K. AZUKAWA AND J. BURBEA
KODAI MATH. J.
7 (1984), 133—152

HESSIAN QUARTIC FORMS AND THE BERGMAN METRIC

BY KAZUO AZUKAWA AND JACOB BURBEA

Contents

§ 0. Introduction and notation. In [7], the *"curvature"* of the Carathéodory metric on a bounded domain in C^m is considered by using the generalized Hessian of this metric; it may be called the *Hessian-curvature.* Referring to this, we define Hessian quartic forms to an arbitrary hermitian metric. These Hessian quartic forms enable us to provide another proof for the following result of Wu [14; Lemmas 1 and 4]: *The holomorphic sectional curvature coincides with the maximum of the Gaussian curvatures to all local one-dimensional submanifolds that contact at the point in the direction under consideration* (Corollary 1.8).

Modifying the construction of the *n-th order Bergman metric* introduced in [6] (also see [5]), we define quantities $\mu_{0,n}$ $(n\in N)$ as follows: We consider a certain linear functional on a specified subspace of square-integrable holomorphic m-forms on a m-dimensional complex manifold and define the quantity μ_n by the square of the operator norm of this functional (Proposition 3.7). We then set $\mu_{0,n} := \mu_n/\mu_0$. The quantity $\mu_{0,n}$ is a $[0, +\infty)$-valued function on the tangent bundle, and is biholomorphic invariant (Theorem 4.2). Especially $\mu_{0,1}$ is the usual *Bergman metric,* and $2(\mu_{0,1})^2-\mu_{0,2}$ is the quartic form defining the holomorphic sectional curvature of the Bergman metric (Theorem 4.4).

Let $\lambda_{0,n}^z$ be the n-th order Bergman metric on a complex manifold, relative to a coordinate z, as introduced in [6]. Then the Hessian quartic form of the Bergman metric coincides with $2(\mu_{0,1})^2-\lambda_{0,2}^z$ (Corollary 5.4). In general, $\lambda_{0,2}^z \geqq \mu_{0,2}$ with an explicit statement as to when equality holds (Proposition 5.5). Finally, we note that the quantity $\lambda_{0,2}^z$ does depend on the coordinate z, by examining a concrete example (Corollary 5.8). One should observe, however, that while the quantity $\lambda_{0,n}^z$ with $n\geqq 2$ is biholomorphic invariant in the weak sense mentioned

Received March 31, 1983

in [5, 6], it is nevertheless dependent on the coordinate z, that is one cannot regard it as a global function on the tangent bundle of the manifold.

NOTATION. The following notation will be used throughout the paper.

0.1. *Matrices.*

(0.1.1) For a positive integer $n \in \boldsymbol{N}$, we put:

$M(n, \boldsymbol{C}) :=$ the set of all (n, n)-matrices over $\boldsymbol{C}$.
$GL(n, \boldsymbol{C}) := \{A \in M(n, \boldsymbol{C});\ \det A \neq 0\}$.
$S(n, \boldsymbol{C}) := \{A \in M(n, \boldsymbol{C});\ A$ is symmetric$\}$.
$H(n, \boldsymbol{C}) := \{A \in M(n, \boldsymbol{C});\ A$ is hermitian$\}$.
$Ps(n, \boldsymbol{C}) := \{A \in H(n, \boldsymbol{C});\ A$ is positive semi-definite$\}$.
$P(n, \boldsymbol{C}) := \{A \in H(n, \boldsymbol{C});\ A$ is positive definite$\}$.

(0.1.2) For $A \in Ps(n, \boldsymbol{C})$, we denote by $A^{1/2}$ the square-root of A in $Ps(n, \boldsymbol{C})$. If $A \in P(n, \boldsymbol{C})$ we put $A^{-1/2} := (A^{-1})^{1/2}$, where A^{-1} is the inverse matrix of A (note that $A^{-1/2} \in P(n, \boldsymbol{C})$).

0.2. *Manifolds.*

(0.2.1) The letter "M" will always mean a paracompact connected complex manifold, while the letter "m" designates its complex dimension. The term "coordinate z" stands for a local coordinate system $z = (z^1, \cdots, z^m)$ in M with defining domain "U_z". We write $\partial_a^z := \partial/\partial z^a$ $(a = 1, \cdots, m)$, for simplicity.

(0.2.2) For a point $p \in M$, we set:

$T_p(M) :=$ the holomorphic tangent space at p.
$T(M) :=$ the holomorphic tangent bundle of M.
$\Lambda_p^{(s,t)}(M) :=$ the space of all (s, t)-forms at p.

(0.2.3) For a pair of coordinates z and w in M with $U_z \cap U_w \neq \phi$, we denote by J_z^w the Jacobian of $w \circ z^{-1}$, i.e. $J_z^w := \det(\partial_a^z . w^b)_{a,b}$.

(0.2.4) For a coordinate $z = (z^1, \cdots, z^m)$, we put $dz := dz^1 \wedge \cdots \wedge dz^m$. The pull-back of the euclidian volume element on $\boldsymbol{C}^m$ by z is given by $(\sqrt{-1}^{m^2}/2^m) dz \wedge \overline{dz}$.

0.3. *Multi-indices.*

Let m be the dimension of M as in (0.2.1).

(0.3.1) Let $MI(n) := \{1, \cdots, m\}^n$, $MII(n) := \{(a_1, \cdots, a_n) \in MI(n);\ a_i \leqq a_{i+1}$ $(i = 1, \cdots, n-1)\}$ $(n \in \boldsymbol{N})$, and $MI(0) := MII(0) = \{\phi\}$. By a multi-index (resp. an increasing multi-index) of length n we mean an element of $MI(n)$ (resp. $MII(n)$).

(0.3.2) For a pair of increasing multi-indices $A = (a_1, \cdots, a_n)$ and $B = (b_1, \cdots, b_{n'})$, we write $A < B$ if $n < n'$ or if $n = n'$ implies that $a_i = b_i$ $(i < i_0)$ and $a_{i_0} < b_{i_0}$ for some $i_0 \in \{1, \cdots, n\}$.

(0.3.3) For a non-negative integer $n \in \boldsymbol{Z}_+$, we denote by $\varphi(n)$ the cardinality of the set $\bigcup_{j=0}^{n} MII(j)$. Thus $\varphi(n)=\binom{m+n}{n}$, while the cardinality of $MII(n)$ is $\varphi(n)-\varphi(n-1)=\binom{m+n-1}{n}$ with $\varphi(-1):=0$.

(0.3.4) We denote by Φ the unique order-preserving bijection from $\boldsymbol{N}$ onto $\bigcup_{n=0}^{\infty} MII(n)$. Thus, for an increasing multi-index A and for $n \in \boldsymbol{N}$ we have $A \in MII(n)$ if and only if $\Phi(\varphi(n-1))<A \leqq \Phi(\varphi(n))$.

0.4. *Local differential operators.*

Let $z=(z^1, \cdots, z^m)$ be a coordinate in M.

(0.4.1) For a constant vector $v=(v^1, \cdots, v^m)$ in $\boldsymbol{C}^m$ we put (see (0.2.1)): $\partial_v^z:=\sum v^a \partial_a^z$, $(\partial_v^z)^0:=1^z$, $(\partial_v^z)^n:=\partial_v^z(\partial_v^z)^{n-1}$ $(n=1, 2, \cdots)$, where 1^z stands for the identity operator on functions on U_z.

(0.4.2) For a multi-index $A=(a_1, \cdots, a_n)$ we put: $\partial_A^z:=\partial_{a_1}^z \cdots \partial_{a_n}^z$ (when $n=0$ we have $\partial_\phi^z=1^z$).

§1. **Hessian quartic form of a hermitian metric.** Let g be an arbitrary hermitian metric on M, and let R be the hermitian curvature tensor to the metric in the sense of Kobayashi and Nomizu [12; pp. 155-159] (cf. also [11; pp. 37-39]). For a coordinate z in M, we put: $g_{z,a\bar{b}}:=g(\partial_a^z, \overline{\partial_b^z})$, $(g_z{}^{\bar{b}a}):=(g_{z,a\bar{b}})^{-1}$, $R_{z,a\bar{b}c\bar{d}}:=g(R(\partial_c^z, \overline{\partial_d^z})\overline{\partial_b^z}, \partial_a^z)$ $(a, b, c, d \in \{1, \cdots, m\})$. Thus,

$$R_{z,a\bar{b}c\bar{d}}=\partial_c^z \overline{\partial_d^z}\, g_{z,a\bar{b}}-\sum_{s,t} g_z^{\bar{t}s}(\partial_c^z\, g_{z,a\bar{t}})(\overline{\partial_d^z}\, g_{z,s\bar{b}})\,.$$

DEFINITION 1.1. For $p \in M$, we define a quartic form $\mathrm{Sec}(p\,;\cdot)$ on $T_p(M)$ by

$$\mathrm{Sec}(p\,;(\partial_v^z)_p):=-\sum R_{z,a\bar{b}c\bar{d}}(p) v^a \bar{v}^b v^c \bar{v}^d\,,$$

where z is a coordinate around p and $v \in \boldsymbol{C}^m$ (see (0.4.1)). Since $\mathrm{Sec}(p\,;X)/g(X, \bar{X})^2$ is the holomorphic sectional curvature of g in the direction $X \in T_p(M)-\{0\}$, we call $\mathrm{Sec}(p\,;\cdot)$ the *curvature quartic form* of g at p.

Remark 1.2. Since $R_{z,a\bar{b}c\bar{d}}$ are components of a tensor, the definition of $\mathrm{Sec}(p\,;\cdot)$ does not depend on the coordinate z around p.

DEFINITION 1.3. For a coordinate z and $v \in \boldsymbol{C}^m-\{0\}$, we set $g_{z,v}:=g(\partial_v^z, \overline{\partial_v^z})>0$. For $p \in U_z$ we define a quartic form $\mathrm{Hess}^z(p\,;\cdot)$ on $T_p(M)$ as follows:

$$\mathrm{Hess}^z(p\,;(\partial_v^z)_p):=\begin{cases} -g_{z,v\bar{v}}(p)\partial_v^z\overline{\partial_v^z}.\log g_{z,v\bar{v}}(p)\,, & v \neq 0 \\ 0\,, & v=0. \end{cases}$$

Since $\partial_v^z\overline{\partial_v^z}$ is a complex Hessian, we call $\mathrm{Hess}^z(p\,;\cdot)$ the *Hessian quartic form* of g, at p, relative to z.

LEMMA 1.4. *Let g be a hermitian metric on M, z a fixed coordinate around p and v a constant vector in $\boldsymbol{C}^m-\{0\}$. We consider the complex line $L:=z(p)+\boldsymbol{C}v$ in the space $\boldsymbol{C}^m$ and the connected component M_1 of $z^{-1}(L)$, containing p, which is a one-dimensional complex submanifold in U_z. We denote by $\mathrm{Gauss}(p, v; \cdot)$ the curvature quartic form, at p, of the metric induced from g on M_1. Then, viewing $T_p(M_1)$ as a subspace of $T_p(M)$,*

$$\mathrm{Hess}^z(p; (\partial_v^z)_p)=\mathrm{Gauss}(p, v; (\partial_v^z)_p).$$

Proof. The mapping $M_1\ni z^{-1}(z(p)+\xi v)\mapsto\xi\in\boldsymbol{C}$, denoted by t, is a coordinate in M_1 around p, while the inclusion mapping $\iota: M_1\to M$ may be represented, under the coordinates t and z, as $\xi\mapsto z(p)+\xi v$. The induced metric ι^*g is given by

$$\iota^*g=2\sum g_{z,a\bar{b}}\circ\iota\, v^a\bar{v}^b\, dt\cdot d\bar{t}=2\, g_{z,v\bar{v}}\circ\iota\, dt\cdot d\bar{t},$$

and the hermitian curvature tensor to ι^*g is

$${}^1R_{t,1\bar{1}1\bar{1}}=\partial^t\partial^{\bar{t}}\cdot g_{z,v\bar{v}}\circ\iota-|\partial^t\cdot g_{z,v\bar{v}}\circ\iota|^2/g_{z,v\bar{v}}\circ\iota.$$

Since $(\partial_v^z)_p=\iota_*(\partial^t)_p=(\partial^t)_p$ by the identification of $T_p(M_1)$ with $\iota_*T_p(M_1)$, we have $\mathrm{Gauss}(p, v; (\partial_v^z)_p)=\mathrm{Gauss}(p, v; (\partial^t)_p)=-{}^1R_{t,1\bar{1}1\bar{1}}(p)=\mathrm{Hess}^z(p; (\partial_v^z)_p)$, and the result follows.

Let $(\ ,\ )_m$ (resp. $\|\ \|_m$) be the canonical hermitian inner product (resp. the induced norm) on $\boldsymbol{C}^m$. Then, for every $p\in U_z$ we have $g_{z,v\bar{v}}(p)=v\, G_z(p)\, v^*=\|v\, G_z(p)^{1/2}\|_m^2$, where $G_z:=(g_{z,a\bar{b}})$ (see (0.1.2)).

PROPOSITION 1.5. *Let g be a hermitian metric on M, and z be a coordinate with $G_z=(g_{z,a\bar{b}})$. Then, for every $(p, v)\in U_z\times(\boldsymbol{C}^m-\{0\})$, we have*

$$\mathrm{Sec}(p; (\partial_v^z)_p)-\mathrm{Hess}^z(p; (\partial_v^z)_p)$$
$$=(\|vA^{1/2}\|_m^2\|vBA^{-1/2}\|_m^2-|(vB, v)_m|^2)/\|vA^{1/2}\|_m^2$$

where $A:=G_z(p)$ and $B:=\partial_v^z.G_z(p)$. In particular, we have

$$\mathrm{Hess}^z(p; (\partial_v^z)_p)\leqq\mathrm{Sec}(p; (\partial_v^z)_p)$$

with equality if and only if

$$v\partial_v^z.G_z(p)=\xi vG_z(p) \tag{1.1}$$

for some scalar $\xi\in\boldsymbol{C}$.

Proof. By Definitions 1.1 and 1.3 we have

$$\mathrm{Sec}(p; (\partial_v^z)_p)-\mathrm{Hess}^z(p; (\partial_v^z)_p)=vBA^{-1}B^*v^*-|vBv^*|^2/vAv^*$$
$$=\|vBA^{-1/2}\|_m^2-|(vB, v)_m|^2/\|vA^{1/2}\|_m^2.$$

The last term is zero if and only if $vBA^{-1/2}=\xi vA^{1/2}$ for some $\xi\in\boldsymbol{C}$. This is equivalent to (1.1) and the proof is complete.

LEMMA 1.6. *Let g be a hermitian metric on M, and let a point $p\in M$ and a tangent vector $X\in T_p(M)-\{0\}$ be given. Then, there exists a coordinate z around p so that condition* (1.1) *holds for $v\in\boldsymbol{C}^m$ with $X=(\partial_v^z)_p$.*

Proof. We arbitrarily fix a coordinate $w=(w^1, \cdots, w^m)$ around p with $w(p)=0$. For every $(\xi_b^a)\in GL(m, \boldsymbol{C})$ and $(\xi_{ab}^c)_{a,b}\in S(m, \boldsymbol{C})$ $(c=1, \cdots, m)$ (see (0.1.1)), the equations

$$w^c=\sum_a\xi_a^c z^a+\sum_{a,b}\xi_{ab}^c z^a z^b \quad (c=1, \cdots, m) \tag{1.2}$$

define a new coordinate $z=(z^1, \cdots, z^m)$ around p with $z(p)=0$ by the inverse mapping theorem. We shall select the numbers ξ_a^c, ξ_{ab}^c so that z satisfies (1.1) for $v\in\boldsymbol{C}^m$ with $X=(\partial_v^z)_p$.

First, we can find a matrix (ξ_a^c) so that

$$v^a=0 \ (a=2, \cdots, m), \quad G_z(p)=1_m, \tag{1.3}$$

where $G_z:=(g_{z,a\bar{b}})$ and 1_m is the identity matrix. Indeed, we set $X_1:=X/g(X, \bar{X})^{1/2}$ and select $X_2, \cdots, X_m\in T_p(M)$ so that $g(X_a, \bar{X}_b)=\delta_{ab}$. If we write $\sum_c\xi_a^c(\partial_c^w)_p:=X_a$, then (ξ_a^c) is the desired matrix.

By virtue of (1.3), condition (1.1) is equivalent to

$$\partial_1^z\cdot g_{z,1\bar{d}}(p)=0 \quad (d=2, \cdots, m). \tag{1.4}$$

Making use of (1.2), condition (1.4) can be rewritten as

$$\sum_{a,b}g_{w,a\bar{b}}(p)\bar{\xi}_d^b\xi_{11}^a=-\frac{1}{2}\sum_{a,b,c}\partial_c^w\cdot g_{w,a\bar{b}}(p)\xi_1^c\xi_1^a\bar{\xi}_d^b \quad (d=2, \cdots, m). \tag{1.5}$$

Since $G_w(p)(\bar{\xi}_b^a)\in GL(m, \boldsymbol{C})$, equations (1.5) with unknowns ξ_{11}^a $(a=1, \cdots, m)$ possess a solution. This concludes the proof.

Combining the last lemma with Proposition 1.5, we obtain the following assertion:

PROPOSITION 1.7. *For $X\in T_p(M)$,* $\mathrm{Sec}(p\,;X)$ *coincides with* $\max\{\mathrm{Hess}^z(p\,;X)\,;$ *z is a coordinate around p*$\}$.

By virtue of Lemma 1.4, this proposition yields the following result which was alluded to in the introduction of this paper.

COROLLARY 1.8. (Wu [14; Lemmas 1 and 4]). *For a tangent vector $X\in T_p(M)-\{0\}$, the holomorphic sectional curvature* $\mathrm{Sec}(p\,;X)/g(X, \bar{X})^2$ *to a hermitian metric g on M coincides with* $\max\{GC_S(p)\,;$ *S is a local one-dimensional submanifold such that $S\ni p$ and $\iota_{S*}T_p(S)=\boldsymbol{C}X\}$, where ι_S is the inclusion mapping* *of S into M, and $GC_S(p)$ is the Gaussian curvature at p to the induced metric $\iota_S{}^*g$.*

Remark 1.9. In [7], a generalized definition of the "Hessian curvature" $\mathrm{Hess}^z(p:X)/g(X,\bar{X})^2$ is used for the square of the Carathéodory metric on a bounded domain in $\boldsymbol{C}^m$.

§ 2. The Bergman form.

We recall the notion of the *Bergman form* of M. For this we follow the description given in [5, 6]. The set of all holomorphic m-forms α on M satisfying $\|\alpha\|^2:=(\sqrt{-1}^{m^2}/2^m)\int_M \alpha\wedge\bar{\alpha}<+\infty$ is denoted by $H(M)$. The space $H(M)$ becomes a pre-Hilbert space over $\boldsymbol{C}$ with an inner product $(\ ,\ )$ inherited from the norm $\|\ \|$.

DEFINITION 2.1. Let α be a $(m, 0)$-form on M, and let z be a coordinate in M. We denote by α_z the function on U_z such that $\alpha|_{U_z}=\alpha_z dz$ (see (0.2.4)).

Applying the Cauchy integral formula to α_z, $\alpha\in H(M)$, we find that $H(M)$ is in fact a separable Hilbert space, and, moreover, for a coordinate z around a point $p\in M$ and for a holomorphic differential operator D^z on U_z, the linear functional $H(M)\ni\alpha\mapsto D^z.\alpha_z(p)\in\boldsymbol{C}$ is bounded (see also Kobayashi [10] and Lichnerowicz [13]). By the Riesz-representation theorem there exists a unique element $\gamma(D^z, p)\in H(M)$ such that

$$D^z.\alpha_z(p)=(\alpha, \gamma(D^z, p))\,, \quad \alpha\in H(M)\,. \tag{2.1}$$

Especially, when $D^z=1^z$ (see (0.4.1)), we set

$$\kappa_{z,p}:=\gamma(1^z, p)\,. \tag{2.2}$$

For another coordinate w around p we have

$$\kappa_{z,p}=\overline{J_z^w(p)}\kappa_{w,p}\,, \tag{2.3}$$

since $\alpha_z=J_z^w\alpha_w$ on $U_z\cap U_w$ for every $\alpha\in H(M)$ (see (0.2.3)).

LEMMA 2.2. *Let $\gamma=\gamma(D^z, p)$ (resp. $\kappa_{z,p}$) be as in* (2.1) (*resp.* (2.2)). *Then,* $D^z.(\kappa_{z,p})_z(p)=\overline{\gamma_z(p)}$.

Proof. By definition $D^z.(\kappa_{z,p})_z(p)=(\kappa_{z,p},\gamma)=\overline{(\gamma,\kappa_{z,p})}=\overline{\gamma_z(p)}$, and the result follows.

Let $\bar{M}$ be the conjugate complex manifold of M, and denote by $M\ni p\mapsto\bar{p}\in\bar{M}$ the conjugation. For a coordinate z in M, we denote by $\bar{z}$ the conjugate coordinate of z with defining domain $\bar{U}_z$, i.e. $\bar{z}(\bar{p}):=\overline{z(p)}$, $p\in U_z$.

DEFINITION 2.3. For $p, q\in M$ we set $K(q,\bar{p}):=\kappa_{z,p}(q)\wedge d\bar{z}_{\bar{p}}$, where z is a coordinate around p. By property (2.3) the quantity K is a well-defined $(2m, 0)$-form on the product manifold $M\times\overline{M}$ of dimension $2m$, and is called the *Bergman form* of M (cf. [5, 6]).

Applying Definition 2.1 for the manifold $M\times\overline{M}$, we find that $K|_{U_w\times\overline{U_z}}=K_{w\times\bar{z}}dw\wedge d\bar{z}$. On the other hand, by Definition 2.3

$$K_{w\times\bar{z}}(\cdot,\ \bar{p})=(\kappa_{z,p})_w \tag{2.4}$$

on U_w, for every $p\in U_z$. It follows from Lemma 2.2 that

$$K_{w\times\bar{z}}(q,\ \bar{p})=\overline{K_{z\times\bar{w}}(p,\ \bar{q})}\,,\quad (p,\ q)\in U_z\times U_w\,. \tag{2.5}$$

By virtue of (2.4) and (2.5), the function $K_{w\times\bar{z}}$ is holomorphic on $U_w\times U_z$ by Hartogs' theorem of holomorphy. Thus, the Bergman form is a holomorphic $2m$-form on $M\times\overline{M}$.

DEFINITION 2.4. Let D be a holomorphic differential operator on a coordinate neighborhood U_z, and let $\omega=\sum_{A\in F}\omega_A dz^A$ be a holomorphic differential form on U_z, where F is a finite subset of $\bigcup_{n=0}^{m}MII(n)$ (see (0.3.1)). Let $dz^A:=dz^{a_1}\wedge\cdots\wedge dz^{a_n}$ for $A=(a_1,\ \cdots,\ a_n)\in F$. We denote by $D.\omega$ the action of D on ω coefficient-wise, i.e. $D.\omega:=\sum_{A\in F}(D.\omega_A)dz^A$. Viewing $\overline{D}$ as a holomorphic differential operator on $M\times\overline{U_z}$, we have $D.K(q,\ \bar{p})=\overline{D}.K_{w\times\bar{z}}(q,\ \bar{p})dw_q\wedge d\bar{z}_{\bar{p}}$, $(q,\ \bar{p})\in U_w\times\overline{U_z}$. We denote by $\overline{D}.K(\cdot,\ \bar{q})/d\bar{z}_{\bar{p}}$ the well-defined holomorphic m-form β on M such that $\beta|_{U_w}=\overline{D}.K_{w\times\bar{z}}(\cdot,\ \bar{p})dw$ for every coordinate w, i.e. $\overline{D}.K(\cdot,\ \bar{p})=(\overline{D}.K(\cdot,\ \bar{p})/d\bar{z}_{\bar{p}})\wedge d\bar{z}_{\bar{p}}$.

PROPOSITION 2.5. ([5; Lemma 1], [6; Lemma 1]). *Let D^z (resp. E^w) be a holomorphic differential operator on a coordinate neighborhood U_z (resp. U_w) of p (resp. q). Let $\gamma(D^z,\ p)$ and $\gamma(E^w,\ q)$ be as in* (2.1). *Then:*

(i) $\overline{D^z}.K(\cdot,\ \bar{p})/d\bar{z}_{\bar{p}}=\gamma(D^z,\ p)\in H(M)$;

(ii) $(\gamma(D^z,\ p),\ \gamma(E^w,\ q))=E^w\overline{D}_z.K_{w\times\bar{z}}(q,\ \bar{p})$.

Proof. (i) Let x be a coordinate, and let $D:=D^z$. Using Lemma 2.2, (2.4) and (2.5) we have for every $r\in U_x$,

$$\begin{aligned}\gamma(D,\ p)_x(r)&=\overline{D.(\kappa_{x,r})_z(p)}\\&=\overline{D.K_{z\times\bar{x}}(p,\ \bar{r})}\\&=\overline{D}.\overline{K_{z\times\bar{x}}(p,\ \bar{r})}\\&=\overline{D}.K_{x\times\bar{z}}(r,\ \bar{p})\,.\end{aligned}$$

Therefore, $\gamma(D,\ p)|_{U_x}=\overline{D}.K_{x\times\bar{z}}(\cdot,\ \bar{p})dx$, as desired.

(ii) By definition and part (i), we have

$$\begin{aligned}(\gamma(D^z, p), \gamma(E^w, q)) &= E^w.\gamma(D^z, p)_w(q)\\ &= E^w.(\overline{D^z}.K_{w\times\bar z}(\cdot, \bar p))(q)\\ &= E^w\overline{D^z}.K_{w\times\bar z}(q, \bar p),\end{aligned}$$

as desired. This concludes the proof.

COROLLARY 2.6. (Characterization of the Bergman form). *The Bergman form K is a unique $(2m, 0)$-form on the product manifold $M\times\overline{M}$ with the reproducing property, in the sense that $K(\cdot, \bar p)\in H(M)\wedge \Lambda_{\bar p}^{(m,0)}(\overline{M})$ for every $p\in M$, and*

$$\alpha_z(p)=(\alpha, K(\cdot, \bar p)/d\bar z_{\bar p}) \tag{2.6}$$

for every $\alpha\in H(M)$, and every pair of p and z with $p\in U_z$.

Proof. The Bergman form K possesses the reproducing property by Definition 2.3 and Proposition 2.5 (i). The uniqueness of K follows from Aronszajn [1; item (2), p. 343].

PROPOSITION 2.7. *Let $(\beta_j)_{j=1}^N$ $(N\in\boldsymbol{Z}_+\cup\{+\infty\})$ be a complete orthonormal system of $H(M)$. If z (resp. w) is a coordinate around a point $p\in M$ (resp. $q\in M$), then the series $\sum_{j=1}^N(\beta_j)_w(q)\overline{(\beta_j)_z(p)}$ converges absolutely to $K_{w\times\bar z}(q, \bar p)$, where K is the Bergman form of M.*

Proof. It follows from (2.6) that the Fourier coefficients ξ_j of $K(\cdot, \bar p)/d\bar z_{\bar p}$ with respect to (β_j) are given by $\xi_j:=(K(\cdot, \bar p)/d\bar z_{\bar p}, \beta_j)=\overline{(\beta_j)_z(p)}$. By the completeness of (β_j) we have $\|\sum_{j=1}^n\xi_j\beta_j-K(\cdot, \bar p)/d\bar z_{\bar p}\|\to 0$ as $n\to N$. Another application of (2.6) gives $\lim_{n\to N}\sum_{j=1}^n\xi_j(\beta_j)_w(q)=K_{w\times\bar z}(q, \bar p)$, and the result follows.

Remark 2.8. By virtue of Proposition 2.7, the Bergman form introduced in Definition 2.3 coincides, up to a multiplicative constant, with the Bergman kernel form given in Kobayashi [10; p. 269] (see also [13]).

§3. **Extremal quantities of the space** $H(M)$. We shall first establish a chain rule for the differential operators ∂_A^z (see (0.4.2)). For $n\in\boldsymbol{Z}_+$, we denote by $\Pi(n)$ the family of all partitions of the set $\{1, \cdots, n\}$ $(\Pi(0)=\{\phi\})$. Given a multi-index $A=(a_1, \cdots, a_n)\in MI(n)$ and a subset $P\subset\{1, \cdots, n\}$, we set $\partial_{A|P}^z:=\prod_{i\in P}\partial_{a_i}^z$ (when $n=0$, we have $\partial_{\phi|\phi}^z=1^z$).

LEMMA 3.1. *Let z and w be coordinates in M with $U_z\cap U_w\neq\phi$, and let $A\in MI(n)$. Then, for every holomorphic function f on $U_z\cap U_w$, we have $\partial_A^z.f=\sum_{\mathcal{P}\in\Pi(n)}f_{A,\mathcal{P}}$, where $f_{A,\mathcal{P}}$ with $\mathcal{P}=\{P_1, \cdots, P_u\}$ is the function given by*

$$\sum_{(b_i)\in MI(u)}(\partial_{A|P_1}^z.w^{b_1})\cdots(\partial_{A|P_u}^z.w^{b_u})(\partial_{(b_i)}^w.f).$$

Proof. The proof is carried by induction on $n\in\boldsymbol{Z}_+$, using the formula

$$\partial^{z}_{a_{n+1}}\cdot f_{A',\mathscr{P}}=\sum_{\nu=1}^{u} f_{A,\mathscr{P}(\nu)}+f_{A,\mathscr{P}'}\,.$$

Here $A=(A', a_{n+1})\in MI(n+1)$, $\mathscr{P}=\{P_1, \cdots, P_u\}\in \Pi(n)$, $\mathscr{P}(\nu):=\{P_1, \cdots, P_\nu\cup\{n+1\}, \cdots, P_u\}$, and $\mathscr{P}':=\{P_1, \cdots, P_u, \{n+1\}\}$. The proof is now complete.

DEFINITION 3.2. For every $n\in \boldsymbol{Z}_+$ and $p\in M$, we define a subspace $H_n(p)$ of $H(M)$ and a condition $(C_n)_p$ as follows:

$$H_n(p):=\{\alpha\in H(M)\,;\,\partial^{z}_{A}\cdot\alpha(p)=0\ (A\in\textstyle\bigcup_{j=0}^{n-1}MI(j))\}\quad (H_0(p)=H(M)),$$

$(C_n)_p$: (For every vector $(\xi^A)_{A\in MII(n)}\in \boldsymbol{C}^N-\{0\}$, there exists a form $\alpha\in H_n(p)$ such that $\sum_A \xi^A\partial^{z}_{A}\cdot\alpha(p)\neq 0$.

Here, z is an arbitrary fixed coordinate around p and $N:=\varphi(n)-\varphi(n-1)$ (see (0.3.3)). Condition (C_n) stands for the collection of all $(C_n)_p$ $(p\in M)$.

By Lemma 3.1, we see that the definitions of $H_n(p)$ and $(C_n)_p$ do not depend on the choice of the coordinate z.

Remark 3.3. Condition (C_0) (resp. (C_1)) coincides with condition (A. 1) (resp. (A. 2)) of Kobayashi [10].

LEMMA 3.4. *Let K be the Bergman form of M, z be a coordinate around a point $p\in M$ and let $n\in\boldsymbol{Z}_+$. Set $S(p, z):=\{\partial^{\bar z}_{A}\cdot K(\cdot, \bar p)/d\bar z_{\bar p}\,;\,A\in\bigcup_{j=0}^{n}MII(j)\}\subset H(M)$. Then:*

(i) *The space $H_{n+1}(M)$ coincides with $S(p, z)^{\perp}$, the orthogonal subspace of the subset $S(p, z)$ in $H(M)$.*

(ii) *Conditions $(C_j)_p$ $(j=0, \cdots, n)$ hold true if and only if the system $S(p, z)$ is linearly independent in $H(M)$.*

Proof. By Proposition 2.5 (i),

$$\partial^{z}_{A}\cdot\alpha_z(p)=(\alpha, \overline{\partial^{z}_{A}}\cdot K(\cdot, \bar p)/d\bar z_{\bar p})\,,\quad \alpha\in H(M)\,. \tag{3.1}$$

Thus, assertion (i) follows immediately from (3.1). To prove part (ii), suppose that $(C_j)_p$ $(j=0, \cdots, n)$ hold true, and let

$$\sum_{j=0}^{n}\sum_{A\in MII(j)}\xi^A\partial^{\bar z}_{A}\cdot K(\cdot, \bar p)/d\bar z_{\bar p}=0$$

for a vector (ξ^A). It follows from (3.1) that

$$\sum_{j=0}^{n}\sum_{A\in MII(j)}\xi^A\partial^{z}_{A}\cdot\alpha_z(p)=0\,,\quad \alpha\in H(M)\,. \tag{3.2}$$

Applying formula (3.2) on $\alpha\in H_n(p)$ and using assumption $(C_n)_p$, we find that $\xi^A=0$ for every $A\in MII(n)$. Similarly and inductively, we conclude that $\xi^A=0$ for every A. Conversely, suppose that

(3.3) $S(p, z)$ is linearly independent in $H(M)$,

and let

(3.4) $$\sum_{A\in MII(j)}\xi^A\partial_A^z.\,\alpha(p)=0 \quad (\alpha\in H_j(p)),$$

where $j\in\{0,\cdots,n\}$ and $\xi^A\in\boldsymbol{C}$. Substituting (3.1) into formula (3.4), we see that $\sum_{A\in MII(j)}\xi^A\overline{\partial_A^z}.\,K(\cdot,\bar{p})/d\bar{z}_{\bar{p}}\in H_j(p)^{\perp}$. From part (i) with j instead of n, assumption (3.3) implies that $\xi^A=0$ for every A. This concludes the proof.

LEMMA 3.5. *Let $X\in T_p(M)$ and $\alpha\in H_n(p)$. If we express $X=(\partial_v^z)_p=(\partial_{v'}^w)_p$ $(v, v'\in\boldsymbol{C}^m)$ with respect to coordinates z and w around p, then $(\partial_v^z)^n.\,\alpha(p)=(\partial_{v'}^w)^n.\,\alpha(p)$; therefore, this form at p may be denoted by $X^n.\,\alpha(p)$.*

Proof. We first note that

(3.5) $$v'^a=\partial_v^z.\,w^a(p) \quad (a=1,\cdots,m),$$

(3.6) $$(\partial_v^z)^n.\,\alpha_z(p)=\sum_{j=0}^n\binom{n}{j}(\partial_v^z)^{n-j}.\,J_z^w(p)(\partial_v^z)^j.\,\alpha_w(p),$$

since $\alpha_z=J_z^w\alpha_w$ (see (0.2.3)). Since $\alpha\in H_n(p)$, it follows from Lemma 3.1 as well as (3.5) that

$$(\partial_v^z)^j.\,\alpha_w(p)=\begin{cases}0, & j\leqq n-1\\ (\partial_{v'}^w)^j.\,\alpha_w(p), & j=n.\end{cases}$$

Substituting these values into formula (3.6), we obtain

$$(\partial_v^z)^n.\,\alpha_z(p)=J_z^w(p)(\partial_{v'}^w)^n.\,\alpha_w(p), \quad \text{or} \quad (\partial_v^z)^n.\,\alpha(p)=(\partial_{v'}^z)^n.\,\alpha(p),$$

as desired.

DEFINITION 3.6. (Kobayashi [10; p. 269]). We define an order relation on the subset $\{\omega\wedge\bar{\omega};\ \omega\in\Lambda_p^{(m,0)}(M)\}\subset\Lambda_p^{(m,m)}(M)$ as follows (see (0.2.2)): We let $\omega\wedge\bar{\omega}\leqq\omega'\wedge\bar{\omega}'$, for $\omega,\omega'\in\Lambda_p^{(m,0)}(M)$, if $|\omega_z|\leqq|\omega'_z|$ for some coordinate z around p, where $\omega=\omega_z dz_p$, $\omega'=\omega'_z dz_p$ $(\omega_z,\omega'_z\in\boldsymbol{C})$.

PROPOSITION 3.7. *For every $X\in T_p(M)$ and every $n\in\boldsymbol{Z}_+$, the maximum*

$$\mu_n(p;X):=\max\{X^n.\,\alpha(p)\wedge\overline{X^n.\,\alpha(p)};\ \alpha\in H_n(p),\ \|\alpha\|=1\}$$

under the order in Definition 3.6 exists and coincides with

$$\max\{|(\beta(z),\alpha)|^2;\ \alpha\in S(z)^{\perp},\ \|\alpha\|=1\}(dz\wedge\overline{dz})_p$$

for every coordinate z around p, where

$$S(z):=\{\overline{\partial_A^z}.\,K(\cdot,\bar{p})/d\bar{z}_{\bar{p}};\ A\in\textstyle\bigcup_{j=0}^{n-1}MII(j)\}\subset H(M)$$

and

$$\beta(z):=\overline{(\partial_v^z)^n}.\,K(\cdot,\bar{p})/d\bar{z}_{\bar{p}}\in H(M), \quad X=(\partial_v^z)_p.$$

Proof. Since $X^n.\,\alpha(p)\wedge\overline{X^n.\,\alpha(p)}=|(\partial_v^z)^n.\,\alpha_z(p)|^2(dz\wedge\overline{dz})_p$ for every $\alpha\in H(M)$, the assertion follows from Proposition 2.5 (i) and Lemma 3.4 (i).

Let $p\in M$. From the definition we deduce the following:

$$(3.7)_1 \qquad \left(\begin{array}{l} \text{When } n=0 \text{ or } 1,\ \mu_n(p\,;X)\neq 0 \text{ for every } X\in T_p(M)-\{0\} \\ \text{if and only if } (C_n)_p \text{ holds true;} \end{array} \right.$$

$$(3.7)_2 \qquad \left(\begin{array}{l} \text{When } n\geqq 2,\ \mu_n(p\,;X)\neq 0 \text{ for every } X\in T_p(M)-\{0\} \\ \text{if } (C_n)_p \text{ holds true.} \end{array} \right.$$

To study the μ_n more precisely, we record a lemma which is valid for any pre-Hilbert space H. We denote by $G(x_1, \cdots, x_n)$ the Gramian of a system $(x_1, \cdots, x_n)$ in H (especially, $G(\phi)=1$), and denote by $G_{ij}(x_1, \cdots, x_n)$ the (i, j)-cofactor of the Gram-matrix of $(x_1, \cdots, x_n)$ (especially, $G_{11}(x_1)=1$).

LEMMA 3.8. *Let $(x_1, \cdots, x_n)$ $(n\in \boldsymbol{Z}_+)$ be a linearly independent system in a pre-Hilbert space H, and let $x_{n+1}\in H$. Then*

$$\max\{|(y, x_{n+1})|^2;\ y\in\{x_1, \cdots, x_n\}^{\perp},\ \|y\|=1\}$$
$$=G(x_1, \cdots, x_{n+1})/G(x_1, \cdots, x_n),$$

and the latter coincides with $\|y^{(n)}\|^2$, where

$$y^{(n)}:=G(x_1, \cdots, x_n)^{-1}\textstyle\sum_{j=1}^{n+1}G_{n+1,j}(x_1, \cdots, x_{n+1})x_j\,.$$

Furthermore, when $y^{(n)}\neq 0$, the above maximum is attained by y if and only if $y=e^{\sqrt{-1}\,\theta}y^{(n)}/\|y^{(n)}\|$ for some real θ.

DEFINITION 3.9. Let K be the Bergman form of M, and let z be a coordinate. Then $K|_{U_z\times\overline{U_z}}=K_{z\times\bar z}dz\wedge d\bar z$. We consider the function k_z on U_z given by

$$k_z(p):=K_{z\times\bar z}(p, \bar p) \quad (p\in U_z),$$

which we call the *Bergman function* of M relative to z.

DEFINITION 3.10. Let φ and Φ be as in (0.3.3) and (0.3.4), respectively. For a coordinate z in M, we set:

$$k_{z,i\bar j}:=\partial^z_{\Phi(i)}\overline{\partial^z_{\Phi(j)}}\,.k_z\,,$$

$$L_z(j_1, \cdots, j_n):=[k_{z,i\bar j}]^{i=j_1,\cdots,j_n}_{j=j_1,\cdots,j_n}\,,$$

$$L_z(j_1, \cdots, j_n)_{s,t}:=\det[k_{z,i\bar j}]^{i=j_1,\cdots,j_n,s}_{j=j_1,\cdots,j_n,t}\,,$$

$$K_{z,\bar i}(p):=\overline{\partial^z_{\Phi(i)}}\,.K(\cdot, \bar p)/d\bar z_{\bar p}\in H(M) \quad (p\in U_z).$$

It follows from Proposition 2.5 (ii) that $k_{z,i\bar j}=(K_{z,\bar j}, K_{z,\bar i})$ on U_z. This means that the matrix $L_z(j_1, \cdots, j_n)(p)$ is the transpose of the Gram-matrix of the system $(K_{z,\overline{j_1}}, \cdots, K_{z,\overline{j_n}})$ in $H(M)$ for every $p\in U_z$. Combining this with Lemma 3.4 (ii) and Lemma 3.8, we obtain the following two results.

PROPOSITION 3.11. *Let z be a coordinate around $p\in M$, and let $n\in \boldsymbol{Z}_+$. Then $L_z(1, \cdots, \varphi(n))(p)\in Ps(\varphi(n), \boldsymbol{C})$ (see (0.1.1)), and the following four conditions are mutually equivalent:*

(a) *Conditions $(C_j)_p$ $(j=0, \cdots, n)$ hold true.*
(b) *The system $(K_{z,\bar{1}}(p), \cdots, K_{z,\overline{\varphi(n)}}(p))$ in $H(M)$ is linearly independent.*
(c) *$L_z(1, \cdots, \varphi(n))(p)\in P(\varphi(n), \boldsymbol{C})$.*
(d) *$\det L_z(1, \cdots, \varphi(n))(p)>0$.*

THEOREM 3.12. *Let z be a coordinate in M and let $f_{n,z}$ be the function on $U_z\times \boldsymbol{C}^m$, defined by*

$$\mu_n(p\,;(\partial_v^z)_p)=f_{n,z}(p, v)(dz\wedge d\bar{z})_p\,, \quad (p, v)\in U_z\times \boldsymbol{C}^m\,.$$

Then, for every $p\in U_z$ and any maximal linearly independent subset $\{K_{z,\bar{j}_1}(p), \cdots, K_{z,\bar{j}_l}(p)\}$ of $\{K_{z,\bar{1}}(p), \cdots K_{z,\overline{\varphi(n-1)}}(p)\}$,

$$f_{n,z}(p, v)=\det L_z(j_1, \cdots, j_l)(p)^{-1}$$
$$\times \textstyle\sum_{\varphi(n-1)<s,t\leq\varphi(n)} C_{\Phi(s)}C_{\Phi(t)}v^{\Phi(s)}\bar{v}^{\Phi(t)}L_z(j_1, \cdots, j_l)_{s,t}(p)\,.$$

Here, $C_A=n!/n_1!\cdots n_m!$, $v^A=v^{a_1}\cdots v^{a_n}$ for $A=(a_1, \cdots, a_n)$ and $v=(v^1, \cdots, v^m)$, where n_ν is the cardinarity of the set $\{j\,;\, a_j=\nu\}$.

COROLLARY 3.13. (Kobayashi [10; Theorem 2.2]). *For $p\in M$,*

$$K(p, \bar{p})=\max\{\alpha(p)\wedge\overline{\alpha(p)}\,;\ \alpha\in H(M),\ \|\alpha\|=1\}\,.$$

If $K(p, \bar{p})\neq 0$, the above maximum is attained by α if and only if $\alpha=e^{\sqrt{-1}\theta}k_z(p)^{-1}K(\cdot, \bar{p})/d\bar{z}_{\bar{p}}$ for some real θ.

Proof. The first assertion follows from Theorem 3.12 with $n=0$, and the latter from Lemma 3.8 with $n=0$.

§ 4. The biholomorphic invariant $\mu_{0,n}$.

In this section we suppose that M satisfies condition (C_0), i.e. M satisfies condition (A.1) of Kobayashi [10] (see Remark 3.3). For every $n\in \boldsymbol{Z}_+$ and every $X\in T_p(M)$, the (n, n)-form

$$\mu_n(p\,;X)=\max\{X^n.\alpha(p)\wedge\overline{X^n.\alpha(p)}\,;\ \alpha\in H_n(p),\ \|\alpha\|=1\} \tag{4.1}$$

at p has been defined in Proposition 3.7. When $n=0$, by Corollary 3.13 together with $(3.7)_1$, we have

$$\mu_0(p\,;X)=k_z(p)(dz\wedge d\bar{z})_p\,, \quad k_z(p)>0\,.$$

DEFINITION 4.1. For every $n\in\boldsymbol{N}$, we let $\mu_{0,n}:=\mu_n/\mu_0$. Thus it follows that $\mu_{0,n}$ is a well-defined $[0, +\infty)$-valued function on the tangent bundle $T(M)$, for which, by (4.1), it possesses the property that for every $X\in T_p(M)$ and every

$\xi\in \boldsymbol{C}$, $\mu_{0,n}(p\,;\xi X)=|\xi|^{2n}\mu_{0,n}(p\,;X)$.

THEOREM 4.2. *The function $\mu_{0,n}$ on $T(M)$ is a biholomorphic invariant, i.e. $\mu_{0,n}(p\,;X)=\mu_{0,n}(f(p)\,;f_{*}X)$ $((p\,;X)\in T(M))$ for every biholomorphic mapping f from M onto the complex manifold $f(M)$.*

Proof. Let $M':=f(M)$ and let $q:=f(p)$. The mapping f induces an isometry f^{*} of the Hilbert space $H(M')$ onto $H(M)$ so that $f^{*}H_{n}(q)=H_{n}(p)$. Let (w, U_{w}) be a chart of M' around q. Then, the function $z:=w\circ f|_{U_{z}}$ with $U_{z}:=f^{-1}(U_{w})$ is a coordinate around p such that

$$z^{a}=w^{a}\circ f \quad \text{on} \quad U_{z} \quad (a=1, \cdots, m). \tag{4.2}$$

Let $X=(\partial_{v}^{z})_{p}\in T_{p}(M)$. Thus, by (4.2), $f_{*}X=(\partial_{v}^{w})_{q}$. Furthermore, by induction on n and by virtue of (4.2), we obtain, for every $\alpha\in H_{n}(q)$,

$$(\partial_{v}^{z})^{n}.(f^{*}\alpha)_{z}=(\partial_{v}^{z})^{n}.(\alpha_{w}\circ f)=((\partial_{v}^{w})^{n}.\alpha_{w})\circ f \quad \text{on } U_{z}\,.$$

Evaluating the above formula at the point p, we obtain that $(\partial_{v}^{z})^{n}.(f^{*}\alpha)_{z}(p)=(\partial_{v}^{w})^{n}.\alpha_{w}(q)$ for every $\alpha\in H_{n}(q)$. It follows from (4.1) that

$$\mu_{n}(p\,;X)/(dz\wedge\overline{dz})_{p}=\mu_{n}(q\,;f_{*}X)/(dw\wedge\overline{dw})_{q}\,.$$

The desired assertion follows now from Definition 4.1.

Remark 4.3. Let $C(p\,;X)$ be the Carathéodory metric on M. Suppose that $(C_{0})_{p}$ holds and $C(p\,;X)>0$ for some $(p\,;X)\in T(M)$. Then the same argument as in the proof in [6; Theorem 1] implies that $C(p\,;X)^{2n}<(n\,!)^{-2}\mu_{0,n}(p\,;X)$ for every $n\in \boldsymbol{N}$.

Now, making use of Theorem 3.13, we have

$$\mu_{0,1}(p\,;X)=\partial_{v}^{z}\partial_{\bar{v}}^{\bar{z}}.\log k_{z}(p)\,, \quad X=(\partial_{v}^{z})_{p}\in T_{p}(M)\,.$$

With the aid of the above formula, one can extend $\mu_{0,1}$ to a unique hermitian pseudo-metric g on M such that $g(X, \bar{X})=\mu_{0,1}(p\,;X)$, $X\in T_{p}(M)$. This pseudo-metric is given by

$$g|_{U_{z}}=2\textstyle\sum_{a,b}\partial_{a}^{z}\partial_{\bar{b}}^{\bar{z}}.\log k_{z}dz^{a}\cdot d\bar{z}^{b}\,,$$

and is called the *Bergman pseudo-metric* on M. We note that the Bergman pseudo-metric g becomes an ordinary metric if and only if M satisfies condition (C_{1}) (see $(3.7)_{1}$), i.e. M satisfies condition (A.2) of Kobayashi [10] (see Remark 3.3).

Assume now that M satisfies condition (C_{1}). It follows from Theorem 3.12 that

$$\mu_{0,2}(p\,;(\partial_{v}^{z})_{p})=k_{z}(p)^{-1}P_{z}(p)^{-1}Q_{z}(p, v)\,, \tag{4.3}$$

where

$$P_z := \det L_z(1, \cdots, \varphi(1))$$

and

$$Q_z(\cdot, v) := \sum_{\varphi(1)<s, t\leq\varphi(2)} C_{\Phi(s)} C_{\Phi(t)} v^{\Phi(s)} \bar{v}^{\Phi(t)} L_z(1, \cdots, \varphi(1))_{s,t}.$$

The following theorem was stated in Fuks [8; p. 525]. For the sake of completeness we give another proof which may have its own interest.

THEOREM 4.4. *Suppose M satisfies conditions (C_0) and (C_1). Let* Sec$(p;\cdot)$ *be the curvature quartic form, at $p\in M$, of the Bergman metric g on M (see Definition* 1.1). *Then,*

$$\mu_{0,2}(p; X) = 2g(X, \bar{X})^2 - \mathrm{Sec}(p; X), \quad X\in T_p(M).$$

Proof. Set $g_{z,a\bar{b}} := \partial_a^z \partial_{\bar{b}}^{\bar{z}} \log k_z$, $G_z := (g_{z,a\bar{b}})$, $(g_z^{\bar{b}a}) := G_z^{-1}$. We compute $\mu_{0,2}(p; (\partial_v^z)_p)$ with the aid of formula (4.3). We first note that

$$P_z = k_z^{m+1} \det G_z,$$

$$Q_z(\cdot, v) = k_z^{m+1} \det \begin{bmatrix} G_z & x_{z,v}^* \\ x_{z,v} & \sigma_{z,v} \end{bmatrix},$$

where $x_{z,v}$ and $\sigma_{z,v}$ are $\boldsymbol{C}^m$-valued and $\boldsymbol{C}$-valued functions on U_z, respectively, given by

$$x_{z,v} := (\partial_{\bar{b}}^{\bar{z}} ((\partial_v^z)^2 . k_z / k_z))_b,$$

$$\sigma_{z,v} := (k_z (\partial_{\bar{v}}^{\bar{z}})^2 (\partial_v^z)^2 . k_z - |(\partial_v^z)^2 . k_z|^2) / k_z^2.$$

It follows that

$$\mu_{0,2}(p; (\partial_v^z)_p) = \sigma_{z,v}(p) - x_{z,v}(p) G_z(p)^{-1} x_{z,v}(p)^*.$$

The desired formula is now obtained from Definition 1.1 (see also [10; p. 275]), and the proof is complete.

COROLLARY 4.5. (Fuks [8; Theorem 1], Kobayashi [10; Theorem 4.4]). *Suppose M satisfies conditions (C_0) and (C_1). Then the holomorphic sectional curvature of the Bergman metric on M is at most* 2. *Let $p\in M$ be fixed. The holomorphic sectional curvature is less than* 2 *for every direction at p if condition $(C_2)_p$ holds.*

Remark 4.6. Concerning the last corollary, the following facts are shown in [2] by means of examples:

(i) There exists a simply connected domain M in $\boldsymbol{C}^2$ such that conditions (C_0) and (C_1) hold true, and such that the holomorphic sectional curvature of the Bergman metric on M is identically 2.

(ii) For every real number ξ with $\xi<2$, there exists a pseudo-convex bounded Reinhardt domain M in $\boldsymbol{C}^2$ such that the holomorphic sectional curvature of the Bergman metric on M is greater than ξ for some direction.

§5. Hessian quartic form of the Bergman metric. We first recall the n-th order Bergman metric introduced in [6]. Let a coordinate z in M be fixed. For $n\in \boldsymbol{Z}_+$ and $(p, v)\in U_z\times \boldsymbol{C}^m$, we set

$$H_n^z(p, v) := \{\alpha\in H(M)\,;\ (\partial_v^z)^j.\,\alpha(p)=0 \quad (j=1, \cdots, n-1)\}$$

and

$$\lambda_n^z(p\,;(\partial_v^z)_p) := \max\{(\partial_v^z)^n.\,\alpha(p)\wedge\overline{(\partial_v^z)^n.\,\alpha(p)}\,;\ \alpha\in H_n^z(p, v),\ \|\alpha\|=1\}$$

(see Definition 3.6). Referring to Definition 3.2, we have

$$H_n^z(p, v)\begin{cases} =H_n(p)\,, & n=0, 1 \\ \supset H_n(p)\,, & n\geqq 2\,. \end{cases} \tag{5.1}$$

In particular,

$$\begin{cases} \lambda_0^z(p\,;\cdot)=\mu_0(p\,;\cdot)=k_z(p)(dz\wedge d\bar{z})_p \\ \lambda_1^z(p\,;\cdot)=\mu_1(p\,;\cdot) \end{cases} \tag{5.2}$$

on $T_p(M)$. When M satisfies condition (C_0), we may consider the $[0, +\infty)$-valued function $\lambda_{0,n}^z$ on $\bigcup_{p\in U_z}T_p(M)$ for every $n\in \boldsymbol{N}$, given by $\lambda_{0,n}^z=\lambda_n^z/\lambda_0^z$. The function $\lambda_{0,n}^z$ is called in [6] the *n-th order Bergman metric* of M. It follows from (5.1) and (5.2) that

$$\lambda_{0,1}^z=\mu_{0,1}\,,\quad \lambda_{0,n}^z\geqq\mu_{0,n}\quad (n\geqq 2)\,. \tag{5.3}$$

Given a vector $v\in \boldsymbol{C}^m$, consider the functions R_n $(n=-1, 0, 1, \cdots)$ on U_z given by

$$R_n := \det[(\partial_v^z)^i\overline{(\partial_v^z)}^j,\ k_z]_{j=0,\cdots,n}^{i=0,\cdots,n}\,, \tag{5.4}$$

the Wronskian of functions $\overline{(\partial_v^z)}^j.\,k_z$ $(j=0, 1, \cdots, n)$ with respect to ∂_v^z (especially, $R_{-1}=1$).

We now recall the Jacobi's formula concerning determinants.

LEMMA 5.1. *Let $A=(\xi_{ij})\in M(n, \boldsymbol{C})$, and let A_{ij} be its (i, j)-cofactor. Then*

$$\det A\,\det(\xi_{ij})_{j=1,\cdots,n-2}^{i=1,\cdots,n-2}=A_{nn}A_{n-1,n-1}-A_{n,n-1}A_{n-1,n}\,.$$

This lemma leads to the following recursive formula for the Wronskians R_n in (5.4).

LEMMA 5.2. *Let z be a coordinate in M, and let $v\in \boldsymbol{C}^m$. Then, for every $n\in \boldsymbol{N}$,*

$$R_nR_{n-2}=R_{n-1}\partial_v^z\overline{\partial_v^z}.\,R_{n-1}-|\partial_v^z.\,R_{n-1}|^2$$

on U_z.

Proof. Let $(R_n)_{ij}$ be the (i, j)-cofactor of the $H(n+1, \boldsymbol{C})$-valued function

$[(\partial_v^z)^i(\overline{\partial_v^z})^j. k_z]_{j=0,\cdots,n}^{i=0,\cdots,n}$. It follows from Lemma 5.1, since R_n is hermitian, that

$$R_n R_{n-2}=(R_n)_{nn}(R_n)_{n+1,n+1}-|(R_n)_{n,n+1}|^2.$$

Moreover, from the derivation properties of the Wronskians we also have $(R_n)_{nn}=R_{n-1}$, $(R_n)_{n,n+1}=-\partial_v^z. R_{n-1}$, and $(R_n)_{n+1,n+1}=\partial_v^z\overline{\partial_v^z}. R_{n-1}$. The proof is now complete.

From Lemma 3.8 together with (5.2) it follows that

$$\lambda_{0,n}^z(p\,;(\partial_v^z)_p)=k_z(p)^{-1}R_{n-1}(p)^{-1}R_n(p), \tag{5.5}$$

provided that $R_{n-1}(p)\neq 0$ (cf. [6; p. 51]).

THEOREM 5.3. *Assume, in addition to the assumptions of Lemma* 5.2, *that M satisfies condition* (C_j) $(j=0, \cdots, n-1)$. *Set*

$$\lambda_{0,j}(p):=\lambda_{0,j}^z(p\,;(\partial_v^z)_p), \quad p\in U_z \quad (j=1, \cdots, n).$$

Then

$$\lambda_{0,n}=\lambda_{0,n-1}(n\lambda_{0,1}+\textstyle\sum_{j=1}^{n-1}\partial_v^z\overline{\partial_v^z}. \log \lambda_{0,j})$$

on U_z, where $\lambda_{0,0}=1$.

Proof. By assumption and Lemma 5.2 we have

$$R_n R_{n-2}=(R_{n-1})^2\partial_v^z\overline{\partial_v^z}. \log R_{n-1}.$$

It follows from (5.5) that

$$\lambda_{0,n}=\lambda_{0,n-1}\partial_v^z\overline{\partial_v^z}. \log R_{n-1}$$

and that

$$R_{n-1}=(k_z)^n\lambda_{0,1}\cdots\lambda_{0,n-1}.$$

The desired result now follows by observing that $\lambda_{0,1}=\partial_v^z\overline{\partial_v^z}. \log k_z$.

As a consequence of this theorem we find an intimate relationship between the quantity $\lambda_{0,2}^z$ and the Hessian quartic form of the Bergman metric.

COROLLARY 5.4. *Suppose that M satisfies conditions* (C_0) *and* (C_1). *Let z be a coordinate in M, and let* $\mathrm{Hess}^z(\cdot\,;\cdot)$ *be the Hessian quartic form of the Bergman metric g on M, relative to z (see Definition* 1.3). *Then, for* $(p, v)\in U_z\times \boldsymbol{C}^m$,

$$\lambda_{0,2}^z(p\,;(\partial_v^z)_p)=2g((\partial_v^z)_p, (\overline{\partial_v^z})_p)^2-\mathrm{Hess}^z(p\,;(\partial_v^z)_p).$$

Combining Theorem 4.3 with Corollary 5.4, we obtain, for $(p, v)\in U_z\times \boldsymbol{C}^m$,

$$\mathrm{Sec}(p\,;(\partial_v^z)_p)-\mathrm{Hess}^z(p\,;(\partial_v^z)_p)=\lambda_{0,2}^z(p\,;(\partial_v^z)_p)-\mu_{0,2}(p\,;(\partial_v^z)_p)\geqq 0. \tag{5.6}$$

The latter inequality follows from Proposition 1.5 or (5.3).

PROPOSITION 5.5. *Suppose that M satisfies conditions* (C_0) *and* (C_1). *Let z be a coordinate in M and let* $\mathrm{Sec}(\cdot\,;\cdot)$ *(resp.* $\mathrm{Hess}^z(\cdot\,;\cdot)$*) be the curvature quartic* *form (resp. Hessian quartic form relative to z) of the Bergman metric g on M. Let* $(p, v)\in U_z\times C^m$ *be fixed. Then, the left hand side of* (5.6) *vanishes if and only if*

$$W_v^z(k_z, \bar{\partial}_a^z.\, k_z, \bar{\partial}_b^z.\, k_z)(p)=0 \quad (a, b\in\{1, \cdots, m\}), \tag{5.7}$$

where $W_v^z(f_0, \cdots, f_n)$ *is the Wronskian of functions* $f_0, \cdots, f_n$ *on* U_z *with respect to* ∂_v^z. *Condition* (5.7) *is equivalent to*

$$\operatorname{rank}\begin{bmatrix}(k_z, \partial_v^z.\, k_z, (\partial_v^z)^2.\, k_z) \\ (\bar{\partial}_a^z.\, k_z, \bar{\partial}_a^z\partial_v^z.\, k_z, \bar{\partial}_a^z(\partial_v^z)^2.\, k_z)_{a=1,\cdots,m}\end{bmatrix}(p)\leqq 2\,. \tag{5.8}$$

Proof. We suppress the dependence on z. Set $g_{a\bar{b}}:=\partial_a\bar{\partial}_b.\log k$ and $G:=(g_{a\bar{b}})$. From Proposition 1.5 it follows that equality in (5.6) holds if and only if $v\partial_v.\, G(p)=\xi G(p)$ for some scalar $\xi\in C$. The latter is equivalent to

$$W_v(\bar{\partial}_a\partial_v.\log k, \bar{\partial}_b\partial_v.\log k)(p)=0 \quad (a, b\in\{1, \cdots, m\}). \tag{5.9}$$

But, using Lemma 5.1 with $n=3$ and standard properties of Wronskians, we arrive at the following identity:

$$W_v(k, \bar{\partial}_a.\, k, \bar{\partial}_b.\, k)=k^3W_v(\bar{\partial}_a\partial_v.\log k, \bar{\partial}_b\partial_v.\log k)\,.$$

It follows that condition (5.9) is equivalent to (5.7).

It remains to show the equivalence of conditions (5.7) and (5.8). Clearly, (5.8) implies (5.7). Assume now that (5.7) holds and $v\neq 0$. Consider the vectors $x:=(k, \partial_v.\, k, (\partial_v)^2.\, k)(p)$, $y:=\bar{\partial}_v.(k, \partial_v.\, k, (\partial_v)^2.\, k)(p)$, $y_a:=\bar{\partial}_a.(k, \partial_v.\, k, (\partial_v)^2.\, k)(p)$ $(a=1, \cdots, m)$ in C^3. Because of condition $(C_1)_p$ which guarantees that $W_v(k, \bar{\partial}_v.\, k)(p)\neq 0$, the set $\{x, y\}$ is linearly independent. It follows, since $y=\sum v^a y_a$, that there exists an $a_0\in\{1, \cdots, m\}$ such that $\{x, y_{a_0}\}$ is linearly independent. Therefore, (5.7) implies that every y_a is spanned by x and y_{a_0}, and hence condition (5.8) holds. The proof is now complete.

We note that condition (5.7) holds true trivially when $m=1$.

EXAMPLE 5.6. Suppose that $M=\{(\xi^1, \xi^2)\in C^2;\ |\xi^1|^2+|\xi^2|^{2/s}<1\}$ for some positive real number s, and that the coordinate z is the inclusion mapping of M into C^2. The Bergman function $k=k_z$ of M is given by

$$k(\xi^1, \xi^2)=c\frac{(1-|\xi^1|^2)^s-r|\xi^2|^2}{((1-|\xi^1|^2)^s-|\xi^2|^2)^3(1-|\xi^1|^2)^{2-s}}\,,$$

where $c:=(1+s)/\pi^2=\mathrm{vol}(M)^{-1}$ and

$$r=r(s):=(1-s)/(1+s) \quad (-1<r<1) \tag{5.10}$$

(cf. Bergman [4; p. 21]). Assume that the point p under consideration is $(0, \xi^2)$ with $|\xi^2|<1$. As in [3] (not Definition 3.10), we use the convenient variable

$$(5.11)\qquad t:=\frac{1-|\xi^2|^2}{1-r|\xi^2|^2}\quad(0<t\leqq 1),\quad\text{or}\quad |\xi^2|^2=\frac{1-t}{1-rt},$$

and the notation $k_a:=\partial_a^z.k$, $k_{a\bar{b}}:=\partial_a^z\partial_{\bar{b}}^{\bar{z}}.k$, etc. Then, we have

$$(5.12)\qquad \begin{cases} k_1/k=0, & k_2/k=x_1\bar{\xi}^2 \\ k_{11}/k=k_{12}/k=0, & k_{22}/k=x_2(\bar{\xi}^2)^2 \\ k_{1\bar{1}}/k=x_3, & k_{1\bar{2}}/k=0, \quad k_{2\bar{2}}/k=x_4 \\ k_{1\bar{1}\bar{1}}/k=0, & k_{1\bar{1}\bar{2}}/k=x_5\xi^2, \quad k_{2\bar{2}\bar{2}}/k=x_6\xi^2 \end{cases}$$

and their corresponding conjugated formulas, where

$$\begin{cases} x_1:=(1-rt)(3-rt)/(1-r)t \\ x_2:=6(1-rt)^2(2-rt)/(1-r)^2t^2 \\ x_3:=(3+rt^2)/(1+r)t \\ x_4:=(1-rt)(12-9(1+r)t+(5+r)rt^2)/(1-r)^2t^2 \\ x_5:=2(1-rt)(6-3rt+rt^2)/(1+r)(1-r)t^2 \\ x_6:=12(1-rt)^2(5-(3+5r)t+(2+r)rt^2)/(1-r)^3t^3. \end{cases}$$

Using (5.12), we find that condition (5.7) is equivalent to

$$(5.13)\qquad \begin{vmatrix} 1 & x_1\xi^2\bar{v}^2 & x_2(\xi^2)^2(\bar{v}^2)^2 \\ 0 & x_3\bar{v}^1 & 2x_5\xi^2\bar{v}^1\bar{v}^2 \\ x_1\bar{\xi}^2 & x_4\bar{v}^2 & x_6\xi^2(\bar{v}^2)^2 \end{vmatrix}=0.$$

If $v^1v^2\xi^2=0$, condition (5.13) holds true trivially. Suppose that $v^1v^2\xi^2\neq 0$. Then (5.13) is equivalent to

$$(5.14)\qquad \begin{vmatrix} |\xi^2|^{-2} & x_1 & x_2 \\ 0 & x_3 & 2x_5 \\ x_1 & x_4 & x_6 \end{vmatrix}=0.$$

Using the values of x_j together with (5.11), and noting that $1-rt>0$ and $t>0$, we find that (5.14) is equivalent to

$$(5.15)\qquad r\{9+9(1-r)t-18rt^2-(1-9r)rt^3+r^2t^4\}=0.$$

Making use of Sturm's method, we can see that the factor in the brace of (5.15) is positive for every $(r, t)\in(-1, 1]\times(0, 1]$ (for Sturm's method, cf., e.g., Isaacson and Keller [9; pp. 126-129]); therefore, (5.15) holds if and only if $r=0$, or by (5.10), if and only if $s=1$. Note that the domain M with $s=1$ is the unit ball in $\boldsymbol{C}^2$.

Summing up the above arguments, we obtain the following assertion.

PROPOSITION 5.7. *Suppose that* M *and* z *are as in Example* 5.6 *with* $s\neq 1$. *Let* Sec *and* Hess^z *be as in Proposition* 5.5, *and let* $X=(\partial_v^z)_p$ *with* $v=(v^1, v^2)\in \boldsymbol{C}^2$ *and* $p=(0, \xi^2)\in M$. *Then,* $\text{Sec}(p\,;X)-\text{Hess}^z(p\,;X)=\lambda_{0,2}^z(p\,;X)-\mu_{0,2}(p\,;X)$ *is positive if and only if* $v^1v^2\xi^2\neq 0$.

It was shown in [6] (see also [5]) that the quantity $\lambda_{0,n}^z$ possesses a certain biholomorphic invariance. This invariance, however, is not an invariance in the ordinary sense and it does not guarantee that for $n\geqq 2$, $\lambda_{0,n}^z$ can be regarded as a global function on the tangent bundle $T(M)$ of M. In fact, as the following corollary of Proposition 5.7 shows, $\lambda_{0,2}^z$ does depend, in general, on the coordinate z.

COROLLARY 5.8. *Let* M, z, Hess^z *be as in Proposition* 5.5 *with* $m=\dim M\geqq 2$. *The quantities* $\lambda_{0,2}^z$ *and* Hess^z, *in general, depend on* z, *i.e. they cannot be considered as global functions on the tangent bundle* $T(M)$.

Proof. It is sufficient to find a manifold M that satisfies (C_0) and (C_1), and in which there exist two coordinates z and w with $U_z\cap U_w\neq\phi$ such that $\lambda_{0,2}^z(p\,;X)\neq\lambda_{0,2}^w(p\,;X)$ for some $p\in U_z\cap U_w$ and $X=(\partial_v^z)_p=(\partial_{v'}^w)_p\in T_p(M)$.

For this, we take as M the domain considered in Example 5.6, and as z the inclusion mapping of M into $\boldsymbol{C}^2$. We also take $p=(0, \xi^2)\in M$ and $v=(v^1, v^2)\in \boldsymbol{C}^2$ so that $v^1v^2\xi^2\neq 0$. Lemma 1.6 guarantees the existence of a coordinate w around p, for which $\text{Hess}^w(p\,;(\partial_{v'}^w)_p)=\text{Sec}(p\,;(\partial_{v'}^w)_p)$ with $(\partial_{v'}^w)_p=(\partial_v^z)_p$. Then, by (5.6) and Proposition 5.7 we have

$$\text{Hess}^z(p\,;(\partial_v^z)_p)<\text{Hess}^w(p\,;(\partial_{v'}^w)_p)\,,$$

$$\lambda_{0,2}^z(p\,;(\partial_v^z)_p)>\lambda_{0,2}^w(p\,;(\partial_{v'}^w)_p)\,,$$

as desired.

REFERENCES

[1] N. ARONSZAJN, Theory of reproducing kernels, Trans. Amer. Math. Soc., 68 (1950), 337-404.

[2] K. AZUKAWA, Square-integrable holomorphic functions on a circular domain in $\boldsymbol{C}^n$ (to appear).

[3] K. AZUKAWA AND M. SUZUKI, The Bergman metric on a Thullen domain, Nagoya Math. J., 89 (1983), 1-11.

[4] S. Bergman, *Sur les Fonctions Orthogonales de Plusieurs Variables Complexes avec les Applications à la Théorie des Fonctions Analytiques*, Mémor Sci. Math., 106, Gauthier-Villars, Paris, 1947.

[5] J. BURBEA, The Carathéodory metric and its majorant metrics, Canad. J. Math., 29 (1977), 771-780.

[6] J. BURBEA, Inequalities between intrinsic metrics, Proc. Amer Math. Soc., 67 (1977), 50-54.

[7] J. BURBEA, On the Hessian of the Carathéodory metric, Rocky Mountain J.

Math., **8** (1978), 555-559.

[8] B. A. FUKS, Ricci curvature of a Bergman metric invariant under biholomorphic mappings, Soviet Math. Dokl., 7 (1966), 525-529.

[9] E. ISAACSON AND H. B. KELLER, *Analysis on Numerical Methods*, John Wiley, New York, 1966.

[10] S. KOBAYASHI, Geometry of bounded domains, Trans. Amer. Math. Soc., **92** (1952), 267-290.

[11] S. KOBAYASHI, *Hyperbolic Manifolds and Holomorphic Mappings*, Marcel Dekker, New York, 1970.

[12] S. KOBAYASHI AND K. NOMIZU, *Foundations of Differential Geometry* (vol. 2), Interscience, New York, 1969.

[13] A. LICHNEROWICZ, Variétés complexes et tenseur de Bergmann, Ann. Inst. Fourier (Grenoble), **15** (1965), 345-408.

[14] H. WU, A remark on holomorphic sectional curvature, Indiana Univ. Math. J., 22 (1973), 1103-1108.

DEPARTMENT OF MATHEMATICS
TOYAMA UNIVERSITY
TOYAMA, 930
JAPAN

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PITTSBURGH
PITTSBURGH, PENNSYLVANIA 15260
U. S. A.