer Veränderlichen , Journal fur Mathematik, X cll (1882) pp. 181-290.
2. Hensel, K. and Landsberg, G., Theorie der Algebraischen Funktionen einer Variablen und ihre Anwendung auf Algebraische Kurven und Abelsche Integrale , (1902).
3. J.C.Field, Algebraic Functions of a Complex Variable, (1906).
4. H.W.E.Jung, Der Riemann-Rochsche Satz Algebraischer Funktionen Zweier Veränderlichen , Jahresb. Deutschen Math.-Ver. 18 (1921) S.S. 267-339.
5. J.C.Fields, Complementary Theorem , American Journal of Math. Vol. XXXII (1910) pp. 1-16.
6. J.C.Fields, Direct derivation of the Complementary Theorem from Elementary Properties of the Rational Functions , Proc. 5, International Math. Congress (1913), pp. 312-326.
7. J.C.Fields, On the Foundations of the Theory of the Algebraic Functions of one Variable , Transactions Lond. Phil. Soc. (A) 212 pp. 339-373, (1913).
8. S.Beatty, Derivation of the Complementary Theorem from the Riemann-Roch Theorem , American Journal Vol.39, pp.257-262, (1917).
9. K.Hensel, Arithmetische Theorie der Algebraischen Funktionen , Encyklo. d. Math. Wiss., II C. 5, 533-650 (1921).
10. S.Beatty, The Algebraic Theory of Algebraic Function of one Variable , L.M.S. Proc. (2) 20 pp. 435-449 (1922).
11. G.A.Bliss, Algebraic Functions and their Divisors , Annals of Math. (2) 26 (1924) pp. 95-124.
12. G.A.Bliss, Algebraic Functions, (1933).
13. F.Klein, Über Riemann's Theorie der Algebraischen Funktionen und ihre Integrale, (1882).
14. Appell and Goursat, Théorie des Fonctions Algébriques et de leurs Integrale, Gauthier-Villars, Paris 1895, revised by Fatou, 1929.
15. Pascal, Die Algebraischen Funktionen und die Abel'schen Integrale, Repertorium der höheren Mathematik, revised by Schep, Teubner, Leipzig (1900-).
16. Wirtinger, Algebraische Funktionen und ihre Integrale , Encyklopädie der Mathematischen Wissenschaften, 11 B 2 (1901), pp. 115-175.
17. Emmy Noether, Die Arithmetischen Theorie der Algebraischen Funktionen einer Veränderlichen in ihrer Beziehung zu den übrigen Theorien und zu der Zahlkörper-Theorie , Jahresbericht der Deutschen Mathematiker-Vereinigung, XXVIII (1919) pp. 182-203.
18. Hensel, K., Neue Begründung der Arithmetischen Theorie der Algebraischen Funktionen einer Variablen , Mathematische Zeitschrift V (1919) pp. 118-131.
19. Jung, Einführung in die Theorie der Algebraischen Funktionen einer Veränderlichen, Berlin and Leipzig, (1923).
20. Jung, Algebraischen Funktionen und ihre Integrale , Pascal's Repertorium I 2 , Teubner, Leipzig (1927).
21. Lefschetz, S., Algebraic Geometry, 2 Vols. (1936-37; 1937-38). Princeton Mathematical Notes.