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ENERGY, TENSION AND FINITE TYPE MAPS

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Abstract

We study the spectral geometry of smooth maps of a compact Riemannian manifold in a Euclidean space, by using the notion of order (introduced by the first author). We give some best possible estimates of energy and total tension of a map in terms of order. Some applications to closed curves and harmonic maps are then obtained. In the last section, we relate the spectral geometry of the Gauss map of a submanifold to its topology and derive some topological obstructions to submanifolds to have a Gauss map of low type.

1. Introduction.

Let M^n be a compact Riemannian manifold and x a smooth map from M^n into the Euclidean space E^{n+m}. To study x, it is natural to consider the spectral decomposition of x with respect to the Laplacian of M^n. This point of view has been adopted by the first author, when x is an isometric immersion [4, 5]. Using the same idea, we define two numbers p and q, canonically associated with x; p is a positive integer, and q is either ∞ or an integer $\geqq p$. The pair $[p, q]$ is called the *order of the map*. A map is called *a finite type map* is q is finite. Thus, we obtain spectral invariants related to the map. From Section 3 to Section 5, we relate the geometric properties of the map to its order and its type. In particular, we give in Section 3 a best possible estimate of the total tension of a map in terms of order, and then, in terms of λ_1 and energy. In Section 4, some relations between moment, energy and order are obtained. In Section 5 the notion of order is applied to obtain a necessary and sufficient condition for a spherical map to be harmonic. As an application of the previous sections, we study the Gauss map associated to a submanifold. We show that the spectral geometry of the Gauss map is related to the topology of the submanifold. In particular, if M^n is a compact submanifold of E^m with nonzero self-intersection number, the type of its Gauss map is "large" ($>n/2$).

The results of the first part of this paper have been announced in [6]. Some classifications of submanifolds with 1 or 2 type Gauss map can be found in [3], [7].

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2. Preliminaries.

Let M be a compact Riemannian manifold of dimension n and Δ the Laplacian of M acting on the space $C^\infty(M)$ of smooth functions. Then Δ has an infinite discrete sequence of eigenvalues:

$$0=\lambda_0<\lambda_1<\lambda_2<\cdots<\lambda_k<\cdots\uparrow\infty .$$

For each $k=0, 1, 2, \cdots$, the eigenspace $V_k=\{f\in C^\infty(M) \mid \Delta f=\lambda_k f\}$ is finite-dimensional. With respect to the inner product $(f, g)=\int_M fg\, dV$ on $C^\infty(M)$, the decomposition $\sum_k V_k$ is orthogonal and dense in $C^\infty(M)$. Therefore, for each $f\in C^\infty(M)$, $f=f_0+\sum_{t\geq 1}f_t$, where f_0 is a constant and f_t is the projection of f into V_t.

For any smooth map $x: M\to E^{n+m}$ of the compact Riemannian manifold M into the Euclidean $(n+m)$-space E^{n+m}, we can apply the above decomposition to the E^{n+m}-valued function x:

$$x=x_0+\sum_{t=1}^{\infty} x_t , \tag{2.1}$$

where x_0 is a constant vector and x_t an eigenvector with $\Delta x_t=\lambda_t x_t$.

If x is a non-constant map, there is a natural number p such that $x_p\neq 0$ and $x=x_0+\sum_{t\geq p}x_t$. If there are infinitely many nonzero x_t's in the decomposition (2.1), we put $q=\infty$. Otherwise, we put q to be the largest integer such that $x_q\neq 0$ in the spectral decomposition (2.1). In any case, we have

$$x=x_0+\sum_{t=p}^{q} x_t . \tag{2.2}$$

As in [4, 5], we call $[p, q]$ the *order of the map* x. Moreover, the map $x: M\to E^{n+m}$ is said to be *of finite type* if q is finite. Otherwise, x is said to be *of infinite type*. More precisely, x is said to be *of k-type* ($k\in \mathbf{N}\cup\{\infty\}$) if there exist exactly k nonzero x_t's ($t\geqq 1$) in the spectral decomposition (2.2).

If M is a compact submanifold of E^{n+m}, then M is a compact Riemannian manifold with respect to the induced Riemannian metric. In this case the submanifold M is said to be of k-type if the immersion is of k-type.

The following result can be proved exactly in the same way as that of Theorem 2.1 of [5, p. 255]. (see, also [1]).

PROPOSITION 2.1. *Let $x: M\to E^{n+m}$ be a non-constant map of a compact*

Riemmanian manifold M into E^{n+m}. Then x is of finite type if and only if there is a non-trivial polynomial $Q(t)$ such that $Q(\Delta)(x-x_0)=0$.

It follows from (2.2) that $x_0=\int_M x\,dV \Big/ \int_M dV$, where dV denotes the volume element of M. This simply says that x_0 is the *center of mass* of x.

If $\varphi: M \to N$ is a map between Riemannian manifolds, the *energy-density* $e(\varphi)$ of φ is the real-valued function on M given by

$$e(\varphi)=\frac{1}{2}\|d\varphi\|^2=\frac{1}{2}\operatorname{trace}(\varphi^* g'), \tag{2.3}$$

where g' is the metric on N and $d\varphi=\varphi^*$. The *energy* $E(\varphi)$ of φ is defined by

$$E(\varphi)=\int_M e(\varphi)dV. \tag{2.4}$$

The Euler-Lagrange operator associated with E shell be written $\tau(\varphi)=\operatorname{div}(d\varphi)$ and called the *tension field* of φ. A map φ is said to be *harmonic* if its tension field vanishes identically.

For the map $x: M \to E^{n+m}$, one has (cf. [8])

$$\Delta x=-\tau(x). \tag{2.5}$$

Similar to Proposition 2.1, we have

PROPOSITION 2.2. *Let $x: M \to E^{n+m}$ be a non-constant map of a compact Riemannian manifold M into E^{n+m}. Then x is of finite type if and only if there is a non-trivial polynomial $Q(t)$ such that $Q(\Delta)\tau=0$, where τ is the tension field of x.*

If $x: M \to E^{n+m}$ is of finite type, there is a monic polynomial $P(t)$ of least degree with $P(\Delta)\tau=0$. The following result follows easily from Proposition 2.2.

PROPOSITION 2.3. *If $x: M \to E^{n+m}$ is a finite type non-constant map, then*

(1) *the polynomial $P(t)$ is unique,*
(2) *if Q is any polynomial with $Q(\Delta)\tau=0$, P is a factor of Q, and*
(3) *x is of k-type if and only if $k=\deg P$.*

The same holds if τ is replaced by $x-x_0$.

The unique polynomial P, associated with the finite type map $x: M \to E^{n+m}$, is called the *minimal polynomial* of x.

If $x: M \to S_c^{n+m-1} \subset E^{n+m}$ is a map of M into a hypersphere S_c^{n+m-1} of E^m, then x is called *mass-symmetric* if the center of mass, x_0, is the center c of the hypersphere in E^{n+m}.

We shall make use of the following convention on the ranges of indices unless mentioned otherwise:

$$1 \leqq i, j, k, \cdots \leqq n ; \quad n+1 \leqq r, s, t, \cdots \leqq n+m ;$$
$$n+1 \leqq \alpha, \beta, \gamma, \cdots \leqq n+m-1 .$$

Remark 1. From (2.5) we know that if $x, \bar{x}: M \rightarrow E^{n+m}$ are two maps from a compact Riemannian manifold M into E^{n+m} such that $x, \bar{x}$ have the same tension field, then x and $\bar{x}$ differ only by a translation.

3. Total Tension and Order.

Let $x: M \rightarrow E^{n+m}$ be a smooth map of a compact Riemannian n-manifold M into E^{n+m}. Denote by $\tau=\tau(x)$ the tension field of x. The *total tension* $\mathcal{T}(x)$ of x is given by

$$\mathcal{T}(x)=\int_M \|\tau\|^2 dV . \tag{3.1}$$

The following result gives a best possible estimate of the total tension in terms of the order.

THEOREM 3.1. *Let $x: M \rightarrow E^{n+m}$ be a non-constant map of a compact Riemannian manifold M into E^{n+m}. Then we have*

$$2\lambda_p E(x) \leqq \int_M \|\tau\|^2 dV \leqq 2\lambda_q E(x) , \tag{3.2}$$

where $[p, q]$ is the order of x. Either equality sign in (3.2) *holds if and only if x is of* 1-*type.*

Proof. Since $[p, q]$ is the order of x, we have

$$x=x_0+\sum_{t=p}^{q} x_t . \tag{3.3}$$

Thus, by (2.5), we find

$$-\tau=\sum_{t=p}^{q} \lambda_t x_t . \tag{3.4}$$

Since $\Delta=d\delta+\delta d$ is a self-adjoint operator on $C^\infty(M)$, we obtain from (2.3), (2.4) and (3.4) that

$$\begin{aligned} 2E(x)&=\int_M \|dx\|^2 dV=(dx, dx) \\ &=(x, \Delta x)=\sum_{t=p}^{q} \lambda_t(x_t, x_t), \end{aligned} \tag{3.5}$$

$$\int_M \|\tau\|^2 dV=(\Delta x, \Delta x)=\sum_{t=p}^{q} \lambda_t^2(x_t, x_t). \tag{3.6}$$

Thus

$$\int_M \|\tau\|^2 dV - 2\lambda_p E(x) = \sum_{t=p}^{q} \lambda_t(\lambda_t - \lambda_p)(x_t, x_t) \geqq 0, \tag{3.7}$$

equality holding if and only if x_p is the only nonzero component. The other inequality is obtained in the same way. (Q. E. D.)

If x is an isometric immersion, Theorem 3.1 is due to [4].

The following corollaries follow immediately from Theorem 3.1.

COROLLARY 3.1. *If $x: M \to E^{n+m}$ is a non-constant map of a compact Riemannian manifold into E^{n+m}, then we have*

$$\int_M \|\tau\|^2 dV \geqq 2\lambda_1 E(x), \tag{3.8}$$

equality holding if and only if x is of order [1, 1].

If x is an isometric immersion, (3.8) is due to [14].

COROLLARY 3.2. *Let $x: C \to E^{m+1}$ be a non-constant map of a closed curve into E^{m+1}. If s denotes the arc length of C, then we have*

$$\int_C \|x''\|^2 ds \geqq \left(\frac{2\pi}{L}\right)^2 \int_C \|x'\|^2 ds, \tag{3.9}$$

where L is the length of C, $x' = dx/ds$, $x'' = d^2x/ds^2$. Equality sign of (3.9) *holds if and only if x is of the form:*

$$x = c_0 + c_1 \cos\frac{2\pi s}{L} + c_2 \sin\frac{2\pi s}{L}, \tag{3.10}$$

for some vectors c_0, c_1, c_2 in E^{n+m}.

This Corollary follows from the fact that the tension field of $x: C \to E^{n+m}$ is given by $-x''$ and λ_1 of C is equal to $(2\pi/L)^2$ with the eigenspace V_1 spanned by $\cos(2\pi s/L)$ and $\sin(2\pi s/L)$.

By applying Corollary 3.2 k times, we obtain.

COROLLARY 3.3. *If $x: C \to E^{m+1}$ is a non-constant map of a closed curve C in to E^{m+1}, then for any positive integers $k > h$, we have*

$$\int_M \|x^{(k)}\|^2 ds \geqq \left(\frac{2\pi}{L}\right)^{2k-2h} \int_C \|x^{(h)}\|^2 ds, \tag{3.11}$$

where $x^{(k)} = d^k x/ds^k$. The equality holds if and only if $x^{(h-1)}$ is of the form (3.10) *for some vectors c_0, c_1, c_2 in E^{m+1}.*

Remark 3.1. If $x: C \to E^{m+1}$ is an isometric immersion, inequality (3.9) reduces to

$$\int_C \kappa^2 ds \geqq \frac{4\pi^2}{L}, \tag{3.12}$$

which is a variant of the famous Fenchel-Borsuk inequality, where κ is the curvature of C in E^{m+1}.

4. Energy, Moment and Order.

Now, we define the moment of a map as follows.

DEFINITION 4.1. Let $x: M \to E^{n+m}$ be a map of a compact Riemannian manifold M into E^{n+m} and c a point in E^{n+m}. The *moment of* x *with respect to* c is defined by

$$\mathcal{M}_c = \mathcal{M}_c(x) = \int_M \langle x-c,\ x-c \rangle dV. \tag{4.1}$$

The moment of x with respect to the center of mass x_0 is simply called the *moment of* the map x. We simply denote it by $\mathcal{M}$, i.e., $\mathcal{M} = \mathcal{M}_{x_0}$.

THEOREM 4.1. *Let* $x: M \to E^{n+m}$ *be a non-constant map of a compact Riemannian manifold into* E^{n+m}. *Then we have*

$$\lambda_p \mathcal{M} \leqq 2E(x) \leqq \lambda_q \mathcal{M}. \tag{4.2}$$

Either equality sign of (4.2) *holds if and only if* x *is of* 1-*type.*

Proof. Since $\Delta = d\delta + \delta d$, we have

$$(x, \Delta x) = (x, \delta dx) = (dx, dx) = 2E(x). \tag{4.3}$$

From (3.3) we find

$$(x, \Delta x) = \sum_{t=p}^{q} \lambda_t (x_t, x_t). \tag{4.4}$$

On the other hand, we have

$$\mathcal{M} = (x - x_0, x - x_0) = \sum_{t=p}^{q} (x_t, x_t). \tag{4.5}$$

Therefore, (4.3), (4.4) and (4.5) imply

$$2E(x) - \lambda_p \mathcal{M} = \sum_{t=p}^{q} (\lambda_t - \lambda_p)(x_t, x_t) \geqq 0,$$

equality holding if and only if $q = p$, i.e., x is of 1-type. The other inequality is obtained in the same way.

If x is spherical, Theorem 4.1 yields the following best possible estimate of the energy.

COROLLARY 4.1. *Let $x: M \to S^{n+m-1} \subset E^{n+m}$ be a mass-symmetric, non-constant map of a compact Riemannian manifold M into a unit hypersphere S^{n+m-1} of E^{n+m}. Then we have*

$$E(x) \geqq \frac{\lambda_1}{2} \mathrm{vol}(M). \tag{4.6}$$

Equality holds if and only if x is of order [1, 1].

Proof. Under the hypothesis, we have $\mathcal{M} = \mathrm{vol}(M)$. Since $p \geqq 1$, (4.6) follows from (4.2). Equality sign of (4.6) holds if and only if x is of 1-type with $p=1$. (Q. E. D.)

For closed curves in E^{m+1}, Theorem 4.1 gives the following best possible estimate of moment.

COROLLARY 4.2. *Let C be a closed curve of length L in E^{m+1}. Then the moment of C satisfies*

$$\mathcal{M} \leqq L^3/4\pi^2. \tag{4.7}$$

Equality holds if and only if C is a plane circle of radius $L/2\pi$.

5. Some Applications to Harmonic Maps.

In this section we apply the notion of order to study harmonic maps.

LEMMA 5.1. *Let $x: M \to S^{n+m-1} \subset E^{n+m}$ be a map of a compact Riemannian manifold M into a hypersphere S^{n+m-1} of E^{n+m}. Then the map $\bar{x}: M \to S^{n+m-1}$ is a harmonic map with positive constant energy density if and only if x is a mass-symmetric, 1-type map.*

Proof. Without loss of generality, we may assume that S^{n+m-1} is a unit hypersphere centered at the origin of E^{n+m}. Denote by j the inclusion of S^{n+m-1} in E^{n+m}. Then the second fundamental forms σ_x, $\sigma_{\bar{x}}$ and σ_j of the maps x, $\bar{x}$ and j respectively satisfy

$$\sigma_x(X, Y) = j_* \sigma_{\bar{x}}(X, Y) + \sigma_j(\bar{x}_* X, \bar{x}_* Y),$$

for X, Y tangent to M. Thus we have

$$\Delta x = -\tau(x) = -j_* \tau(\bar{x}) - \sum_{j=1}^{n} \sigma_j(\bar{x}_* e_j, \bar{x}_* e_j) \tag{5.1}$$

where $e_1, \cdots, e_n$ is an orthonormal local frame on M. Since j is totally umbilical, (5.2) yields

$$\Delta x = -\jmath_* \tau(\bar{x}) + 2e(\bar{x})x\,, \tag{5.2}$$

where $e(\bar{x})$ is the energy density of $\bar{x}$.

If x is a mass-symmetric 1-type map, we have $x=x_p$ and $\Delta x=\lambda_p x$, where $[p, p]$ is the order of x. Hence, (5.2) gives $\tau(\bar{x})=0$ and $e(\bar{x})=\lambda_p/2$. Since x is a non-constant map, $\lambda_p>0$. Thus, $\bar{x}$ is a harmonic map with constant energy density.

Conversely, if $\bar{x}$ is a harmonic map with constant energy density, then from (5.2) we find $\Delta x=2e(\bar{x})x$. This implies x is a mass-symmetric, 1-type map. (Q. E. D.)

By using Lemma 5.1 we have the following.

PROPOSITION 5.1. *Let $x':(M, g)\to S^{m+1}$ be a map from a Riemannian surface (M, g) into S^{m+1} $(\subset E^{m+2})$. If x' has positive energy-density $e'=e(x')$, then x' is a harmonic map if and only if the composition:*

$$x: (M, e'g) \xrightarrow{id} (M, g) \xrightarrow{x'} S^{m+1}\subset E^{m+2}$$

is a mass-symmetric, 1-type map.

Proof. Let $e_1, \cdots, e_n$ be an orthonormal frame on $(M, e'g)$. Then $\varepsilon e_1, \cdots, \varepsilon e_n$ $(\varepsilon=\sqrt{e'})$ is an orthonormal frame for (M, g). Thus, the map $x:(M, e'g)\to S^{m-1}$ has constant energy-density 1. If $x':(M, g)\to S^{m+1}$ is harmonic, then it is known that the composition

$$(M, e'g) \xrightarrow{id} (M, g) \xrightarrow{x'} S^{m+1}$$

is also harmonic (cf. [8]). Thus, by applying Lemma 5.1, we conclude that x is a mass-symmetric, 1-type map.

Conversely, if x is a mass-symmetric, 1-type map, then, by Lemma 5.1, the composition:

$$(M, e'g) \xrightarrow{id} (M, g) \xrightarrow{x'} S^{m+1}$$

is a harmonic map. Thus, $x'=x'\cdot id\cdot id^{-1}$ is also harmonic. (Q. E. D.)

LEMMA 5.1 implies immediately the following: *A compact Riemannian manifold M admits a harmonic map into a m-sphere with constant positive energy-density if and only if there is an eigenspace V_k of Δ on M which contains $m+1$ functions $f_1, \cdots, f_{m+1}$ with $f_1^2+\cdots+f_{m+1}^2=c$ for some nonzero constant c.*

Remark 5.1. Some special cases of Proposition 5.1 were obtained in [12, 15].

6. Topological Obstruction.

Let V be an oriented m-plane in E^{n+m}. Denote by $e_{n+1}, \cdots, e_{n+m}$ an oriented orthonormal basis of V. Then $e_{n+1}\wedge\cdots\wedge e_{n+m}$ is a decomposable m-vector of norm 1 and $e_{n+1}\wedge\cdots\wedge e_{n+m}$ gives the orientation of V. Conversely, a decomposable m-vector of norm 1 determines a unique oriented m-plane in E^{n+m}. Consequently, if we denote by $G(m, n)$ the Grassmannian of oriented m-planes in E^{n+m}, then $G(m, n)$ can be identified with decomposable m-vectors of norm 1. This shows that $G(m, n)$ can be regarded as an nm-dimensional submanifold of the unit hypersphere S^{N-1} centered at the origin of $E^N=\Lambda^m E^{n+m}$, $N=\binom{n+m}{m}$ in a natural way. Thus, we have the following canonical inclusions:

$$G(m, n)\subset S^{N-1}\subset E^N=\Lambda^m E^{n+m}. \tag{6.1}$$

Let $x: M\to E^{n+m}$ be an isometric immersion of a compact oriented n-dimensional Riemannian manifold M into E^{n+m}. For a vector X tangent to M we identify X with its image under the differential $x*$ of x. If $e_{n+1}, \cdots, e_{n+m}$ is an oriented orthonormal normal frame on M, then the Gauss map ν:

$$\nu: M\to G(m, n)\subset S^{N-1}\subset E^N=\Lambda^m E^{n+m} \tag{6.2}$$

can be defined by $\nu(p)=(e_{n+1}\wedge\cdots\wedge e_{n+m})(p)$.

The following result is known.

LEMMA 6.1. *For a compact oriented submanifold M in E^{n+m}, the Gauss map $\nu: M\to G(n, m)\subset S^{N-1}=E^N=\Lambda^m E^{n+m}$ is mass-symmetric in S^{N-1}, $N=\binom{n+m}{n}$.*

Let ∇ and ∇' be the Levi-Civita connections of M and E^{n+m}, respectively. Denote by h, A and D the second fundamental form, the Weingarten map and the normal connection of M in E^{n+m}, respectively. For the second fundamental form h, we define the covariant derivative $\bar{\nabla}h$ of h by

$$(\bar{\nabla}_X h)(Y, Z)=D_X h(Y, Z)-h(\nabla_X Y, Z)-h(Y, \nabla_X Z). \tag{6.3}$$

Let $\varphi: M\to N$ be a map between Riemannian manifolds. For vector fields X, Y tangent to M, the symmetric bilinear map $\sigma: TM\times TM\to TN$ defined by

$$\sigma(X, Y)=\bar{\nabla}'_X f_* Y-f_*\nabla_X Y \tag{6.3}$$

is called the *second fundamental form of the map* φ, where $\bar{\nabla}'$ is the φ-induced connection on $\varphi^{-1}(TN)$.

In the following, we choose an oriented orthonormal local frame $e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+m}$ such that $e_1, \cdots, e_n$ is an oriented orthonormal local frame tangent to M and $e_{n+1}, \cdots, e_{n+m}$ an oriented orthonormal local frame normal to M in E^{n+m}. We denote by $h^r{}_{ij}=\langle h(e_i, e_j), e_r\rangle$ the coefficients of h and by R^D the normal curvature tensor with coefficients $K^r{}_{sij}=\langle R^D(e_i, e_j)e_r, e_s\rangle$. Then we have [3]

$$e_i\nu=-\sum h^r{}_{ij}e_{n+1}\wedge\cdots\wedge e_j^r\wedge\cdots\wedge e_{n+m}\,, \tag{6.4}$$

$$\Delta\nu=\|h\|^2\nu+n\sum_r e_{n+1}\wedge\cdots\wedge\nabla H_r\wedge\cdots\wedge e_{n+m} \tag{6.5}$$
$$-\sum_{r\neq s}\sum_{j<k} K^r{}_{sjk}e_{n+1}\wedge\cdots\wedge e_j^r\wedge\cdots\wedge e_k^s\wedge\cdots\wedge e_{n+m}\,,$$

where H_r and ∇H_r denote the mean curvature and gradient of the mean curvature in the direction of e_r and e_j^r means to replace e_r by e_j.

By applying Proposition 2.3 we have the following.

THEOREM 6.1. *Let M be a compact oriented n-dimensional manifold immersed in E^{n+m}. If the Euler class $e(T^\perp M)$ of the normal bundle is nontrivial, then the Gauss map of M in E^{n+m} is of k-type with $k>m/2$.*

Proof. If either m is odd or $m>n$, then the Euler class of normal bundle vanishes automatically. Thus we have $m\leqq n$ and $m=2\delta$ is an even integer.

For any positive integer $l\leqq m/2$, let V_l be the subspace of $\Lambda^m E^{n+m}$ spanned by

$$\{e_{i_1}\wedge\cdots\wedge e_{i_{2l}}\wedge e_{r_1}\wedge\cdots\wedge e_{r_{m-2l}}: 1\leqq i_1, \cdots, i_{2l}\leqq n,\ n+1\leqq r_1, \cdots, r_{m-2l}\leqq n+m\}.$$

Denote by $\pi_l: \Lambda^m E^{n+m}\to V_l$ the canonical projection. Then from (6.5) we have

$$\pi_\alpha(\nu)=0\,, \qquad \alpha\geqq 1\,, \tag{6.6}$$

$$\pi_\alpha(\Delta\nu)=0\,, \qquad \alpha\geqq 2\,, \tag{6.7}$$

$$\pi_1(\Delta\nu)=-\sum K^r{}_{sjk}e_{n+1}\wedge\cdots\wedge e_j^r\wedge\cdots\wedge e_k^s\wedge\cdots\wedge e_{n+m}\,. \tag{6.8}$$

Now, assume that the Gauss map ν is of k-type for some $k\leqq\delta$. Then, Proposition 2.3 and Lemma 6.1 imply that there is a monic polynomial P of degree k such that $P(\Delta)\nu=0$. Since $k\leqq\delta$, $Q(t)=t^{\delta-k}P(t)$ is a monic polynomial of degree δ such that $Q(\Delta)\nu=0$. In particular, we have

$$\pi_\delta(Q(\Delta)\nu)=0\,. \tag{6.9}$$

By direct computation we have

$$\pi_\delta(\Delta^l\nu)=0 \qquad \text{for}\quad l<\delta\,. \tag{6.10}$$

Thus we have $\pi_\delta(\Delta^q\nu)=\pi_\delta(Q(\Delta)\nu)=0$. On the other hand, by direct computation, we may find

$$\pi_q(\Delta^\delta\nu)=(-1)^\delta\sum K^{r_1}{}_{r_2j_1j_2}\cdots K^{r_{m-1}}{}_{r_mj_{m-1}j_m}\cdot \tag{6.11}$$
$$e_{j_1}^{r_1}\wedge e_{j_2}^{r_2}\wedge\cdots\wedge e_{j_m}^{r_m}=0\,.$$

Thus we find

$$\sum K^{r_1}{}_{r_2 j_1 j_2} \cdots K^{r_{m-1}}{}_{r_m j_{m-1} j_m} e^{r_1}_{j_1} \wedge \cdots \wedge e^{r_m}_{j_m} = 0. \tag{6.12}$$

This is equivalent to

$$\sum \varepsilon_{r_1 \cdots r_m} K^{r_1}{}_{r_2 j_1 j_2} \cdots K^{r_{m-1}}{}_{r_m j_{m-1} j_m} \omega^{j_1} \wedge \cdots \wedge \omega^{j_m} = 0, \tag{6.13}$$

where $\omega^1, \cdots, \omega^n$ is the dual frame of $e_1, \cdots, e_n$ and $\varepsilon_{r_1 \cdots r_m}$ is 1 or -1 according as $(r_1, \cdots, r_m)$ is an even or odd permutation of $(n+1, \cdots, n+m)$. Now, we put

$$\Omega^r_s = \frac{1}{2} \sum K^r{}_{sij} \omega^i \wedge \omega^j, \tag{6.14}$$

where $\omega^1, \cdots, \omega^n$ is the dual frame of $e_1, \cdots, e_n$. Then (6.13) gives

$$\gamma = \frac{(-1)^\delta}{2^{2\delta} \pi^\delta \delta!} \sum \varepsilon_{r_1 \cdots r_{2\delta}} \Omega^{r_1}_{r_2} \wedge \cdots \wedge \Omega^{r_{2\delta-1}}_{r_{2\delta}} = 0. \tag{6.15}$$

Since γ represents the Euler class of the normal bundle, we obtain $e(T^\perp M) = 0$. (Q. E. D.)

For a compact, oriented n-dimensional submanifold M immersed in E^{2n}, the Euler number $\chi(T^\perp M)$ of the normal bundle is equal to twice of the self-intersection number [10]. Thus, from Theorem 6.1, we have the following.

COROLLARY 6.1. *Let M be a compact, oriented, n-dimensional manifold immersed in E^{2n}. If the self-intersection number of M in E^{2n} is non-zero, then the Gauss map ν is of k-type with $k > n/2$.*

It is well-known that the self-intersection number is a regular homotopic invariant. From Theorem 6.1 we also have the following.

COROLLARY 6.2. *Let $x: M \to E^{2n}$ be an immersion of a compact, oriented, n-dimensional manifold M in E^{2n}. If the Euler class $e(T^\perp M)$ of the normal bundle of x is nontrivial, then x cannot be deformed regularly to an immersion with k-type Gauss map for $k \leqq n/2$.*

Example 6.1. Although the standard immersion of S^{2n} in $E^{2n+1} \subset E^{4n}$ has 1-type Gauss map, the Whitney immersion w of S^{2n} in E^{4n} cannot be deformed regularly to an immersion with k-type Gauss map of $k \leqq n$. The Whitney immersion w is defined as follows.

Let $f: E^{2n+1} \to E^{4n}$ be a map of E^{2n+1} into E^{4n} defined by

$$f(x_0, x_1, \cdots, x_{2n}) = (x_1, \cdots, x_{2n}, 2x_0x_1, \cdots, 2x_0x_{2n}).$$

Then f induces an immersion $w: S^{2n} \to E^{4n}$, called the Whitney immersion, which has a unique self-intersection point $f(-1, 0, \cdots, 0) = f(1, 0, \cdots, 0)$. The self-intersection number $I(w)$ is one. Corollary 6.2 shows that w cannot be deformed regularly to any immersion of S^{2n} in E^{4n} with k-type Gauss map for $k \leqq n/2$.

If $x: M \to C^n$ is a totally real immersion, then the tangent bundle is isomorphic to the normal bundle. Thus, by Theorem 6.1, we have the following.

COROLLARY 6.3. *Let M be a compact, oriented, n-dimensional, totally real submanifold of $\boldsymbol{C}^n$. If the Euler number $\chi(M)$ of M is nontrivial, then the Gauss map of M in $\boldsymbol{C}^n$ is of k-type with $k > n/2$.*

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