ON COMPACT RIEMANNIAN MANIFOLDS ADMITTING A CERTAIN FUNCTION

BY KAZUHIRO MATSUSHITA

1. Introduction.

Let (M, g) be an n -dimensional complete Riemannian manifold. Then it is known that the existence of a function satisfying some differential equations on (M, g) sometimes determines differential geometric properties of (M, g) . For example, if (M, g) admits a non-trivial solution f of the system of differential equations

$$(1.1) \quad \nabla(df) + kfg = 0,$$

for some positive constant k , then it is isometric to an n -dimensional Euclidean sphere $S^n(k)$ of radius $1/\sqrt{k}$ (S. Ishihara and Y. Tashiro [5], M. Obata [7], [8]). In this paper, concerning with this result, we shall consider the problem: “Perturbing the above equality (1.1), can we restrict the topological structure of (M, g) ?”, and prove the following

THEOREM A. *Let (M, g) be an n -dimensional compact connected Riemannian manifold with the diameter $d(M)$ satisfying the inequality $\pi/2 < d(M) < 3\pi/2$. Assume that (M, g) admits a non-constant function f satisfying following condition*

$$(*) \quad |\nabla(df)(X, X) + fg(X, X)| < \varepsilon(d(M), f(P_+), f(P_-)) \|X\|^2,$$

for any $X (\neq 0) \in M_x (x \in M)$, where

$$0 < \varepsilon(d(M), f(P_+), f(P_-)) \\ = \min \left\{ \frac{f(P_+)}{4}, -\frac{f(P_-)}{4}, -\frac{f(P_+)\cos d(M)}{\cos d(M)+3}, \frac{f(P_-)\cos d(M)}{\cos d(M)+3} \right\},$$

and $f(P_+)$ ($f(P_-)$, resp.) $= \max_{p \in M} \{f(p)\}$ ($\min_{p \in M} \{f(p)\}$, resp.). Then M is homeomorphic to an n -dimensional sphere.

PROPOSITION B. *Let (M, g) be an n -dimensional compact connected Riemannian manifold and $H_1(M, g)$ the set of all the first eigen functions of the Laplace operator Δ of (M, g) . Assume that there exists a function $f \in H_1(M, g)$ satisfying*

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the following condition

$$(**) \quad |\nabla(df)(X, X) + fg(X, X)| < \varepsilon(f(P_+), f(P_-))\|X\|^2$$

for all $X (\neq 0) \in M_x$ ($x \in M$), where

$$0 < \varepsilon(f(P_+), f(P_-)) < \min \left\{ \frac{f(P_+)}{4}, -\frac{f(P_-)}{4} \right\}.$$

Then M is homeomorphic an n -dimensional sphere.

The above problem has also been discussed by M. A. Pinsky [8] and S. Gallot [4]. The author wishes to express his deep gratitude to Prof. K. Sekigawa for useful comments and kind encouragement.

2. Some estimations.

In order to show our Theorem A in the next section, we have to prove several lemmas on the estimation. The proofs of the lemmas are essentially the same as those of M. A. Pinsky [8]. Let (M, g) be an n -dimensional complete Riemannian manifold and assume that it admits a function f satisfying (*). Let γ be a normal geodesic. We put

$$(2.1) \quad \delta_\gamma(t) = \frac{d^2(f \circ \gamma)}{dt^2}(t) + (f \circ \gamma)'(t). \quad \text{along } \gamma.$$

Then by the condition (*) we get

$$(2.2) \quad |\delta_\gamma(t)| < \varepsilon, \quad \text{along } \gamma.$$

From (2.1), we get easily

$$(2.3) \quad (f \circ \gamma)(t) = (f \circ \gamma)(0) \cos t + (f \circ \gamma)'(0) \sin t + \int_0^t \delta_\gamma(u) \sin(t-u) du.$$

In particular, if $\gamma(0)$ is a critical point of f , (2.3) reduces to

$$(2.4) \quad (f \circ \gamma)(t) = (f \circ \gamma)(0) \cos t + \int_0^t \delta_\gamma(u) \sin(t-u) du.$$

Then taking account of (2.2) and (2.4), we get easily

$$(2.5) \quad \{f(P_+) + \varepsilon\} \cos t - \varepsilon \leq (f \circ \gamma)(t) \leq \{f(P_+) - \varepsilon\} \cos t + \varepsilon, \quad \text{for } 0 \leq t \leq \pi,$$

$$(2.5)' \quad \{f(P_-) + \varepsilon\} \cos t - \varepsilon \leq (f \circ \gamma)(t) \leq \{f(P_-) - \varepsilon\} \cos t + \varepsilon, \quad \text{for } 0 \leq t \leq \pi,$$

$$(2.6) \quad \{f(P_+) - \varepsilon\} \cos t - 3\varepsilon < (f \circ \gamma)(t) < \{f(P_+) + \varepsilon\} \cos t + 3\varepsilon, \quad \text{for } \pi \leq t \leq 2\pi,$$

$$(2.6)' \quad \{f(P_-) - \varepsilon\} \cos t - 3\varepsilon < (f \circ \gamma)(t) < \{f(P_-) + \varepsilon\} \cos t + 3\varepsilon, \quad \text{for } \pi \leq t \leq 2\pi.$$

From (2.4), we get

$$(2.7) \quad (f \circ \gamma)'(t) = -f(P_+) \sin t + \int_0^t \delta_{\gamma}(u) \cos(t-u) du,$$

$$(2.7)' \quad (f \circ \gamma)'(t) = -f(P_-) \sin t + \int_0^t \delta_{\gamma}(u) \cos(t-u) du.$$

Taking account of (2.2), (2.7) and (2.7)', we get

$$(2.8) \quad -\{f(P_+) + \varepsilon\} \sin t \leq (f \circ \gamma)'(t) \leq \{-f(P_+) + \varepsilon\} \sin t, \quad \text{for } 0 \leq t \leq \pi/2,$$

$$(2.8)' \quad -\{f(P_-) + \varepsilon\} \sin t \leq (f \circ \gamma)'(t) \leq \{-f(P_-) + \varepsilon\} \sin t, \quad \text{for } 0 \leq t \leq \pi/2,$$

$$(2.9) \quad \{-f(P_+) + \varepsilon\} \sin t - 2\varepsilon < (f \circ \gamma)'(t) < \{f(P_+) + \varepsilon\} \sin t + 2\varepsilon, \\ \text{for } \pi/2 \leq t \leq 3\pi/2,$$

$$(2.9)' \quad \{-f(P_-) + \varepsilon\} \sin t - 2\varepsilon < (f \circ \gamma)'(t) < \{-f(P_-) + \varepsilon\} \sin t + 2\varepsilon, \\ \text{for } \pi/2 \leq t \leq 3\pi/2.$$

We put

$$t_0(\gamma) = \min \{t > 0; (f \circ \gamma)'(t) = -(f \circ \gamma)'(0)\},$$

$$\bar{t}_0(\gamma) = \min \{t > 0; (f \circ \gamma)'(t) = (f \circ \gamma)'(0)\},$$

$$t'_0(\gamma) = \min \{t > 0; (f \circ \gamma)'(t) = 0\}.$$

Then we get the following

LEMMA 2.1. *Let γ be any normal geodesic starting from P_+ (P_- , resp.). Then we have*

$$(2.10) \quad \bar{t}_0(\gamma) > \pi, \quad t_0(\gamma) > \pi - \cos^{-1} \frac{f(P_+) - \varepsilon}{f(P_+) + \varepsilon},$$

$$\text{and} \quad t'_0(\gamma) > \pi - \sin^{-1} \frac{2\varepsilon}{f(P_+) + \varepsilon} > \pi - \cos^{-1} \frac{f(P_+) - \varepsilon}{f(P_+) + \varepsilon},$$

$$\left((2.10)' \right) \quad \bar{t}_0(\gamma) > \pi, \quad t_0(\gamma) > \pi - \cos^{-1} \frac{f(P_-) + \varepsilon}{f(P_-) - \varepsilon},$$

$$\text{and} \quad t'_0(\gamma) > \pi - \sin^{-1} \frac{2\varepsilon}{-f(P_-) + \varepsilon} > \pi - \cos^{-1} \frac{f(P_-) + \varepsilon}{f(P_-) - \varepsilon}, \text{ resp. } \Big).$$

Proof. First, we show the first inequality of (2.10). By (2.5), we get

$$(2.11) \quad \cos t - \frac{\varepsilon}{f(P_+)} (1 - \cos t) \leq \frac{(f \circ \gamma)'(t)}{f(P_+)} \leq \cos t + \frac{\varepsilon}{f(P_+)} (1 - \cos t) < 1, \\ \text{for } 0 < t < \pi.$$

Then, we see that $\bar{t}_0(\gamma) > \pi$. Next, we show the second inequality of (2.10). We have only to consider the case $0 < t_0(\gamma) < \pi$. From (2.11), we get

Fig. II

From (2.5), (2.6), (2.8)~(2.10), we have the figures I, II. Moreover, from (2.5)', (2.6)', (2.8)'~(2.10)', we have also the similar figures. However, we omit to write them.

LEMMA 2.2. *Let γ be any normal geodesic starting from P_+ (P_- , resp.), and $\gamma(t_c)$ the conjugate point of $P_+=\gamma(0)$ ($P_-=\gamma(0)$, resp.) along γ . Then we have*

$$(2.14) \quad t_c > \pi - \sin^{-1} \frac{2\varepsilon}{f(P_+) + \varepsilon},$$

$$\left((2.14)' \quad t_c > \pi - \sin^{-1} \frac{2\varepsilon}{-f(P_-) + \varepsilon}, \text{ resp.} \right).$$

Proof. Let Y be a Jacobi field along γ such that $Y(0)=0$, $\langle Y, \dot{\gamma} \rangle=0$ and $Y(t_c)=0$. First, we assume that $\|Y(t_0)\|=1$ for some t_0 ($0 < t_0 < t_c$). Let $\sigma_{t_0}(s)$ be the normal geodesic starting from $\gamma(t_0)$ such that $\delta_{t_0}(0)=Y(t_0)$, and let $\gamma_s(t)$ be the normal geodesic from P_+ to $\sigma_{t_0}(s)$ such that $\alpha(t, s)=\gamma_s(t)$ is a smooth variation of γ . Then we may easily see that the associated variational field $Z(t)=d\alpha\left(\frac{\partial}{\partial s}\Big|_{(t,0)}\right)$ along γ coincides with the given Jacobi field Y . By L_s we denote the length of the geodesic γ_s . First of all, we get

$$(2.15) \quad \frac{dL_s}{ds}\Big|_{s=0}=0.$$

In view of the second variational formula, we get

$$(2.16) \quad \left. \frac{d^2 L_s}{ds^2} \right|_{s=0} = \langle Y'(t_0), Y(t_0) \rangle,$$

where $Y' = \nabla_{\dot{\gamma}} Y$. From (2.4), we get

$$(2.17) \quad f(\sigma_{t_0}(s)) = f(\gamma_s(L_s)) = (f \circ \gamma)(0) \cos L_s + \int_0^{L_s} \delta_{\gamma_s}(u) \sin(L_s - u) du.$$

From (2.17), taking account of (2.7) and (2.15), we get

$$(2.18) \quad \begin{aligned} \left. \frac{d^2 L_s}{ds^2} \right|_{s=0} (f \circ \gamma)'(t_0) + \int_0^{t_0} \left(\left. \frac{d^2 \delta_{\gamma_s}(u)}{ds^2} \right|_{s=0} \right) \sin(t_0 - u) du \\ = \delta_{\sigma_{t_0}}(0) - \delta_{\dot{\gamma}}(t_0) + (f \circ \gamma)''(t_0). \end{aligned}$$

We now assume that $(f \circ \gamma)'(t_0) \neq 0$. Then, from (2.18), we get

$$(2.19) \quad \left. \frac{d^2 L_s}{ds^2} \right|_{s=0} = \frac{F(t_0)}{(f \circ \gamma)'(t_0)},$$

where

$$F(t) = (f \circ \gamma)''(t) - \delta_{\dot{\gamma}}(t) + \delta_{\sigma_t}(0) - \int_0^t \left(\left. \frac{d^2 \delta_{\gamma_s}(u)}{ds^2} \right|_{s=0} \right) \sin(t - u) du.$$

Next, we assume that $\|Y(t_0)\| \neq 1$. We put $\tilde{Y}(t) = \frac{1}{\|Y(t_0)\|} Y(t)$. Then we get

$$(2.20) \quad \langle \tilde{Y}'(t_0), \tilde{Y}(t_0) \rangle = \frac{\langle Y'(t_0), Y(t_0) \rangle}{\langle Y(t_0), Y(t_0) \rangle}.$$

Therefore, applying the formulas (2.16), (2.19) and (2.20), we have

$$(2.21) \quad \frac{\langle Y'(t), Y(t) \rangle}{\langle Y(t), Y(t) \rangle} = \frac{F(t)}{(f \circ \gamma)'(t)}.$$

The rest of the proof is similar to the one of Lemma 5 in [8]. Similarly, we may also show (2.14)'. Q.E.D.

By $C(P_+)$ ($C(P_-)$, resp.) we denote the cut locus of P_+ (P_- , resp.). Then we have the following

LEMMA 2.3. *Let $d(P_+, C(P_+))$ ($d(P_-, C(P_-))$, resp.) be the distance from P_+ (P_- , resp.) to the cut locus $C(P_+)$ ($C(P_-)$, resp.). Then*

$$(2.22) \quad d(P_+, C(P_+)) > \pi - \sin^{-1} \frac{2\varepsilon}{f(P_+) + \varepsilon},$$

$$((2.22)') \quad d(P_-, C(P_-)) > \pi - \sin^{-1} \frac{2\varepsilon}{-f(P_-) + \varepsilon}, \text{ resp.}.$$

The proof of this Lemma is similar to the one of Lemma 6 in [8]. So, we omit it. Furthermore, we have the following

LEMMA 2.4. Let $d(M)$ be the diameter of (M, g) . Then we have

$$(2.23) \quad \pi - \sin^{-1} \frac{2\varepsilon}{f(P_+) + \varepsilon} < d(P_+, P_-) \leq d(M) < 2\pi - \cos^{-1} \frac{-3\varepsilon}{f(P_+) + \varepsilon}$$

and

$$(2.24) \quad \pi - \sin^{-1} \frac{2\varepsilon}{-f(P_-) + \varepsilon} < d(P_+, P_-) \leq d(M) < 2\pi - \cos^{-1} \frac{3\varepsilon}{f(P_-) - \varepsilon}.$$

Proof. The first inequality of (2.23) follows immediately from (2.10), and the second one of (2.23) is trivial. If $d(M) \leq \pi$, the third inequality is trivial because of $\pi \leq 2\pi - \cos^{-1} \frac{-3\varepsilon}{f(P_+) + \varepsilon}$ (cf. Fig. I). So we assume that $d(M) > \pi$. From the condition (*) in Theorem A, it follows that

$$\varepsilon < \frac{-f(P_+) \cos d(M)}{\cos d(M) + 3},$$

and hence

$$-\frac{3\varepsilon}{f(P_+) + \varepsilon} > \cos d(M) = \cos(2\pi - d(M)).$$

Thus, we have finally $\cos^{-1} \frac{-3\varepsilon}{f(P_+) + \varepsilon} < 2\pi - d(M)$. Similarly, we may also show
(2.24). Q. E. D.

LEMMA 2.5. We have the following inequalities:

$$(2.25) \quad \cos^{-1} \frac{-\varepsilon}{f(P_+) - \varepsilon} < \pi - \cos^{-1} \frac{f(P_+) - \varepsilon}{f(P_+) + \varepsilon} \left(< \pi - \sin^{-1} \frac{2\varepsilon}{f(P_+) - \varepsilon} \right),$$

$$\cos^{-1} \frac{-\varepsilon}{f(P_+) - \varepsilon} < \pi - \sin^{-1} \frac{2\varepsilon}{-f(P_-) + \varepsilon},$$

$$(2.26) \quad \cos^{-1} \frac{\varepsilon}{f(P_-) + \varepsilon} < \pi - \cos^{-1} \frac{f(P_-) + \varepsilon}{f(P_-) - \varepsilon} \left(< \pi - \sin^{-1} \frac{2\varepsilon}{-f(P_-) + \varepsilon} \right),$$

$$\cos^{-1} \frac{\varepsilon}{f(P_-) + \varepsilon} < \pi - \sin^{-1} \frac{2\varepsilon}{f(P_+) + \varepsilon}.$$

Proof. First, we show the first inequality of (2.25). Let us define a function $g(t)$ by

$$g(t) = \{f(P_+) - \varepsilon\} \cos t + \varepsilon \quad (0 \leq t \leq \pi),$$

which is easily found to be a monotonously decreasing function satisfying

$$g\left(\cos^{-1} \frac{-\varepsilon}{f(P_+) - \varepsilon}\right) = 0, \quad g(\pi) = -f(P_+) + 2\varepsilon < 0.$$

By the definition of $g(t)$ and the condition (*), we get immediately

$$\begin{aligned}
g\left(\pi - \cos^{-1} \frac{f(P_+) - \varepsilon}{f(P_+) + \varepsilon}\right) &= \{f(P_+) - \varepsilon\} \cos\left(\pi - \cos^{-1} \frac{f(P_+) - \varepsilon}{f(P_+) + \varepsilon}\right) + \varepsilon \\
&= -\frac{\{f(P_+) - \varepsilon\}^2}{\{f(P_+) + \varepsilon\}} + \varepsilon > 0,
\end{aligned}$$

and hence

$$(2.27) \quad \cos^{-1} \frac{-\varepsilon}{f(P_+) - \varepsilon} < \pi - \cos^{-1} \frac{f(P_+) - \varepsilon}{f(P_+) + \varepsilon}.$$

Thus, we have the first inequality of (2.25). Next, we show the second inequality of (2.25). Taking account of Fig. I, we get

$$(2.28) \quad f(P_+) + 2\varepsilon > -f(P_-) > f(P_+) - 2\varepsilon.$$

From (2.28), we get

$$(2.29) \quad \pi - \sin^{-1} \frac{2\varepsilon}{f(P_+) + 3\varepsilon} > \pi - \sin^{-1} \frac{2\varepsilon}{-f(P_-) + \varepsilon} > \pi - \sin^{-1} \frac{2\varepsilon}{f(P_+) - \varepsilon}.$$

On the other hand, by the condition (*), we get

$$(2.30) \quad \frac{\sqrt{\{f(P_+) - 3\varepsilon\} \{f(P_+) + \varepsilon\}}}{f(P_+) - \varepsilon} > \frac{\varepsilon}{f(P_+) - \varepsilon} \quad (> 0).$$

From (2.30), we get

$$\begin{aligned}
&\cos\left(\pi - \cos^{-1} \frac{\sqrt{\{f(P_+) - 3\varepsilon\} \{f(P_+) + \varepsilon\}}}{f(P_+) - \varepsilon}\right) \\
&= -\frac{\sqrt{\{f(P_+) - 3\varepsilon\} \{f(P_+) + \varepsilon\}}}{f(P_+) - \varepsilon} < -\frac{\varepsilon}{f(P_+) - \varepsilon} \\
&= \cos\left(\cos^{-1} \frac{-\varepsilon}{f(P_+) - \varepsilon}\right),
\end{aligned}$$

and hence

$$(2.31) \quad \pi - \cos^{-1} \left(\frac{\sqrt{\{f(P_+) - 3\varepsilon\} \{f(P_+) + \varepsilon\}}}{f(P_+) - \varepsilon} \right) > \cos^{-1} \frac{-\varepsilon}{f(P_+) - \varepsilon}.$$

Finally the formula (2.31) together with

$$1 - \left(\frac{2\varepsilon}{f(P_+) - \varepsilon} \right)^2 = \left\{ \frac{\sqrt{\{f(P_+) - 3\varepsilon\} \{f(P_+) + \varepsilon\}}}{f(P_+) - \varepsilon} \right\}^2,$$

implies

$$\pi - \sin^{-1} \frac{2\varepsilon}{f(P_+) - \varepsilon} = \pi - \cos^{-1} \frac{\sqrt{\{f(P_+) - 3\varepsilon\} \{f(P_+) + \varepsilon\}}}{f(P_+) - \varepsilon} > \cos^{-1} \frac{-\varepsilon}{f(P_+) - \varepsilon}.$$

Similarly, we may also show (2.26).

Q. E. D.

3. Proof of Theorem A.

In this section, we give a proof of Theorem A based on the estimations obtained in § 2.

LEMMA 3.1. *There is a positive number r such that $M = B_r(P_+) \cup B_r(P_-)$, and*

$$\begin{aligned}
 (3.1) \quad \cos^{-1} \frac{-\varepsilon}{f(P_+) - \varepsilon} &\leq r \\
 &\leq \min \left\{ \pi - \sin^{-1} \frac{2\varepsilon}{f(P_+) + \varepsilon}, \pi - \sin^{-1} \frac{2\varepsilon}{-f(P_-) + \varepsilon} \right\}, \\
 \cos^{-1} \frac{\varepsilon}{f(P_-) + \varepsilon} &\leq r \\
 &\leq \min \left\{ \pi - \sin^{-1} \frac{2\varepsilon}{-f(P_-) + \varepsilon}, \pi - \sin^{-1} \frac{2\varepsilon}{f(P_+) + \varepsilon} \right\},
 \end{aligned}$$

where $B_r(P_+)$ ($B_r(P_-)$, resp.) denotes the open ball of the center P_+ (P_- , resp.) and of radius r .

Proof. From Lemmas 2.3 and 2.5, we may easily see that there exists a positive number r satisfying the condition (3.1). Let γ (σ , resp.) be any normal geodesic starting from P_+ (P_- , resp.). From Lemma 2.4, taking account of Fig. I, we get

$$\begin{aligned}
 (f \circ \gamma)(t) < 0 \quad \text{for} \quad \cos^{-1} \frac{-\varepsilon}{f(P_+) - \varepsilon} \leq t \leq d(M) \\
 < 2\pi - \cos^{-1} \frac{-3\varepsilon}{f(P_+) + \varepsilon}.
 \end{aligned}$$

Similarly, we get

$$\begin{aligned}
 (f \circ \sigma)(s) > 0 \quad \text{for} \quad \cos^{-1} \frac{\varepsilon}{f(P_-) + \varepsilon} \leq s \leq d(M) \\
 < 2\pi - \cos^{-1} \frac{3\varepsilon}{f(P_-) - \varepsilon}.
 \end{aligned}$$

We suppose that $M - (B_r(P_+) \cup B_r(P_-)) \neq \emptyset$. Let $x \in M - (B_r(P_+) \cup B_r(P_-))$ and γ (σ , resp.) be the minimal geodesic from P_+ (P_- , resp.) to $x = \gamma(t_0)$ ($x = \sigma(s_0)$, resp.). Then, by the choice of x , we see that

$$r \leq t_0 < 2\pi - \cos^{-1} \frac{-3\varepsilon}{f(P_+) + \varepsilon}$$

and

$$r \leq s_0 < 2\pi - \cos^{-1} \frac{3\varepsilon}{f(P_-) - \varepsilon}.$$

Thus, we have

$$f(x)=(f \circ \gamma)(t_0) < 0 \quad \text{and} \quad f(x)=(f \circ \sigma)(s_0) > 0,$$

which is a contradiction.

Q. E. D.

LEMMA 3.2. *Suppose that r is a positive number satisfying the condition in Lemma 3.1, and γ any normal geodesic starting from P_+ . Then there is a unique point x on γ such that $d(P_+, x)=d(P_-, x)<r$.*

Proof. We put

$$h(t)=d(P_+, \gamma(t))-d(P_-, \gamma(t)) \quad \text{for} \quad 0 \leq t \leq \pi - \sin^{-1} \frac{2\varepsilon}{f(P_+) + \varepsilon}.$$

Then, by Lemma 3.1, we get

$$\begin{aligned} h(0) &= -d(P_-, P_+) < 0, \\ h\left(\pi - \sin^{-1} \frac{2\varepsilon}{f(P_+) + \varepsilon}\right) &= \pi - \sin^{-1} \frac{2\varepsilon}{f(P_+) + \varepsilon} \\ &\quad - d\left(P_-, \gamma\left(\pi - \sin^{-1} \frac{2\varepsilon}{f(P_+) + \varepsilon}\right)\right) > 0. \end{aligned}$$

Thus, by the intermediate value theorem, there is a point x such that $d(P_+, x)=d(P_-, x)$. First, we assume that $d(P_+, x)>r$. Then, by Lemma 3.1, we see that $d(P_-, x)<r$, which contradicts to $d(P_+, x)=d(P_-, x)$. Thus, $d(P_+, x)<r$. Next, we assume that there exist x_1, x_2 on γ ($x_1 \neq x_2$) such that $d(P_+, x_i)=d(P_-, x_i)$, $i=1, 2$. Moreover, we may assume that $d(P_+, x_1)<d(P_+, x_2)$. Then, we get

$$\begin{aligned} (3.2) \quad d(P_-, x_2) &= d(P_+, x_2) = d(P_+, x_1) + d(x_1, x_2) \\ &= d(P_-, x_1) + d(x_1, x_2). \end{aligned}$$

Let σ be the minimal geodesic from x_1 to P_- . Since $r < \pi - \sin^{-1} \frac{2\varepsilon}{f(P_+) + \varepsilon}$, there exists a unique minimal geodesic from x_1 to x_2 . Furthermore, by (3.2), we see that $\sigma \cup (-\gamma|_{[x_1, x_2]})$ is a minimal geodesic from x_2 to P_- , and hence $\sigma \cup (-\gamma|_{[x_1, x_2]}) = -\gamma$. Therefore, it follows that $P_+ = P_-$. But, this is a contradiction.

Q. E. D.

Now, we are in a position to prove Theorem A. Let S^n be an n -dimensional unit sphere, and (p, q) ($p, q \in S^n$) be an antipodal pair. Let $I: S_p^n \rightarrow M_{P_+}$ be a linear isometry. By Lemma 3.2, there is a real number t_0 such that $d(P_+, \exp_{P_+}(t_0 v)) = d(P_-, \exp_{P_+}(t_0 v))$. Then we put $k(v) = t_0 v \in M_{P_+} - \{0\}$. By means of $k(v)$, we may define a map $\phi: S^n \rightarrow M$ by

$$(3.3) \quad \phi(x) = \begin{cases} P_+, & \text{for } x = p, \\ \exp_{P_+} \left(\frac{2\tilde{d}(x, p)}{\pi} (k \circ I \circ \exp_p^{-1}(x)) \right), & \text{for } x \in \overline{B_{\pi/2}(p)} - \{p\}, \\ \exp_{P_-} \left(\frac{2\tilde{d}(x, q)}{\pi} (\exp_{P_-}^{-1} \circ \exp_{P_+} \circ k \circ I \circ \exp_p^{-1}(x)) \right), & \text{for } x \in \overline{B_{\pi/2}(q)} - \{q\}, \\ P_-, & \text{for } x = q, \end{cases}$$

where $\tilde{d}$ denotes the distance on S^n .

By the similar arguments as those of [2] (Theorem 6.1), we may show that ϕ is continuous, injective and surjective, and hence ϕ is a homeomorphism from S^n to M by the compactness of S^n and M . This completes the proof of Theorem A.

Lastly, we give an outline of the proof of Proposition B. Let f be a first eigen function of the Laplace operator Δ of (M, g) satisfying the condition $(**)$ in Proposition B. Then, by Courant's nodal domain theorem ([3]), we see that the number of the connected components of $M - f^{-1}(0)$ is 2. From Lemma 3.1, we see that f is negative (positive, resp.) on $(B_\alpha(P_+))^c$, $\alpha = \cos^{-1} \frac{-\varepsilon}{f(P_+) - \varepsilon}$ ($(B_\beta(P_-))^c$, $\beta = \cos^{-1} \frac{\varepsilon}{f(P_-) + \varepsilon}$, resp.), and hence $M = B_r(P_-) \cup B_r(P_+)$. The rest of the proof is similar as above.

Remark. Here we give a remark on the sign of $f(P_-)$ in the Theorem A. By (2.5) for $t = \pi$ we get $(f \circ \gamma)(\pi) < -f(P_+) + 2\varepsilon$. Therefore, $f(P_+) > 0$ and $\varepsilon < f(P_+)/2$ imply $f(P_-) < 0$.

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DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY