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KODAI MATH. J.
7 (1984), 192—202

ON THE BOUNDARY LIMITS OF POLYHARMONIC FUNCTIONS IN A HALF SPACE

Dedicated to Professor Mitsuru Ozawa on the occasion of his 60th birthday

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1. Introduction and statement of result.

Let R^n be the n-dimensional Euclidean space ($n \geqq 2$), and set

$$R_+^n = \{x=(x', x_n) \in R^{n-1} \times R^1 ;\ x_n > 0\}.$$

For $\xi \in \partial R_+^n$, $\gamma \geqq 1$ and $a>0$, define

$$T_\gamma(\xi, a) = \{(x', x_n) \in R_+^n ;\ |(x', 0)-\xi| < a x_n^{1/\gamma}\}.$$

Recently Cruzeiro [2] proved the existence of $\lim u(x)$ as $x \to \xi$, $x \in T_\gamma(\xi, a)$, for a harmonic function u with gradient in $L^n(R_+^n)$. In this note we are concerned with polyharmonic functions in R_+^n, and our purpose is to give a generalization of her result to the polyharmonic case.

For a nonnegative integer m, denote by Δ^m the Laplace operator iterated m times; in particular, Δ^0 denotes the identity operator. A function $u \in C^\infty(R_+^n)$ is said to be polyharmonic of order m in R_+^n if

$$\Delta^m u = 0 \qquad \text{on} \quad R_+^n.$$

For $u \in C^m(R_+^n)$ and $x=(x_1, \cdots, x_n) \in R_+^n$, define

$$|\nabla_m u(x)| = \{\sum_{|\lambda|=m} |D^\lambda u(x)|^2\}^{1/2},$$

where $\lambda=(\lambda_1, \cdots, \lambda_n)$ denotes a multi-index with length $|\lambda| = \lambda_1 + \cdots + \lambda_n$ and $D^\lambda = (\partial/\partial x_1)^{\lambda_1} \cdots (\partial/\partial x_n)^{\lambda_n}$.

THEOREM. *Let m be a positive integer and u be a function which is polyharmonic of order $m+1$ in R_+^n and satisfies*

$$(1) \qquad \int_G |\nabla_m u(x)|^p x_n^\alpha dx < \infty, \qquad p>1, \quad \alpha < mp-1,$$

for any bounded open set G in R_+^n. Suppose $(\alpha+1)/p$ is not a positive integer.

Received April 21, 1983

(i) *If* $n-mp+\alpha>0$, *then for each* $\gamma>1$ *there exists a set* $E_\gamma\subset\partial R_+^n$ *such that* $H_{\gamma(n-mp+\alpha)}(E_\gamma)=0$ *and*

$$\lim_{x\to\xi,\, x\in T_\gamma(\xi, a)} u(x) \tag{2}$$

exists and is finite for any $a>0$ *and any* $\xi\in\partial R_+^n-E_\gamma$.

(ii) *If* $n-mp+\alpha=0$, *then there exists a set* $E\subset\partial R_+^n$ *such that* $B_{n/p,p}(E)=0$ *and* (2) *exists and is finite for any* $a>0$, *any* $\gamma>1$ *and any* $\xi\in\partial R_+^n-E$.

(iii) *If* $n-mp+\alpha<0$, *then* $\lim_{x\to\xi,\, x\in R_+^n} u(x)$ *exists and is finite for any* $\xi\in\partial R_+^n$.

Here H_l denotes the l-dimensional Hausdorff measure, and $B_{l,p}$ the Bessel capacity of index (l, p) (see Meyers [4]). Note the following results (cf. [4]):

(a) If $H_{n-l}(E)<\infty$, then $B_{l/p,p}(E)=0$ for any $p>1$;

(b) If $B_{l/p,p}(E)=0$ for some $p>1$, then $H_{l'}(E)=0$ for any $l'>n-l$.

In the case where $(\alpha+1)/p$ is a positive integer, we have the next theorem.

THEOREM$'$. *Let* u *be a function which is polyharmonic of order* $m+1$ *in* R_+^n *and satisfies* (1) *for any bounded open set* G *in* R_+^n, *where* $p>1$ *and* $(\alpha+1)/p$ *is a positive integer smaller than* m.

(i) *If* $n-mp+\alpha>0$, *then for each* $\gamma>1$ *there exists a set* $E_\gamma\subset\partial R_+^n$ *such that* E_γ *has Hausdorff dimension at most* $\gamma(n-mp+\alpha)$ *and* (2) *exists and is finite for any* $a>0$ *and any* $\xi\in\partial R_+^n-E_\gamma$.

(ii) *If* $n-mp+\alpha=0$, *then there exists a set* $E\subset\partial R_+^n$ *such that* E *has Hausdorff dimension* 0 *and* (2) *exists and is finite for any* $a>0$, *any* $\gamma>1$ *and any* $\xi\in\partial R_+^n-E$.

(iii) *If* $n-mp+\alpha<0$, *then* $\lim_{x\to\xi,\, x\in R_+^n} u(x)$ *exists and is finite for any* $\xi\in\partial R_+^n$.

If $\lim_{x\to\xi,\, x\in T_1(\xi, a)} u(x)$ exists and is finite for any $a>0$, then u is said to have a nontangential limit at ξ. If u is a function which is polyharmonic of order $m+1$ in R_+^n and satisfies (1) with $p>1$ and $\alpha<mp-1$ for any bounded open set G in R_+^n, then u has a nontangential limit at any $\xi\in\partial R_+^n$ except for those in a set E with $B_{m-\alpha/p,p}(E)=0$; this result is best possible as to the size of the exceptional set in the following sense: If $E\subset\partial R_+^n$, $B_{m-\alpha/p,p}(E)=0$ and $-1<\alpha<mp-1$, then we can find a harmonic function u in R_+^n which satisfies (1) with $G=R_+^n$ such that $\lim_{x\to\xi,\, x\in R_+^n} u(x)=\infty$ for any $\xi\in E$ (see [8; Theorems 1 and 2]). Thus (ii) of the theorem gives an improvement of [8; Theorem 1], and also the best possible result as to the size of the exceptional set.

2. Lemmas.

First we prepare several properties of polyharmonic functions. Let $B(x, r)$ denote the open ball with center at x and radius r. For $E\subset R^n$, denote the closure of E by $\bar{E}$.

LEMMA 1. *Let u be a function which is polyharmonic of order $m+1$ in R_+^n. Then there exist constants c_i independent of u such that*

$$r^{1-n}\int_{\partial B(x,r)} \Delta u(y)dS(y) = \sum_{i=1}^{m} c_i r^{2i-2}\Delta^i u(x)$$

whenever $\overline{B(x, r)} \subset R_+^n$.

Proof. By a result in [9; p. 189], there exist harmonic functions v_i in $B(x, r')$ such that

$$\Delta u(y) = \sum_{i=1}^{m} |y-x|^{2i-2} v_i(y) \qquad \text{on} \quad B(x, r'),$$

where $\overline{B(x, r')} \subset R_+^n$. Then we note that $\Delta^i u(x) = c_i' v_i(x)$, so that

$$r^{1-n}\int_{\partial B(x,r)} \Delta u(y)dS(y) = \sum_{i=1}^{m} c_i'' r^{2i-2} v_i(x) = \sum_{i=1}^{m} c_i r^{2i-2}\Delta^i u(x)$$

for r with $0<r<r'$. The constants c_i', c_i'' and c_i depend only on i and the dimension n.

LEMMA 2. *Let u be a function which is polyharmonic of order $m+1$ in R_+^n, and let $\overline{B(x, r)} \subset R_+^n$. Then for each nonnegative integer i, $i \leqq m$, there exist constants $a_\lambda^{(i)}$ independent of u, x and r such that*

$$\Delta^i u(x) = r^{-n-2i} \sum_{0<|\lambda| \leqq m} a_\lambda^{(i)} \int_{B(x,r)} (y-x)^\lambda D^\lambda u(y)dy. \tag{3}$$

Proof. In view of [3; (15)],

$$\Delta^i u(x) = \sum_{k=0}^{m-i} a_k \rho^k \int_{\partial B(0,1)} \left(\frac{\partial}{\partial \rho}\right)^k \Delta^i u(x+\rho\sigma)dS(\sigma)$$

with constants a_k. We introduce a differential operator

$$\nu = \sum_{j=1}^{n} (y_j - x_j)\frac{\partial}{\partial y_j}.$$

Letting I denote the identity operator, we note that

$$\nu^k \Delta^i = \Delta^i(\nu - 2iI)^k,$$

so that

$$\rho^{n-1}\Delta^i u(x) = \sum_{k=0}^{m-i} a_k \int_{\partial B(x,\rho)} \nu^k \Delta^i u(y)dS(y)$$

$$= \sum_{k=0}^{m-i} a_k \int_{\partial B(x,\rho)} \Delta^i(\nu-2iI)^k u(y)dS(y).$$

Integrating both sides with respect to ρ over the interval $(0, r)$, we obtain

$$\Delta^i u(x)=r^{-n}\sum_{k=0}^{m-i}a'_k\int_{B(x,r)}\Delta^i(\nu-2iI)^k u(y)dy$$

$$=r^{-n-1}\sum_{k=0}^{m-i}a'_k\int_{\partial B(x,r)}\nu\Delta^{i-1}(\nu-2iI)^k u(y)dS(y)$$

$$=r^{-n-1}\sum_{k=0}^{m-i}a'_k\int_{\partial B(x,r)}\Delta^{i-1}(\nu-2(i-1)I)(\nu-2iI)^k u(y)dS(y)\,.$$

Repeating this process, we finally obtain

$$\Delta^i u(x)=r^{-n-2i}\sum_{k=0}^{m-i}a''_k\int_{B(x,r)}\nu(\nu-2I)\cdots(\nu-2(i-1)I)(\nu-2iI)^k u(y)dy\,,$$

which is of the form (3).

The following fact can be proved easily (cf. [6; Lemma 5]).

LEMMA 3. *Let u be a function in $C^1(R^n_+)$ such that*

$$\int_G|\nabla_1 u(x)|^p x_n^\alpha dx<\infty\,,\qquad p>1\,,$$

for any bounded open set G in R^n_+. Then

$$\int_G|u(x)|^p x_n^\beta dx<\infty$$

for any bounded open set G in R^n_+, where $\beta=\alpha-p$ if $\alpha>p-1$ and $\beta>-1$ if $\alpha=p-1$.

By [6; Lemma 4] we have

LEMMA 4. *Let k be a positive integer, $p>1$ and $\beta<p-1$. Let u be a function in $C^k(R^n_+)$ such that*

$$\int_G|\nabla_k u(x)|^p x_n^\beta dx<\infty$$

for any bounded open set G in R^n_+. If we set

$$A=\left\{\xi\in\partial R^n_+\,;\int_{B(\xi,1)\cap R^n_+}|\xi-x|^{k-n}|\nabla_k u(y)|dy=\infty\right\},$$

then $B_{k-\beta/p,p}(A)=0$.

LEMMA 5. *Let f be a nonnegative measurable function on R^n_+ such that $\int_G f(y)dy<\infty$ for any bounded open set G in R^n_+, and define*

$$B_\delta=\left\{\xi\in\partial R^n_+\,;\int_{B(\xi,1)\cap R^n_+}(|\xi'-y'|^{2\gamma}+y_n^2)^{-(l+\delta)/2}f(y)y_n^\delta dy=\infty\right\},$$

where $l \geqq 0$ *and* $\gamma \geqq 1$. *Then* $H_{\gamma l}(B_\delta)=0$ *for any* $\delta>0$; *in case* $l=0$, *this implies that* B_δ *is empty.*

Proof. Suppose $H_{\gamma l}(B_\delta)>0$. Then by [1; Theorems 1 and 3 in § II] we can find a nonnegative measure μ with compact support in ∂R_+^n such that $\mu(B_\delta)>0$ and

$$\mu(B(x, r)) \leqq r^{\gamma l} \qquad \text{for any } x \text{ and } r.$$

Then $\int(|\xi'-y'|^{2\gamma}+y_n^2)^{-(l+\delta)/2} d\mu(\xi) \leqq \text{const.}\, y_n^{-\delta}$ for $y \in R_+^n$. Hence

$$\infty=\int\left\{\int_{B(\xi,1)\cap R_+^n}(|\xi'-y'|^{2\gamma}+y_n^2)^{-(l+\delta)/2} f(y) y_n^\delta dy\right\} d\mu(\xi)$$

$$\leqq \int_G\left\{\int(|\xi'-y'|^{2\gamma}+y_n^2)^{-(l+\delta)/2} d\mu(\xi)\right\} f(y) y_n^\delta dy$$

$$\leqq \text{const.} \int_G f(y) dy<\infty,$$

which is a contradiction. Here $G=\bigcup_{\xi \in \operatorname{supp} \mu} B(\xi, 1) \cap R_+^n$.

LEMMA 6. *Let k be a positive integer, $p>1$ and $\beta<p-1$. Let K be a Borel measurable function on R^n such that $|\nabla_l K(x)| \leqq |x|^{k-l-n}$ on $R^n-\{0\}$ for $l=0, 1, \cdots, k-1$, and define*

$$u(x)=\int K(x-y) f(y) dy$$

for a nonnegative measurable function f on R^n such that $\int|x-y|^{k-n} f(y) dy \not\equiv \infty$ and $\int_G f(y)^p|y_n|^\beta dy<\infty$ for any bounded open set $G \subset R^n$. Set

$$E_{l,\gamma}=\left\{\xi \in \partial R_+^n ;\ \limsup_{x \to \xi,\, x \in T_\gamma(\xi, a)} \int_{B(x, x_n/2)}|\nabla_l u(y)|^p y_n^{lp-n} dy>0 \quad \textit{for some } a>0\right\}$$

for $\gamma \geqq 1$ and $l=1, \cdots, k-1$. Then $H_{\gamma(n-kp+\beta)}(E_{l,\gamma})=0$ if $n-kp+\beta>0$, and $E_{l,\gamma}$ is empty if $n-kp+\beta \leqq 0$.

Proof. Define

$$E_\gamma=\left\{\xi \in \partial R_+^n ;\ \limsup_{r \downarrow 0} r^{\gamma(kp-\beta-n)} \int_{B(\xi, r)} f(y)^p|y_n|^\beta dy>0\right\}$$

for $\gamma \geqq 1$. Then, in view of [7; Lemma 2], we see that $H_{\gamma(n-kp+\beta)}(E_\gamma)=0$ if $n-kp+\beta>0$ and E_γ is empty if $n-kp+\beta \leqq 0$.

Let l be a positive integer such that $l<k$. Then for almost every x,

$$|\nabla_l u(x)| \leqq \int|x-y|^{k-l-n} f(y) dy=U_1(x)+U_2(x)+U_3(x),$$

where

$$U_1(x)=\int_{B(x,cx_n)}|x-y|^{k-l-n}f(y)dy\,, \qquad 0<c<1/3\,,$$

$$U_2(x)=\int_{B(\xi,2|x-\xi|)-B(x,cx_n)}|x-y|^{k-l-n}f(y)dy\,,$$

$$U_3(x)=\int_{R^n-B(\xi,2|x-\xi|)}|x-y|^{k-l-n}f(y)dy\,.$$

We first note from Hölder's inequality that

$$\lim_{r\downarrow 0} r^{k-n}\int_{B(\xi,r)}f(y)dy=0$$

if $\xi\in\partial R^n_+-E_1$ and hence if $\xi\in\partial R^n_+-E_\gamma$. Setting $\varepsilon(\eta)=\sup_{0<r\leqq\eta} r^{k-n}\int_{B(\xi,r)}f(y)dy$ for $\eta>0$, we have

$$U_3(x)\leqq\text{const.}\int_{R^n-B(\xi,2|x-\xi|)}|y-\xi|^{k-l-n}f(y)dy$$

$$\leqq\text{const.}\left\{\int_{R^n-B(\xi,\eta)}|y-\xi|^{k-l-n}f(y)dy+\varepsilon(\eta)|x-\xi|^{-l}\right\}.$$

Consequently, $\lim_{z\to\xi,\,z\in R^n_+}\sup\int_{B(z,z_n/2)}U_3(x)^p x_n^{lp-n}dx\leqq\text{const.}\,\varepsilon(\eta)^p$. This implies that

$$\lim_{z\to\xi,\,z\in R^n_+}\int_{B(z,z_n/2)}U_3(x)^p x_n^{lp-n}dx=0\,.$$

By Hölder's inequality,

$$U_1(x)\leqq\text{const.}\,x_n^{(k-l)/p'}\left\{\int_{B(x,cx_n)}|x-y|^{k-l-n}f(y)^p dy\right\}^{1/p},$$

so that

$$\int_{B(z,z_n/2)}U_1(x)^p x_n^{lp-n}dx$$

$$\leqq\text{const.}\,z_n^{(k-l)p/p'+lp-n}\int_{B(z,(1+3c)z_n/2)}f(y)^p\left\{\int_{B(z,z_n/2)}|x-y|^{k-l-n}dx\right\}dy$$

$$\leqq\text{const.}\,z_n^{kp-\beta-n}\int_{B(\xi,2|z-\xi|)}f(y)^p|y_n|^\beta dy\,.$$

Therefore if $n-kp+\beta>0$ and $\xi\in\partial R^n_+-E_\gamma$, then

$$\lim_{z\to\xi,\,z\in T_\gamma(\xi,a)}\int_{B(z,z_n/2)}U_1(x)^p x_n^{lp-n}dx=0\,;$$

if $n-kp+\beta\leqq 0$, then

$$\lim_{z\to\xi,\,z\in R^n_+}\int_{B(z,z_n/2)}U_1(x)^p x_n^{lp-n}dx=0\,.$$

Letting $\eta=2|x-\xi|$ and $M=\int_{B(\xi,\eta)} f(y)^p|y_n|^\beta dy$, we have by [7; Lemma 5],

$$U_2(x)^p \leqq \text{const.} \begin{cases} x_n^{(k-l)p-\beta-n}M & \text{if } (k-l)p-\beta-n<0, \\ [\log(\eta x_n^{-1}+2)]^{p-1}M & \text{if } (k-l)p-\beta-n=0, \\ \eta^{(k-l)p-\beta-n}M & \text{if } (k-l)p-\beta-n>0. \end{cases}$$

If $z\in T_\gamma(\xi, a)\cap B(\xi, 1)$ and $x\in B(z, z_n/2)$, then there exists $a'>0$ such that $x\in T_\gamma(\xi, a')$. Hence we obtain

$$\int_{B(z,z_n/2)} U_2(x)^p x_n^{lp-n}dx \leqq \text{const.} \begin{cases} z_n^{kp-\beta-n}M & \text{if } (k-l)p-\beta-n<0, \\ z_n^{lp}[\log(\eta z_n^{-1}+2)]^{p-1}M & \text{if } (k-l)p-\beta-n=0, \\ z_n^{lp}\eta^{(k-l)p-\beta-n}M & \text{if } (k-l)p-\beta-n>0, \end{cases}$$

which tends to zero as $z\to\xi$, $z\in T_\gamma(\xi, a)$, if $kp-\beta-n<0$ and $\xi\notin E_\gamma$, and as $z\to\xi$ if $kp-\beta-n\geqq 0$. Thus we proved that $E_{l,\gamma}\subset E_\gamma$ if $n-kp+\beta>0$ and $E_{l,\gamma}$ is empty if $n-kp+\beta\leqq 0$. The proof is now complete.

COROLLARY. *Let k, p and β be as in the lemma. Let u be a function in $C^k(R_+^n)$ such that $\int_G|\nabla_k u(x)|^p x_n^\beta dx<\infty$ for any bounded open set G in R_+^n, and define $E_{l,\gamma}$ as in the lemma. Then $H_{\gamma(n-kp+\beta)}(E_{l,\gamma})=0$ if $n-kp+\beta>0$ and $E_{l,\gamma}$ is empty if $n-kp+\beta\leqq 0$.*

Proof. Let $q=p$ if $\beta\leqq 0$ and $1<q<p/(\beta+1)$ if $\beta>0$. By Hölder's inequality we have

$$\int_G|\nabla_k u(x)|^q dx<\infty$$

for any bounded open set G in R_+^n. By Theorem 5 and its proof in [10; Chap. VI], we can find a function $v\in L_{\text{loc}}^q(R^n)$ such that $v=u$ a.e. on R_+^n,

$$\int_G|\nabla_k v(x)|^q dx<\infty \quad \text{and} \quad \int_G|\nabla_k v(x)|^p|x_n|^\beta dx<\infty$$

for any bounded open set G in R^n, where the derivatives are taken in the sense of distributions.

We shall show that $H_{\gamma(n-kp+\beta)}(E_{l,\gamma}\cap B(0, r))=0$ if $n-kp+\beta>0$ and $E_{l,\gamma}\cap B(0, r)$ is empty if $n-kp+\beta\leqq 0$ for any $r>0$. Let $r>0$ be fixed, and take a function $\phi\in C_0^\infty(R^n)$ such that $\phi=1$ on $B(0, 2r)$. Set $w=\phi v$. Then by [5; Theorem 4.1],

$$w(x)=\sum_{|\lambda|=k} a_\lambda\int\frac{(x-y)^\lambda}{|x-y|^n}D^\lambda w(y)dy \qquad \text{a.e. on } R^n.$$

Since w is considered to be continuously k times differentiable on R_+^n, the right hand side is also continuously k times differentiable on R_+^n and the equality is considered to hold at every point of R_+^n. Further,

$$\int_{B(0,r)} |\nabla_k w(y)|^p |y_n|^\beta dy < \infty .$$

Thus the proof of Lemma 6 shows that $H_{\gamma(n-kp+\beta)}(E_{l,\gamma} \cap B(0, r))=0$ if $n-kp+\beta>0$ and $E_{l,\gamma} \cap B(0, r)$ is empty if $n-kp+\beta \leqq 0$. By noting the arbitrariness of r, we conclude the proof.

3. Proof of the theorem.

Let u be as in the theorem. If $\alpha<p-1$, we let $k=1$, and if $\alpha \geqq p-1$, then we let k be a positive integer such that $(k-1)p-1<\alpha<kp-1$. Define $\beta=\alpha-(k-1)p$. Then $\beta<p-1$, and, in view of Lemma 3,

$$\int_G |\nabla_{m-l} u(x)|^p x_n^{\alpha-lp} dx < \infty$$

for any bounded open set G in R_+^n and $l=0, 1, \cdots, k-1$.

Let $q=p$ if $\beta \leqq 0$ and $1<q<p/(\beta+1)$ if $\beta>0$. By Hölder's inequality we have

$$\int_G |\nabla_{m-k+1} u(x)|^q dx < \infty$$

for any bounded open set G in R_+^n. As in the proof of the corollary to Lemma 6, we can find a function $v \in L_{\mathrm{loc}}^q(R^n)$ such that $v=u$ a.e. on R_+^n,

$$\int_G |\nabla_{m-k+1} v(x)|^q dx < \infty$$

and

$$\int_G |\nabla_{m-k+1} v(x)|^p |x_n|^\beta dx < \infty$$

for any bounded open set G in R^n.

Define

$$A=\left\{\xi \in \partial R_+^n ; \int_{B(\xi,1)} |\xi-y|^{m-k+1-n} |\nabla_{m-k+1} v(y)| dy=\infty\right\},$$

$$E_{l,\gamma}=\left\{\xi \in \partial R_+^n ; \limsup_{x \to \xi,\, x \in T_\gamma(\xi, a)} \int_{B(x, x_n/2)} |\nabla_l u(y)|^p y_n^{lp-n} dy>0 \quad \text{for some } a>0\right\},$$

$$F_\eta=\left\{\xi \in \partial R_+^n ; \limsup_{r \downarrow 0} r^{-\eta} \int_{B(\xi, r)} |\nabla_{m-k+1} v(y)|^p |y_n|^\beta dy>0\right\} \quad \text{for} \quad \eta>0,$$

$$F_0=\left\{\xi \in \partial R_+^n ; \limsup_{r \downarrow 0} (\log r^{-1})^{p-1} \int_{B(\xi, r)} |\nabla_{m-k+1} v(y)|^p |y_n|^\beta dy>0\right\}$$

and

$$E_\gamma=A \cup\left(\bigcup_{l=1}^m E_{l,\gamma}\right) \cup F_{\gamma(n-mp+\alpha)} \qquad \text{for} \quad n-mp+\alpha \geqq 0 .$$

We shall show below that $\lim_{x\to\xi,\, x\in T_\gamma(\xi,a)} u(x)$ exists and is finite for any $\xi\in\partial R_+^n - E_\gamma$ and any $a>0$; in case $n-mp+\alpha<0$, our proof below shows that $u(x)$ has a finite limit as $x\to\xi$, $x\in R_+^n$, for any $\xi\in\partial R_+^n$.

By Lemma 4, $B_{m-\alpha/p,p}(A)=0$. In view of Lemma 5,

$$\int_{T_\gamma(\xi,a)\cap B(\xi,1)} |\nabla_l u(x)|^p x_n^{lp-n} dx<\infty\,, \qquad l=m-k+1,\cdots,m\,,$$

for any $a>0$ and any $\xi\in\partial R_+^n$ except for a set B_γ such that $H_{\gamma(n-mp+\alpha)}(B_\gamma)=0$ if $n-mp+\alpha>0$ and B_γ is empty if $n-mp+\alpha\leqq 0$, so that

$$H_{\gamma(n-mp+\alpha)}(E_{l,\gamma})=0 \quad \text{if} \quad n-mp+\alpha\geqq 0 \quad \text{and} \quad l=m-k+1,\cdots,m\,.$$

The corollary to Lemma 6 implies that $H_{\gamma(n-mp+\alpha)}(E_{l,\gamma})=0$ if $n-mp+\alpha>0$ and $l=1,\cdots,m-k$, and $E_{l,\gamma}$ is empty if $n-mp+\alpha\leqq 0$ and $l=1,\cdots,m-k$. Thus, with the aid of [7; Lemmas 2 and 3], we see that $H_{\gamma(n-mp+\alpha)}(E_\gamma)=0$ if $n-mp+\alpha>0$, and $B_{n/p,p}(E_\infty)=0$ if $n-mp+\alpha=0$, where $E_\infty\equiv\bigcup_{\gamma>1}E_\gamma=A\cup F_0$.

Let $\xi\in\partial R_+^n-E_\gamma$, and take a function $\phi\in C_0^\infty(R^n)$ such that $\phi=1$ on $B(\xi,2)$. Write $m-k+1=2s+s^*$, where s and s^* are nonnegative integers such that $0\leqq s^*\leqq 1$. Setting $w=\phi v$, we have the following integral representation (cf. [5; Theorems 4.1 and 4.2]):

$$w(x)=U(x\,;\,w)\equiv\begin{cases}\displaystyle\int K_{2s}(x-y)\Delta^s w(y)dy & \text{if} \quad s^*=0\,,\\ \displaystyle\sum_{j=1}^n\int\frac{\partial K_{2s+2}}{\partial x_j}(x-y)\Big(\frac{\partial}{\partial y_j}\Delta^s w(y)\Big)dy & \text{if} \quad s^*=1\,,\end{cases}$$

holds for almost every $x\in R^n$, where $K_{2l}(x)=C_l|x|^{2l-n}$ if $2l<n$ or n is odd, and $K_{2l}(x)=C_l|x|^{2l-n}\log|x|$ if $2l\geqq n$ and n is even; the constants C_l are chosen so that $U(x\,;\,\phi)=\phi$ for any $\phi\in C_0^\infty(R^n)$. Since w is infinitely differentiable on R_+^n, $U(x\,;\,w)$ is continuous on R_+^n and $w(x)=U(x\,;\,w)$ holds for any $x\in R_+^n$.

We shall prove the theorem only in the case $s^*=1$; the case $s^*=0$ can be proved similarly. Write $U(x\,;\,w)=U_1(x)+U_2(x)+U_3(x)$, where

$$U_1(x)=\sum_{j=1}^n\int_{B(x,x_n/2)}\frac{\partial K_{2s+2}}{\partial x_j}(x-y)\Big(\frac{\partial}{\partial y_j}\Delta^s w(y)\Big)dy\,,$$

$$U_2(x)=\sum_{j=1}^n\int_{B(x,|x-\xi|/2)-B(x,x_n/2)}\frac{\partial K_{2s+2}}{\partial x_j}(x-y)\Big(\frac{\partial}{\partial y_j}\Delta^s w(y)\Big)dy\,,$$

$$U_3(x)=\sum_{j=1}^n\int_{R^n-B(x,|x-\xi|/2)}\frac{\partial K_{2s+2}}{\partial x_j}(x-y)\Big(\frac{\partial}{\partial y_j}\Delta^s w(y)\Big)dy\,.$$

Since $\xi\notin A$ by our assumption, $\int|\nabla_1 K_{2s+2}(\xi-y)|\,|\nabla_{2s+1}w(y)|\,dy<\infty$, so that Lebesgue's dominated convergence theorem implies that $\lim U_3(x)$ exists and is finite as $x\to\xi$, $x\in R_+^n$.

Define $W(x)=\int_{B(\xi,2|x-\xi|)}|\nabla_{2s+1}w(y)|^p|y_n|^\beta dy$. As in [7; Lemma 5], we have

$$|U_2(x)|^p \leqq \text{const.}\begin{cases} x_n^{mp-\alpha-n}W(x) & \text{if } n-mp+\alpha>0, \\ \left\{\log\left(\frac{|x-\xi|}{x_n}+2\right)\right\}^{p-1}W(x) & \text{if } n-mp+\alpha=0, \\ |x-\xi|^{mp-\alpha-n}[\log(|x-\xi|^{-1}+2)]^{p-1}W(x) & \text{if } n-mp+\alpha<0. \end{cases}$$

Since $w(x)=v(x)$ on $B(\xi, 1)$, $\lim_{x\to\xi,\, x\in T_\gamma(\xi,a)} U_2(x)=0$ for any $a>0$.

Set $k_s(r)=K_{2s+2}(x)$, where $r=|x|$. If $n-mp+\alpha\geqq 0$, then $2s+1=m-k+1<n$. First suppose $2s+2\leqq n$. Then

$$\begin{aligned} U_1(x) &= -\sum_{j=1}^n k_s(x_n/2)\int_{\partial B(x,x_n/2)}\frac{\partial}{\partial y_j}\Delta^s u(y)\frac{y_j-x_j}{|y-x|}dS(y) \\ &\quad +\int_{B(x,x_n/2)}K_{2s+2}(x-y)\Delta^{s+1}u(y)dy \\ &= -\int_{B(x,x_n/2)}\{k_s(x_n/2)-K_{2s+2}(x-y)\}\Delta^{s+1}u(y)dy \\ &= -\int_0^{x_n/2}\{k_s(x_n/2)-k_s(r)\}\left\{\int_{\partial B(x,r)}\Delta^{s+1}u(y)dS(y)\right\}dr \\ &= -\sum_{i=1}^{m-s}c_i\Delta^{i+s}u(x)\int_0^{x_n/2}\{k_s(x_n/2)-k_s(r)\}r^{n-1+2i-2}dr \\ &= -\sum_{i=1}^{m-s}c_i'\Delta^{i+s}u(x)x_n^{2i+2s} \\ &= x_n^{-n}\sum_{0<|\lambda|\leqq m}c_\lambda\int_{B(x,x_n/2)}(y-x)^\lambda D^\lambda u(y)dy \end{aligned}$$

by Lemmas 1 and 2, where $x\in B(\xi,1)\cap R_+^n$, so that $u(x)=w(x)$ there. Hence it follows from Hölder's inequality that

$$|U_1(x)|\leqq \text{const.}\sum_{l=1}^m\left(\int_{B(x,x_n/2)}|\nabla_l u(y)|^p y_n^{lp-n}dy\right)^{1/p},$$

which tends to zero as $x\to\xi$, $x\in T_\gamma(\xi,a)$, since $\xi\notin\bigcup_{l=1}^m E_{l,\gamma}$. Thus the proof of the theorem is complete.

4. Further results and remarks.

Let D be a special Lipschitz domain as defined in Stein [10; Chap. VI]. Then similar results can be shown to hold for u which is polyharmonic of order $m+1$ in D and satisfies

$$\int_D |\nabla_m u(x)|^p d(x)^\alpha dx < \infty, \qquad p>1, \ \alpha < mp-1,$$

if we replace $T_\gamma(\xi, a)$ by the set $\{x \in D;\ |x-\xi| < ad(x)^{1/\gamma}\}$. Here $d(x)$ denotes the distance from x to the boundary ∂D.

Finally we give an open problem: If u is a function which is polyharmonic of order $m+1$ in R_+^n and satisfies (1) with $p>1$ and $\alpha=mp-1$ for any bounded open set G in R_+^n, then does there exist a set E such that $H_{n-1}(E)=0$ and u has a nontangential limit at any $\xi \in \partial R_+^n - E$? By a well known result [10; Theorem 4 in Chap. VII], this is true for a harmonic function u in R_+^n satisfying (1) with $1<p\leqq 2$ and $\alpha=p-1$ for any bounded open set G in R_+^n. In view of the proofs of [8; Theorem 1] and our theorem, we have the following result: If u is a function which is polyharmonic of order $m+1$ in R_+^n and satisfies (1) with $p>1$ and $\alpha=mp-1$ for any bounded open set G in R_+^n, then there exists a set $E \subset \partial R_+^n$ such that $H_{n-1}(E)=0$ and

$$C(\xi;\ u,\ l_\xi)=C(\xi;\ u,\ T_1(\xi,\ a))$$

for any $a>0$ and any $\xi \in \partial R_+^n - E$, where $C(\xi;\ u,\ F)=\bigcap_{r>0} \overline{u(F \cap B(\xi,\ r))}$ for a set $F \subset R_+^n$ and $l_\xi=\{\xi+(0, \cdots, 0, t);\ t>0\}$.

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