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ON COMPLEX OSCILLATION AND A PROBLEM OF OZAWA

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Abstract

It is shown that if $Q(z)$ is a non-constant polynomial, then all non-trivial solutions of

$$y''+(e^z+Q(z))y=0$$

have zeros with infinite exponent of convergence. Similar methods are used to settle a problem of M. Ozawa: if $P(z)$ is a non-constant polynomial, all non-trivial solutions of

$$y''+e^{-z}y'+P(z)y=0$$

have infinite order.

1. Introduction.

We are concerned with the order of growth of solutions of the differential equation

$$y''+A(z)y'+B(z)y=0 \tag{1.1}$$

where $A(z)$ and $B(z)$ are entire functions of finite order, not both polynomials. Denoting by $\sigma(g)$ the order of an entire function g, we note that if $\sigma(B)>\sigma(A)$ then it follows at once from the lemma of the logarithmic derivative (see [11]) that all non-trivial solutions of (1.1) are entire functions of infinite order. However, if $\sigma(B)<\sigma(A)$ it may be difficult to determine whether (1.1) can have non-trivial solutions of finite order, although since a finite order solution cannot grow large when $A(z)$ is close to its maximum modulus in this case, certain equations (for example if $A(z)=\sin(z^n)$) are easily dealt with.

Ozawa [13] considered the equation

$$y''+e^{-z}y'+P(z)y=0 \tag{1.2}$$

where $P(z)$ is a polynomial. A result of Frei [7] states that if P is a constant, then (1.2) has a solution of finite order if and only if $P=-n^2$, where n is an integer. (See also [3] for a result on the oscillation of solutions of (1.2) when P is a constant). The case where P is non-constant is more difficult to resolve and results of Ozawa [13], Amemiya and Ozawa [1], and Gundersen [9] may be summarised as follows:

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THEOREM A. *Let* $P(z)=a_n z^n+a_p z^p+\cdots+a_0$ *be a non-constant polynomial. Then all non-trivial solutions of*

$$y''+e^{-z}y'+P(z)y=0$$

have infinite order if $P(z)$ *satisfies any of the following*:

(a) n *is odd*;

(b) n *is even, and* $n\geqq 2p+3$;

(c) $n=2$, *and* $a_1=a_0=0$;

(d) n *is even, and* $a_n(-1)^{n/2}$ *is not real and negative.*

Now the problem of the growth of solutions of (1.1) is related to that of the oscillation of solutions of

$$y''+A(z)y=0 \tag{1.3}$$

where $A(z)$ is a transcendental entire function of finite order. Denoting by $\lambda(g)$ the exponent of convergence of the zeros of g, it is shown in [2] that the product $E(z)$ of two linearly independent solutions f_1 and f_2 of (1.3) itself satisfies

$$4A=(E'/E)^2-2(E''/E)-(1/E^2) \tag{1.4}$$

after a normalisation which makes the Wronskian of f_1 and f_2 equal to 1. Now (1.4) implies that if $\lambda(E)$ is finite then so is $\sigma(E)$ and moreover E is small where A is large so that if, for example, $\sigma(A)<\frac{1}{2}$ then $\lambda(E)$ must be infinite (see [2]). It is conjectured that if $\lambda(E)$ is finite then $\sigma(A)$ must be a positive integer. On the other hand examples of integer order do exist—there are pairs of polynomials P and Q (see [5]) whose degrees d_P, d_Q satisfy

$$d_Q+2=2d_P$$

such that the equation

$$y''+(e^P+Q)y=0 \tag{1.5}$$

has two linearly independent non-vanishing solutions. These examples make the following result sharp [5]:

THEOREM B. *Let* $A(z)$ *be a transcendental entire function of finite order* ρ *with the following property*: *there exists a set* $H\subseteqq \boldsymbol{R}$, *of measure zero, such that for each real* θ *not in* H, *either*

(i) $r^{-N}|A(re^{i\theta})|\to\infty$ *as* $r\to\infty$, *for each* $N>0$,

or

(ii) $\int_0^\infty r|A(re^{i\theta})|dr<\infty$,

or

(iii) *there exists* $n\geqq 0$, *possibly depending on* θ, *such that* $(n+2)<2\rho$ *and*

$$A(re^{i\theta})=O(r^n) \qquad as \quad r\to\infty.$$

Then if f_1, f_2 *are linearly independent solutions of*

$$y''+A(z)y=0$$

we have $\max\{\lambda(f_1), \lambda(f_2)\}=\infty$.

Now if (1.4) has a solution E of finite order, which must be the product of two solutions of (1.3) (see [4]), then E can only grow large where the growth of $A(z)$ is bounded by some power of $|z|$. In view of this fact, Theorem B and the examples preceding it, it is of interest to consider equations of the form (1.5), and the following was proved in [5]:

THEOREM C. *Let* $K\in\boldsymbol{C}$ *and suppose that*

$$f''+(e^z-K)f=0 \tag{1.6}$$

has a non-trivial solution with $\lambda(f)<\infty$.

Then

$$K=q^2/16 \tag{1.7}$$

where q *is an odd integer. Conversely if* K *satisfies* (1.7) *with* q *odd, then* (1.6) *has two linearly independent solutions* f_1, f_2 *with* $\lambda(f_1f_2)\leqq 1$.

We remark that this result gives examples where $\sigma(E)$ is finite, but E has zeros, although it seems worth noting that $\sigma(E)=\sigma(A)$ and the author is unaware of any examples of solutions of (1.4) for which A is transcendental and

$$\sigma(A)<\sigma(E)<\infty\,.$$

In the case where P is linear and Q is non-constant we are able to prove the following:

THEOREM 1. *Let* $Q(z)$ *be a non-constant polynomial, and let* $\alpha\in\boldsymbol{C}$. *Then every non-trivial solution* f *of*

$$y''+(e^{z+\alpha}+Q(z))y=0 \tag{1.8}$$

satisfies $\lambda(f)=\infty$.

Now it turns out that (1.8) can only have a solution f with $\lambda(f)<\infty$ if the equation

$$y''+\left(2ce^{z/2}-\frac{1}{2}\right)y'+\left(Q(z)+\frac{1}{16}\right)y=0,$$

where $c\in\boldsymbol{C}\setminus\{0\}$, has a solution of finite order (see §3 for a proof of this fact) and of course this equation is very close to (1.2). By similar methods to those of Theorem 1 we are able to settle Ozawa's problem:

THEOREM 2. *Let* $Q(z)$ *be a non-constant polynomial. Then all non-trivial solutions of*

$$y''+Ae^{-z}y'+Q(z)y=0 \tag{1.9}$$

have infinite order, for any $A\in\boldsymbol{C}\backslash\{0\}$.

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2. Notation and Preliminary Lemmas.

We define an R-set to be a countable set of discs in the plane the sum of whose radii is finite, and remark, following Hayman [10], that the set of θ for which the ray $re^{i\theta}$ meets infinitely many discs of a given R-set has measure zero.

A key role is played by the following lemma:

LEMMA 1. *Let* S *be the strip*

$$z=x+iy, \qquad x\geqq x_0, \qquad |y|\leqq 4.$$

Suppose that in S

$$Q(z)=a_n z^n+O(|z|^{n-2})$$

where n *is a positive integer and* $a_n>0$. *Then there exists a path* Γ *tending to infinity in* S *such that all solutions of*

$$y''+Qy=0$$

tend to zero on Γ.

Proof. We set $N=(n+2)/2$ and

$$Z=\int_a^z Q(t)^{1/2}dt$$

for some large a in S. Then Z satisfies

$$\begin{aligned} Z&=\int_a^z k_1 t^{n/2}(1+O(t^{-2}))dt \\ &=k_2 z^N+o(|z|^{N-1}) \end{aligned} \tag{2.1}$$

in the smaller strip S_1 given by

$$x\geqq x_1, \qquad |y|\leqq 2.$$

Here k_1, k_2, $\cdots$ denote positive constants.

We assert that Z is univalent in the strip S_2 given by

$$x\geqq x_2, \qquad |y|\leqq 1$$

for some large x_2. For suppose that z and z_1 are in S_2. Then since

$$z^N - z_1^N = (z^N + z_1^N)^{-1}(z-z_1)z_1^{2N-1}\left(1+\cdots+\left(\frac{z}{z_1}\right)^{2N-1}\right)$$

we have

$$|z^N - z_1^N| \geqq k_3 |z-z_1| \max\{|z|^{N-1}, |z_1|^{N-1}\} \tag{2.2}$$

using the fact that $Re((z/z_1)^j)>0$ for $j=1, \cdots, 2N-1$ if x_2 is large enough. So (2.1) and (2.2) imply that if z_1 and z_2 are large with $Z(z_1)=Z(z_2)$, then $|z_1-z_2| <1/2$. But then applying Rouché's theorem on the circle $|z-z_1|=1/2$, and using (2.1) and (2.2) we see that $Z(z)-Z(z_1)$ has but one zero in $|z-z_1|<1/2$.

We assert now that if X is large and positive, $Z(z)$ takes the value X at some point in S_2. This follows again from Rouché's theorem. Set $x_3=X^{1/N}$. Then on the circle $|z-x_3|=1/2$, using (2.2), we have

$$Z(z)-X=(1+o(1))(z^N-x_3^N).$$

We now choose Γ to be the preimage in S_2 of the half line $\mathcal{L}$ given by

$$Z=X, \qquad X \geqq X_0,$$

with X_0 large. We make the standard transformation

$$y(z)=Q(z)^{-(1/4)}U(Z) \tag{2.3}$$

so that $U(Z)$ satisfies

$$\frac{d^2U}{dZ^2}+(1-F(Z))U=0$$

where

$$F(Z)=\frac{1}{4}\frac{Q''(z)}{Q(z)^2}-\frac{5}{16}\frac{Q'(z)^2}{Q(z)^3}.$$

Hence, for large z in S_2 and $Z=Z(z)$,

$$\begin{aligned}F(Z)&=O(|z|^{-(n+2)})\\&=O(|Z|^{-2})\end{aligned}$$

so that writing

$$U(Z)=U_1(Z)+\int_{X_0}^{Z}\sin(Z-t)F(t)U(t)\,dt$$

where U_1 satisfies

$$U_1''+U_1=0$$

we obtain

$$|U| \leqq k_4+k_5\int_{X_0}^{Z}|F(t)U(t)|\,dt$$

for Z on $\mathcal{L}$ so that U is bounded on $\mathcal{L}$ by Gronwall's lemma ([6], p 35). By (2.3) we see that $y(z)\to 0$ on Γ.

We require the growth estimate below, deduced in [5] from Herold's comparison theorem [12]:

LEMMA A. *Suppose that $A(z)$ is analytic in a sector containing the ray $\mathcal{L}$: $re^{i\theta}$ and that as $r\to\infty$,*

$$A(re^{i\theta})=O(r^{n})$$

for some $n\geqq 0$. Then all solutions of

$$y''+A(z)y=0$$

satisfy

$$\log^{+}|y(re^{i\theta})|=O(r^{(n+2)/2})$$

on $\mathcal{L}$.

3. Proof of Theorem 1.

We assume that (1.8) has a solution $y(z)$ with $\lambda(y)<\infty$. By a translation we may assume that

$$Q(z)=a_{n}z^{n}+a_{n-2}z^{n-2}+\cdots \tag{3.1}$$

if $n\geqq 2$ or

$$Q(z)=a_{1}z \tag{3.2}$$

if $n=1$. We define the critical rays for Q as those rays $re^{i\theta}$ for which

$$\operatorname{Arg} a_{n}+(n+2)\theta=0 \pmod{2\pi}$$

and remark that the substitution $z=xe^{i\theta}$ transforms the equation (1.8) into

$$\frac{d^{2}y}{dx^{2}}+(e^{2i\theta}e^{\alpha+xe^{i\theta}}+Q_{1}(x))y=0,$$

where $Q_{1}(x)=\alpha_{1}x^{n}+O(x^{n-2})$ and $\alpha_{1}>0$. By Lemma 1, for any critical θ lying in $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ there exists a path Γ_{θ} tending to infinity, such that $\arg z\to\theta$ on Γ_{θ} while $y(z)\to 0$ there.

A representation for $y(z)$.

Since $\lambda(y)<\infty$ we may write $y=\Pi e^{h}$ where $\sigma(\Pi)$ is finite, so that

$$\frac{\Pi''}{\Pi}+\frac{2\Pi'}{\Pi}h'+h''+h'^{2}+e^{z+\alpha}+Q=0. \tag{3.3}$$

We differentiate (3.3) and subtract (3.3) from the differentiated equation, and find that

$$h'(2h''-h')=P(h')$$

where $P(h')$ is a differential polynomial in h', linear in $h', h'', \cdots$, and with coefficients which are differential polynomials in Π'/Π. Clunie's Lemma applies ([11], p 68) and we deduce that

$$2h''-h'=P_1(z),$$

with P_1 a polynomial, so that

$$h'=ce^{z/2}+P_2(z), \tag{3.4}$$

where c is a constant and P_2 is a polynomial. We then set

$$y=W\exp\left(2ce^{z/2}-\frac{z}{4}\right) \tag{3.5}$$

and substitute in the equation (1.8). (This device is due to S. Bank.) We remark that by (3.4) and (3.5), W has finite order since Π has.

Now W satisfies

$$W''+\left(2ce^{z/2}-\frac{1}{2}\right)W'+\left(c^2e^z+e^{z+\alpha}+Q(z)+\frac{1}{16}\right)W=0.$$

Setting $V=W'/W$ we have

$$V'+V^2+\left(2ce^{z/2}-\frac{1}{2}\right)V+\left(c^2e^z+e^{z+\alpha}+Q(z)+\frac{1}{16}\right)=0. \tag{3.6}$$

Since

$$|V'|+|V|=O(|z|^M)$$

outside an R-set U we deduce from (3.6) that $c^2=-e^\alpha$. Moreover, if $\phi\in\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is such that the ray $re^{i\phi}$ meets only finitely many discs of U we see that $V=o(|z|^{-2})$ as z tends to infinity on this ray and hence that W tends to a finite, non-zero limit. Applying this reasoning to a set of ϕ outside a set of zero measure we deduce by the Phragmén-Lindelöf principle that with no loss of generality, if ε is positive, then

$$W(re^{i\theta})\longrightarrow 1 \tag{3.7}$$

as $r\to\infty$ with $|\theta|<\frac{\pi}{2}-\varepsilon$. We deduce also from (3.5) that $W\to 0$ along the paths Γ_θ determined by the critical rays in the sector $\frac{\pi}{2}<\theta<\frac{3\pi}{2}$.

The order of W.

By Lemma A, $y(z)$ satisfies

$$\log^+|y(re^{i\theta})|=O(r^{(n+2)/2}) \tag{3.8}$$

as $r\to\infty$, for any θ in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Thus (3.5), (3.7), (3.8) and the fact that ε is arbitrary imply that, by the Phragmén-Lindelöf principle,

$$\sigma(W) \leqq \frac{n+2}{2}. \tag{3.9}$$

Conclusion of the proof.

If $n=1$, at least one critical θ for $Q(z)$ must lie in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. But this implies the existence of an unbounded domain of angular measure at most $\frac{\pi}{3}+\varepsilon$, bounded by a path on which $W(z)\to 0$ and a ray on which $W(z)\to 1$. But then the Phragmén-Lindelöf Theorem (see eg. [8], p 104) implies that $\sigma(W)\geqq 3$, since ε is arbitrary, and this contradicts (3.9).

In the case $n\geqq 2$ we claim that $\theta=\frac{\pi}{2}$ is a critical ray for $Q(z)$. For otherwise there exists a critical θ for Q in

$$\frac{\pi}{2}<\theta<\frac{\pi}{2}+\frac{2\pi}{n+2}$$

and the same reasoning as above, with

$$\varepsilon=\frac{1}{2}\left(\frac{\pi}{2}+\frac{2\pi}{n+2}-\theta\right)$$

implies that $\sigma(W)>\frac{n+2}{2}$, contradicting (3.9). But now by Lemma 1, $W(z)\to 0$ on a path $\Gamma_{\pi/2}$ on which $\arg z\to\frac{\pi}{2}$ and this combined with (3.7) provides a contradiction.

4. Proof of Theorem 2.

Since the proof is very similar to that of Theorem 1 we present only a sketch. Assuming that (1.9) has a solution $y(z)$ of finite order and that $Q(z)$ satisfies (3.1), or (3.2) if Q is linear, we deduce as in the proof of Theorem 1 that, given any position ε, with no loss of generality

$$y(z) \longrightarrow 1 \tag{4.1}$$

as $z=re^{i\theta}\to\infty$ with $\frac{\pi}{2}+\varepsilon<\theta<\frac{3\pi}{2}-\varepsilon$.

We now set

$$y=uv \tag{4.2}$$

where

$$v=\exp\left(\frac{A}{2}e^{-z}\right) \tag{4.3}$$

and u satisfies

$$u''+\left(Q(z)-\frac{A^2}{4}e^{-2z}+\frac{A}{2}e^{-z}\right)u=0. \tag{4.4}$$

Now, as $z\to\infty$ with $|\arg z|<\frac{\pi}{2}$, we see that $v(z)\to 1$, while on the strip

$$z=x+iy, \qquad |x|\leqq 4$$

$|v(z)|$ is bounded above and below. Defining again the critical rays for $Q(z)$ as those rays $re^{i\theta}$ for which

$$\operatorname{Arg} a_n+(n+2)\theta=0 \qquad (\bmod 2\pi)$$

we deduce from (4.2), (4.3), (4.4) and Lemma 1 that for any critical ray with θ lying in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ there exists a path Γ_θ on which $u(z)$, and moreover $y(z)$, tend to zero, and on which arg $z\to\theta$, while the same is true if $n\geqq 2$ and $\theta=\pm\frac{\pi}{2}$ is a critical ray.

As in the proof of Theorem 1 we obtain from Lemma A the estimate

$$\sigma(y)\leqq\frac{n+2}{2}, \tag{4.5}$$

and proceed to a contradiction as follows. If $n=1$ some critical ray for Q must lie in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and we find that $y(z)\to 1$ on a ray and $y(z)\to 0$ on a path which together bound a region of angular measure at most $\frac{\pi}{3}+\varepsilon$, implying that $\sigma(y)>\frac{3}{2}$, which contradicts (4.5). On the other hand if $n\geqq 2$ the same reasoning implies that $\theta=\frac{\pi}{2}$ is a critical ray for $Q(z)$ and again we find that $y(z)$ tends to zero on a path $\Gamma_{\pi/2}$. Since y has finite order and ε is arbitrary, this and (4.1) provide a contradiction.

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