ON A NON-ABELIAN THEORY OF ALGEBRAIC FUNCTIONS

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Contents

Introduction

The theory of Abelian functions of Riemann and Weierstrass realized that great scientific prophecy enunciated in Abel's Theorem. That theory of Abelian functions and the theory of automorphic functions of F. Klein and H. Poincaré, which can be regarded as the most beautiful and profound in the last century, came however to the standstill since the appearance of H. Weyl's admirable book on Riemann surface. After E. Artin's research on quadratic function fields over finite constant field, the tendency of investigations is directed rather towards the methodological purification and abstractization. Chiefly by the efforts of H. Hasse, F. K. Schmidt, M. Deuring and E. Witt this branch was raised to the high level of perfection and culminated in A. Weil's proof of the Riemann conjecture. But all these theories remain within the limit of "Abelian mathematics", in which the commutativity of the underlying groups plays the central role.

The first step into the "non-Abelian mathematics" was made by A. Weil in his pioneering work on the generalization of Abelian functions, which pointed out to us for the first time the possibility of the theory of hyperabelian functions. In his work the non-commutative fundamental group appeared instead of the commutative Betti group, and in accordance therewith the notion of divisors is generalized. One may regard the generalization of Riemann-Roch's theorem and the analogue of Abel's Theorem as the chief foundations of his theory.

In this article I will develop a non-Abelian theory of algebraic functions after the model of A. Weil. In Chapter I the normal form of divisors is obtained and by this normalization the algebraic and arithmetical structure of divisors and divisors classes is investigated. Chapter II deals with the algebraico-geometrical properties of the set of representations, which should find some important applications in the following Chapters. Chapter III is devoted to the description of hyperabelian integrals and by means of this new notion the correspondence between divisors and representations is explicitly realized. In Chapter IV we will obtain the normal form of divisor classes under some restrictions, and the significance of logarithmic differentials in our theory is pointed out. Chapter V is devoted to the existence proof of logarithmic differentials, which constitutes one of the chief difficulties of our theory. In Chapter VI we will prove the non-Abelian extension of Jacobi's inversion problem. In Chapter VII the properties of unitary representations are discussed and we understand that there exist familiar connections with the usual theory of representations. Chapter VIII is devoted to the duality theorem of the fundamental group.

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Square brackets in the foot-note refer to the bibliography placed in the next.

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Chapter I. Generalized divisors

Let $\mathfrak{F}$ be a closed Riemann surface of an algebraic function field K of genus p with complex numbers as its constant field. Throughout the present article p is assumed as greater than unity, or in other words, rational and elliptic cases are excluded, for these cases seem to present numerous peculiarities.

A signature $N(\mathfrak{p})$ is definied for every point $\mathfrak{p}$ on $\mathfrak{F}$, such that its value is greater than 1 only in finite number of points $q_1, q_2, \cdots q_\ell$, which are called branch points. Then by the wellknown theorem of uniformization there exists one and only one maximal covering surface $\tilde{\mathfrak{F}}$ with that signature $N(\mathfrak{p})$. This surface $\tilde{\mathfrak{F}}$ is no more closed, but on the other hand simply connected. When $N(\mathfrak{p})$ is everywhere 1 and there exists no branch points, the maximal surface $\tilde{\mathfrak{F}}$ is the universal covering surface of $\mathfrak{F}$.

The group of covering transformations G of $\tilde{\mathfrak{F}}$ with respect to $\mathfrak{F}$, which is the fundamental group of $\mathfrak{F}$, is an infinite discrete group with $2p+\ell$ generators $a_1, a_2, \cdots a_p, b_1, \cdots b_p, C_1, \cdots C_\ell$ which are subject to the following relations:

$$\left(\prod_{i=1}^{p} a_i b_i a_i^{-1} b_i^{-1}\right) C_1 C_2 \cdots C_\ell = e$$

$$C_1^{n_1} = C_2^{n_2} = \cdots = C_\ell^{n_\ell} = e$$

$$(n_i = N(q_i), \; i = 1, 2, \cdots \ell)$$

where $a_1, a_2, \cdots a_p, b_1, b_2, \cdots b_p$ represent $2p$ period loops and $C_1, C_2, \cdots C_\ell$ are circuits around the branch points $q_1, \cdots q_\ell$.

As is well known, between $\mathfrak{F}$ and $\tilde{\mathfrak{F}}$ lies the so-called covering surface of homologies $\hat{\mathfrak{F}}$, which corresponds to the commutator subgroup of the unramified G. Therefore the group of covering transformations is isomorphic to the Betti group of $\mathfrak{F}$. H. Weyl pointed out that $\hat{\mathfrak{F}}$ is the maximal unramified relatively Abelian covering surface of $\mathfrak{F}$, which is the exact analogue of the Hilbert class field in the algebraic number theory.[37].

When we map $\tilde{\mathfrak{F}}$ conformally onto the unit circle the field K becomes the field of automorphic functions belonging to the Fuchsian group isomorphic to G, and $\mathfrak{F}$ can be regarded as a fundamental domain of that Fuchsian group. Hereafter we regard $\mathfrak{F}$, $\tilde{\mathfrak{F}}$ and G chiefly from this point of view.

Let $\mathfrak{p}$ be a point on $\mathfrak{F}$ and t be a local uniformising parameter in the neighborhood of $\mathfrak{p}$. When $N(q) > 1$, that is, q is a branch point of $\mathfrak{F}$, $\tau = t^{\frac{1}{n}}$ $(n = N(q))$ is a local uniformising parameter of $\tilde{\mathfrak{F}}$. We denote by $\tilde{K}_\mathfrak{p}$ the field of all functions meromorphic in the neighborhood of $\mathfrak{p}$, which possess a Laurent expansion with respect to τ with only a finite number of negative powers. $\mathfrak{M}_\mathfrak{p}$ denotes a set of all r-order non-singular square matrices, whose elements belong to $\tilde{K}_\mathfrak{p}$. Such a set forms a group with ordinary matrix-multiplication. If both matrices $U_\mathfrak{p}$ and $U_\mathfrak{p}^{-1}$ belonging to $\mathfrak{M}_\mathfrak{p}$ have integral elements, $U_\mathfrak{p}$ is called a unit function matrix. Then the set $\mathfrak{U}_\mathfrak{p}$ of all $U_\mathfrak{p}$ makes a subgroup of $\mathfrak{M}$. When $\mathfrak{p}$ is not a branch point, a coset of $\mathfrak{U}_\mathfrak{p}$ in $\mathfrak{M}_\mathfrak{p}$ is named a local divisor. Hereafter the local divisor is expressed by a big German letter $\mathfrak{P}$. If q_i be a branch point, i.e. $N(q_i) > 1$, moreover the invariance of $\mathfrak{P}$ as a set for the substitution C_i is assumed. The point is called the base point of $\mathfrak{P}$.

As to the local divisor $\mathfrak{P}$ the following theorem of normal forms holds:

Theorem 1. Every local divisor $\mathfrak{P}$ contains a uniquely determined canonical matrix with the form:

$$\begin{pmatrix} \tau^{d_1} & & & \\ & \tau^{d_2} & & 0 \\ & & \ddots & \\ 0 & & & \tau^{d_r} \end{pmatrix} \begin{pmatrix} t^{d_1} & \theta_{12} & \cdots & \theta_{1r} \\ & t^{d_2} & & \vdots \\ & & \ddots & \vdots \\ 0 & & & t^{d_r} \end{pmatrix}$$

where $0 \leqq d_i < n = N(\mathfrak{p})$ and θ_{ik} is a polynomial of t only with order $\leqq d_k$, when $d_i < d_k$ and $< d_k$ when $d_i \geqq d_k$.

Proof. At first we show the existence of such a matrix. If we can obtain such a matrix by left multiplication of suitable unit function matrix to a given matrix, then the so obtained matrix belongs to the divisor.

Let θ be a matrix belonging to $\mathfrak{P}$:

$$\theta = \begin{pmatrix} \theta_{11} & \theta_{12} & \cdots & \theta_{1r} \\ \theta_{21} & \theta_{22} & \cdots & \theta_{2r} \\ \vdots & \vdots & & \vdots \\ \theta_{r1} & \cdots & \cdots & \theta_{rr} \end{pmatrix}$$

In the first column $\theta_{11}, \theta_{21} \cdots \theta_{r1}$ we select a element of the lowest order and bring it to the first row by a left multiplication of a suitable permutation matrix P, which evidently belongs to $\mathfrak{U}_\mathfrak{p}$

$$\theta_1 = P\theta$$

$$\theta_1 = \begin{pmatrix} \theta_{11} & * & & \\ * & * & & \\ * & & & \\ & & & \end{pmatrix}$$

As $\theta_{11} = \tau^{\alpha_1}(a_0 + at + \cdots), (a_0 \neq 0)$ is of the lowest order, θ_{i1}/θ_{11}

are integral functions of τ. Then

$$U_\beta \theta_1 = \begin{pmatrix} \tau^{\alpha_1} & * & \cdots & * \\ 0 & & & \\ \vdots & & & \\ 0 & * & \cdots & * \end{pmatrix}$$

Again, if we apply the same procedure to the second column, then we obtain the matrix

$$\theta_2 = \begin{pmatrix} \tau^{\alpha_1} & \theta_{12} & & \\ 0 & \tau^{\alpha_2} & & \\ \vdots & & \ddots & \\ 0 & 0 & & \end{pmatrix}$$

By the Euclidean algorithm we get

$$\theta_{12} = g\tau^{\alpha_2} + \theta_{12}^*$$

the order of θ_{12}^* being $< \alpha_2$

By the left multiplication of U_β

$$U_\beta \theta_2 = \begin{pmatrix} 1 & -g & & \\ & 1 & 0 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \tau^{\alpha_1} & \theta_{12} & \\ & \tau^{\alpha_2} & \\ 0 & & \ddots \end{pmatrix} = \begin{pmatrix} \tau^{\alpha_1} & \theta_{12} & \\ & \tau^{\alpha_2} & \\ 0 & & \ddots \end{pmatrix}$$

Similarly we can apply the same method to the 3rd,.....rth columns successively and obtain finally the matrix

$$\begin{pmatrix} \tau^{\alpha_1} & \theta_{12}^* & \cdots & \theta_{1r}^* \\ & \tau^{\alpha_2} & & \vdots \\ 0 & & \ddots & \vdots \\ & & & \tau^{\alpha_r} \end{pmatrix}$$

where θ_{ik} is a polynomial of order $< \alpha_k$ If $\dot{\theta}$ and θ' belong to the same β

$$\theta = U\theta', \quad \theta = \begin{pmatrix} \tau^{\alpha_1} & \theta_{12} & \cdots \\ & \tau^{\alpha_2} & \\ 0 & & \tau^{\alpha_r} \end{pmatrix} \quad \theta' = \begin{pmatrix} \tau^{\alpha_1'} & \theta_{12}' & \theta_{1r}' \\ & \tau^{\alpha_2'} & \\ 0 & & \ddots \quad \tau^{\alpha_r'} \end{pmatrix}$$

then the unit function matrix U is canonical, because both θ and θ' are canonical.

We get $\tau^{\alpha_i} = u_{ii}\tau^{\alpha_i'}$ $(i = 1, \cdots r)$

so $u_{ii} = 1$, $\alpha_i = \alpha_i'$

As to the (12) element

$$\theta_{12} = \theta_{12}' + u_{12}\tau^{\alpha_2}$$

As the orders of θ_{12} and θ_{12}' is $< \alpha_2$, we conclude

$$u_{12} = 0 \quad \theta_{12} = \theta_{12}'$$

Similarly we obtain successively

$$\theta_{13} = \theta_{13}', \ \theta_{14} = \theta_{14}', \cdots \ \theta_{1r} = \theta_{1r}'$$

and $u_{13} = u_{14} = \cdots = u_{1r} = 0$

Finally we get

$$\theta = \theta', \quad U = E_r$$

Thus the uniqueness of the normal *form* is proved. The exponents $\alpha_1, \alpha_2, \cdots \alpha_r$ in the main diagonal have the following meaning.

Let τ^{β_k} be the greatest common divisor of all the k-order subdeterminants in the matrix

$$\begin{pmatrix} \theta_{11} & \theta_{12} & \cdots & \theta_{1r} \\ \theta_{21} & & & \vdots \\ \vdots & & & \vdots \\ \theta_{r1} & \cdots & \cdots & \theta_{rr} \end{pmatrix} \quad ($$

then

$$\alpha_1 = \beta_1, \ \alpha_2 = \beta_2 - \beta_1, \cdots \ \alpha_r = \beta_r - \beta_{r-1}$$

If the point p is a branch point q_i the divisor β must be invariant with respect to C_i $(p = q_i)$ according to Weil's definition

$$\theta^{C_i} = U\theta$$

where U is the unit function matrix

$$\begin{pmatrix} \zeta^{\alpha_1}\tau^{\alpha_1} & \theta_{12}^{C_i} & \cdots & \theta_{1r}^{C_i} \\ & \zeta^{\alpha_2}\tau^{\alpha_2} & & \vdots \\ 0 & & \ddots & \zeta^{\alpha_r}\tau^{\alpha_r} \end{pmatrix} = \begin{pmatrix} \zeta^{\alpha_1} & u_{12} & \\ & \zeta^{\alpha_2} & \\ & & \ddots \ \zeta^{\alpha_r} \end{pmatrix} \begin{pmatrix} \tau^{\alpha_1} & \theta_{12} \cdots & \theta_{1r} \\ & \tau^{\alpha_2} & \vdots \\ 0 & & \ddots \ \tau^{\alpha_r} \end{pmatrix}$$

Comparing the (1,2)-element, we get

$$\theta_{12}^{c_1} = \zeta^{\alpha_1}\theta_{12} + u_{12}\tau^{\alpha_2}$$

Because the degree of $\theta_{12}^{c_1}$ and θ_{12} is smaller than α_2,

$$\theta_{12}^{c_i} = \zeta^{\alpha_1}\theta_{12}, \quad u_{12} = 0$$

We denote the fractional part of a real number x by $\langle x \rangle$ and the integral part by $[\,x\,]$

$$x = [x] + \langle x \rangle .$$

Now we set

$$d_i = \langle \frac{\alpha_i}{n} \rangle n, \quad \alpha_i' = [\frac{\alpha_i}{n}] n,$$

$$0 \leqq d_i < n$$

Then $\zeta^{\alpha_1} = \zeta^{d_1}$, and the function $\theta_{12}/\tau^{\alpha_1}$ remains invariant. Therefore the function $\theta_{12}^* = \theta_{12}/\tau^{\alpha_1}$ is a polynomial of t only. Let the degree of θ_{12} with respect to τ be g_{12} and that of θ_{12}^* with respect to t be g_{12}^*. Then

$$g_{12} < d_2 + n\alpha_2$$

$$g_{12} = d_1 + n g_{12}^*$$

By subtraction

$$d_1 - d_2 < (\alpha_2' - g_{12}^*)$$

From $d_1 \geqq d_2$ follows

$$\alpha_2' > g_{12}^*$$

and from $d_1 < d_2$ follows

$$\alpha_2' \geqq g_{12}^*$$

This argument applies easily to every g_{ik}. Thus the theorem is completely proved. q.e.d.

For convenience we denote the diagonal matrix on the left hand by

$$\langle \beta \rangle = \begin{pmatrix} \tau^{d_1} & & & \\ & \tau^{d_2} & & 0 \\ & & \ddots & \\ 0 & & & \tau^{d_r} \end{pmatrix}$$

and the canonical matrix on the right hand by

$$[\beta] = \begin{pmatrix} t^{\alpha_1} & \theta_{12} & \cdots & \theta_{1r} \\ & t^{\alpha_2} & & \vdots \\ & & \ddots & \\ 0 & & & t^{\alpha_r} \end{pmatrix}$$

We call the former $\langle \beta \rangle$ the fractional part and the latter the integral part of the local divisor β. Our Theorem 1 asserts the unique decomposition of β in the form $\langle \beta \rangle [\beta]$.

In the next theorem we prove the fact, which may be looked upon as the somewhat weakened form of the Theorem 1.

Theorem 2. If a matrix $\Omega \cdot \theta(t)$ belongs to local divisor β, where

$$\Omega = \begin{pmatrix} \tau^{d_1} & & & \\ & \tau^{d_2} & & \\ & & \ddots & \\ & & & \tau^{d_r} \end{pmatrix}, \quad 0 \leqq d_i < n$$

and $\theta(t)$ is a matrix belonging to K, then Ω is written as follows:

$$\Omega = P \langle \beta \rangle P^{-1}$$

where P is a permutation matrix.

Proof. We write $\Omega\theta$ in a normal form:

$$\Omega\theta = U \langle \beta \rangle [\beta].$$

Operating C on both sides, we obtain

$$\Omega^c \theta^c = U^c \langle \beta \rangle^c [\beta]^c$$

and

$$\Delta' \Omega \theta = U^c \Delta \langle \beta \rangle [\beta]$$

where

$$\Delta' = \begin{pmatrix} \zeta^{d_1'} & & & \\ & \zeta^{d_2'} & & \\ & & \ddots & \\ & & & \zeta^{d_r'} \end{pmatrix}, \quad \Delta = \begin{pmatrix} \zeta^{d_1} & & & \\ & \zeta^{d_2} & & \\ & & & \\ & & & \zeta^{d_r} \end{pmatrix}$$

From (1) and (2) we have

$$\Delta' = U^c \Delta U^{-1}$$

Expanding U in the power series of τ

$$U = U_0 + U_1 \tau + \cdots$$

$$U^{-1} = U_0^{-1} + U_1' \tau + \cdots$$

Then

$$\Delta' = (U_0 + U_1 \zeta \tau + \cdots)\Delta(U_0^{-1} + \cdots)$$

$$\Delta' = U_0 \Delta U_0^{-1}$$

We see that Δ and Δ' have the same characteristic values, differing only in the order. Finally we can conclude the existence of the permutation matrix P. such that

$$\Omega = P \langle \mathfrak{P} \rangle P^{-1}$$

q.e.d.

Next we go to the view-point in the large by the introduction of the divisor.

To every point $\mathfrak{p}$ of $\mathfrak{F}$ we suppose defined a local divisor of r-th order, which has $\mathfrak{p}$ as its base point, and among these only the finite number of them is different from E_r. Such a set of local divisors is called simply a divisor $\mathfrak{D}$. We will denote such a divison $\mathfrak{D}$ by

$$\mathfrak{D} = (\mathfrak{P}_1, \mathfrak{P}_2, \cdots \mathfrak{P}_n),$$

writing explicitly only local divisors different from E_r. Here the order of $\mathfrak{P}_i$ is unessential.

This expression corresponds to the decomposition of divisors in primary ones. But the above notation does not mean multiplication.

From the divisor $\mathfrak{D}$ we can construct the divisor of order 1, $|\mathfrak{D}|$, which we shall call the norm divisor

$$|\mathfrak{D}| = (|\mathfrak{P}_1|, |\mathfrak{P}_2|, \cdots |\mathfrak{P}_n|)$$

As in the case of local divisor we shall call the divisor

$$\langle \mathfrak{D} \rangle = (\langle \mathfrak{P}_1 \rangle, \langle \mathfrak{P}_2 \rangle, \cdots, \langle \mathfrak{P}_n \rangle)$$

and

$$[\mathfrak{D}] = ([\mathfrak{P}_1], [\mathfrak{P}_2], \cdots, [\mathfrak{P}_n])$$

the fractional and integral part of $\mathfrak{D}$ respectively. The degree of $\mathfrak{D}$ is defined as follows:

$$\deg(\mathfrak{D}) = \sum_{i=1}^{n} \deg |\mathfrak{P}_i|$$

By the Theorem 1 we obtain easily:

$$\deg(\mathfrak{D}) = \deg\langle \mathfrak{D} \rangle + \deg[\mathfrak{D}]$$

$\deg \langle \mathfrak{D} \rangle$ is called the ramification degree of $\mathfrak{D}$. The divisor $\mathfrak{D}$, for which $\langle \mathfrak{D} \rangle = E$ is called unramified.

Because of the above definition of local divisor $\mathfrak{D}$ the set of local function matrices $(\theta^T)^{-1}$ forms also a divisor which we shall call the contragredient divisor and designate by $\mathfrak{D}^K$. A remark should be made, that the set of inverse matrices θ^{-1} ($\theta \in \mathfrak{D}$) does not make a divisor in our sense.

Theorem 3. $\deg(\mathfrak{D}^K) = -\deg(\mathfrak{D})$ and

$$\langle \mathfrak{D}^K \rangle = P \langle\langle \mathfrak{D} \rangle^{-1} \rangle P^{-1}$$

where P is a permutation matrix, and

$$\deg \langle \mathfrak{D}^K \rangle = \deg \langle\langle \mathfrak{D} \rangle^{-1} \rangle$$

Proof. Considering $|\mathfrak{D}^K| = |\mathfrak{D}^{-1}| = |\mathfrak{D}|^{-1}$, we get

$$\deg(\mathfrak{D}^K) = -\deg \mathfrak{D}$$

By the Theorem 1

$$\mathfrak{D} = \langle \mathfrak{D} \rangle [\mathfrak{D}]$$

$$\mathfrak{D}^K = \langle\langle \mathfrak{D} \rangle^{-1} \rangle [\mathfrak{D}]^K$$

On the other hand

$$\mathfrak{D}^K = \langle \mathfrak{D}^K \rangle [\mathfrak{D}^K]$$

We obtain, using the Theorem 2

$$\langle \mathfrak{D}^K \rangle = P \langle\langle \mathfrak{D} \rangle^{-1} \rangle P^{-1}$$

and

$$\deg \langle \mathfrak{D}^K \rangle = \deg \langle\langle \mathfrak{D} \rangle^{-1} \rangle$$

q.e.d.

In our definition of divisors the ordinary multiplication seems to be difficult to introduce. But the direct or Kronecker multiplication can be defined as follows:

Let $\mathfrak{D}_1$ resp. $\mathfrak{D}_2$ be a divisor of order r_1, resp. r_2. If θ resp. θ' belongs to $\mathfrak{D}_1$ resp. $\mathfrak{D}_2$,

$$\theta = \begin{pmatrix} \theta_{11} & \cdots & \theta_{1r_1} \\ \theta_{21} & & \\ \vdots & & \\ \theta_{r_1 1} & \cdots & \theta_{r_1 r_1} \end{pmatrix}, \quad \theta' = \begin{pmatrix} \theta'_{11} & \cdots & \theta'_{1r_2} \\ \vdots & & \vdots \\ \theta'_{r_2 1} & & \theta'_{r_2 r_2} \end{pmatrix}$$

We form a matrix of $r_1 r_2$ order, that is the left direct product.

$$\theta \cdot \times \theta' = \begin{pmatrix} \theta\,\theta'_{11} & \cdots & \theta\,\theta'_{1r_2} \\ \vdots & & \vdots \\ \theta\,\theta'_{r_2 1} & & \theta\,\theta'_{r_2 r_2} \end{pmatrix}$$

employing Macduffee's notation.[11].

The $(r_1 r_2)$-divisor, which contains all such $\theta \cdot \times \theta'$ is called the left direct product of $\mathfrak{D}_1$ and $\mathfrak{D}_2$ and denoted by $\mathfrak{D}_1 \cdot \times \mathfrak{D}_2$. We see easily:

Theorem 4. $\deg(\mathfrak{D}_1 \cdot \times \mathfrak{D}_2) = r_2 \deg \mathfrak{D}_1 + r_1 \deg \mathfrak{D}_2$

and

$$\langle \mathfrak{D}_1 \cdot \times \mathfrak{D}_2 \rangle = P \langle\langle \mathfrak{D}_1 \rangle \cdot \times \langle \mathfrak{D}_2 \rangle\rangle P^{-1}$$

where P is a permutation matrix.

Then

$$\deg \langle \mathfrak{D}_1 \cdot \times \mathfrak{D}_2 \rangle = \deg \langle\langle \mathfrak{D}_1 \rangle \cdot \times \langle \mathfrak{D}_2 \rangle\rangle$$

Proof. By the well-known formula in the determinant theory

$$|\mathfrak{D}_1 \cdot \times \mathfrak{D}_2| = |\mathfrak{D}_1|^{r_2} \cdot |\mathfrak{D}_2|^{r_1}$$

We obtain easily

$$\deg(\mathfrak{D}_1 \cdot \times \mathfrak{D}_2) = r_2 \deg \mathfrak{D}_1 + r_1 \times \deg \mathfrak{D}_2.$$

As for the second formula

$$\mathfrak{D}_1 = \langle \mathfrak{D}_1 \rangle [\mathfrak{D}_1], \quad \mathfrak{D}_2 = \langle \mathfrak{D}_2 \rangle [\mathfrak{D}_2]$$

so

$$\mathfrak{D}_1 \cdot \times \mathfrak{D}_2 = (\langle \mathfrak{D}_1 \rangle \cdot \times \langle \mathfrak{D}_2 \rangle)([\mathfrak{D}_1] \cdot \times [\mathfrak{D}_2])$$

We set

$$\langle \mathfrak{D}_1 \rangle \cdot \times \langle \mathfrak{D}_2 \rangle = \langle\langle \mathfrak{D}_1 \rangle \cdot \times \langle \mathfrak{D}_2 \rangle\rangle \cdot \theta$$

where θ is a function of t only. Then by Theorem 2

$$\langle \mathfrak{D}_1 \cdot \times \mathfrak{D}_2 \rangle = P \langle\langle \mathfrak{D}_1 \cdot \times \mathfrak{D}_2 \rangle\rangle P^{-1}$$

From this we get

$$\deg \langle \mathfrak{D}_1 \cdot \times \mathfrak{D}_2 \rangle = \deg \langle\langle \mathfrak{D}_1 \rangle \times \langle \mathfrak{D}_2 \rangle\rangle$$

As a natural counterpart of direct product we may define the direct sum

$$\mathfrak{D}_1 \dotplus \mathfrak{D}_2 = \begin{pmatrix} \mathfrak{D}_1 & 0 \\ 0 & \mathfrak{D}_2 \end{pmatrix}$$

Still more easily we can prove for the direct sum

Theorem 5.

$$\deg(\mathfrak{D}_1 \dotplus \mathfrak{D}_2) = \deg(\mathfrak{D}_1) + \deg(\mathfrak{D}_2)$$

$$\langle \mathfrak{D}_1 \dotplus \mathfrak{D}_2 \rangle = \langle \mathfrak{D}_1 \rangle \dotplus \langle \mathfrak{D}_2 \rangle$$

$$[\mathfrak{D}_1 \dotplus \mathfrak{D}_2] = [\mathfrak{D}_1] \dotplus [\mathfrak{D}_2].$$

As in the classical theory we can introduce the notion of divisor class. Two divisors $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of the same order r, which can be obtained from each other by the right multiplication of a non-singular matrix Φ belonging to K, is said to belong to the same divisor class, that is,

$$\mathfrak{D}_1 = \mathfrak{D}_2 \Phi.$$

Of course the direct product and sum of divisors are not commutative, but those of divisor classes are commutative, for

$$\begin{pmatrix} 0 & E_{r_1} \\ E_{r_2} & 0 \end{pmatrix} \begin{pmatrix} \mathfrak{D}_1 & 0 \\ 0 & \mathfrak{D}_2 \end{pmatrix} \begin{pmatrix} 0 & E_{r_2} \\ E_{r_1} & 0 \end{pmatrix} = \begin{pmatrix} \mathfrak{D}_2 & 0 \\ 0 & \mathfrak{D}_1 \end{pmatrix}$$

and

$$P(\mathfrak{D}_1 \cdot \times \mathfrak{D}_2) P^{-1} = \mathfrak{D}_2 \cdot \times \mathfrak{D}_1$$

Moreover considering the distributive law:

$$\mathfrak{D}_1 \cdot \times (\mathfrak{D}_2 \dotplus \mathfrak{D}_3) = (\mathfrak{D}_1 \cdot \times \mathfrak{D}_2) \dotplus (\mathfrak{D}_1 \cdot \times \mathfrak{D}_3)$$

The following theorem holds

Theorem 6. The set of all divisor classes is additive and multiplicatively closed, and addition and multiplication are both commutative, associative and distributive.

This algebraic system of divisor classes, which is denoted by D, is a natural generalization of divisor class group in the classical theory, and the subsystem D_0 of divisor classes of degree 0 is also important.

Theorem 7. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ belong to the same class,

$$P\langle\mathfrak{D}_1\rangle P^{-1} = \langle\mathfrak{D}_2\rangle$$

Proof.

$$\mathfrak{D}_1 \Phi = \mathfrak{D}_2$$

$$\langle\mathfrak{D}_1\rangle[\mathfrak{D}_1]\Phi = U\langle\mathfrak{D}_2\rangle[\mathfrak{D}_2].$$

By the Theorem 2,

$$P\langle\mathfrak{D}_1\rangle P^{-1} = \langle\mathfrak{D}_2\rangle \qquad \text{q.e.d.}$$

This theorem asserts that the exponents d_i $(i = 1,2,\ldots,r)$ of τ in $\langle\mathfrak{D}\rangle$ at the base point $\mathfrak{q}_\mu$ are invariant, when we replace $\mathfrak{D}$ by any divisor equivalent to $\mathfrak{D}$, if we disregard their ordering. Let $N_{\mu\alpha}$ be a number of d_i which are equal to α at $\mathfrak{q}_\mu$, then $N_{\mu\alpha}$ $(\mu = 1,2,\ldots \ell)$, called the ramification indices, are class invariants of divisor classes. By means of $N_{\mu\alpha}$ we can express:

$$\deg\langle\mathfrak{D}\rangle = \sum_\mu \sum_{\alpha=0}^{n_\mu - 1} \frac{\alpha N_{\mu\alpha}}{n_\mu}$$

For any divisor $\mathfrak{D}$, there exist functions Φ belonging to K such that $\mathfrak{D}\Phi$ are everywhere finite. The set $\{\mathfrak{D}\}$ of all such Φ makes a linear system. The dimension of such $\{\mathfrak{D}\}$ is called the dimension of $\mathfrak{D}$ and denoted by $\dim \mathfrak{D}$. Together with $\deg \mathfrak{D}$ is also $\dim \mathfrak{D}$ class invariant of $\mathfrak{D}$.

Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be divisors and r_1 and r_2 be their orders, then there may be a (r_1, r_2)-matrix Φ belonging to K, such that $\mathfrak{D}_1 \Phi \mathfrak{D}_2^{-1}$ is everywhere finite. We see immediately that these Φ belongs to $\{\mathfrak{D}_1 \cdot \times \mathfrak{D}_2^{K}\}$.

By the above notation we can write Riemann-Roch's theorem generalised by Weil as follows:

$$\dim(\mathfrak{D}_1 \cdot \times \mathfrak{D}_2^{K}) = \dim(\mathfrak{D}_2 \cdot \times \mathfrak{D}_1 \times w)$$
$$+ r_2 \deg \mathfrak{D}_1 - r_1 \deg \mathfrak{D}_2$$
$$- \deg\langle\langle\mathfrak{D}_1\rangle \times \langle\mathfrak{D}_2\rangle^{-1}\rangle - r_1 r_2 (p-1),$$

where w is a differential divisor. This formula we shall call the Riemann-Roch-Weil's theorem, which plays the fundamental role in our whole theory.

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